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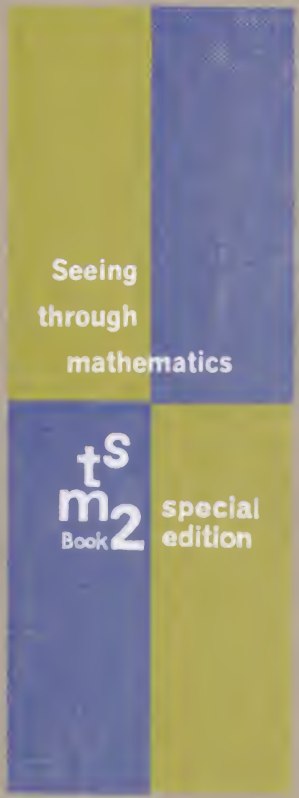

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Seeing through mathematics

special edition

Book 2

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This is the eighth book of
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Welcome back! We hope that what you learned in *Seeing Through Mathematics 1* has made you eager to learn more mathematics. Each of you has probably become absorbed in a picture puzzle on some occasion, and you know the kind of satisfaction you feel when you are able to fit a new piece into the puzzle. In a sense, the study of mathematics is somewhat like putting together a complicated picture puzzle. Learning a new mathematical idea that "fits in" with—and perhaps makes clearer—ideas that you studied earlier gives you the same sense of accomplishment as finding another piece for a puzzle. In the same way that you get a more complete picture as you put more pieces in the puzzle, you get a more complete "picture" of mathematics as you learn more mathematics.

Sometimes, when a mathematician encounters a new idea, it does not seem to fit in with earlier ideas. It is then very important for him to determine whether the new idea can be expressed in such a way that it does fit in, or whether he must change some of his previous ideas. This is one reason why the mathematician finds the idea of a *set* so useful. By using the set idea, he can often see relationships among ideas that earlier may have seemed unrelated. In this way, he can enlarge and unify his picture of mathematics.

In *Book 2*, you will begin by studying geometry involving measurement. You will then

review and increase your knowledge of simple and compound conditions in one, two, and three variables. You will also review the use of such conditions in problem solving. Next you will study again some new geometric ideas. Then you will study two new number systems —the system of rational numbers and the system of real numbers. You will study the operations and their properties for these new sets of numbers, and you will learn how to solve problems that involve these numbers.

Following the work on number systems, you will study direct and indirect measurement, and you will learn how to solve problems that involve measures. In the next unit, you will study certain conditions, called formulas, that are useful for solving many problems in science, business, and geometry. In the last unit of *Book 2*, you will be introduced to probability and statistics.

In some of the early lessons of *Book 2*, you will find "Reminders" to ideas that you studied earlier. Often these reminders will contain all the information that you need. But, sometimes you may need to turn to the lesson, often in *Book 1*, mentioned in the reminder, to help you recall an idea. In *Book 2*, blue question marks label certain exercises, responses for which are found in the back of the book.

We have again tried to avoid *telling* you about mathematics. Instead, we have attempted to *ask* you questions that will help you discover ideas for yourself because we know that this makes mathematics more interesting and meaningful. After all, putting a puzzle together isn't much fun or much of a challenge if someone else tells you where the pieces belong. We hope that the ideas in *Book 2* will help you fit several more pieces into your picture of mathematics.

THE AUTHORS

Unit 8

Geometry and measurement

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Measure of a segment

In unit 1 of *Book I*, you compared segments by using the ideas of congruence, of "less than," and of "greater than." Now you will compare segments by measuring them.

A You can use a number line like the one represented in D1 to help you think about measuring segments. What number is associated with point A? With point B?

B $\overline{BD} \cong \overline{AB}$ and $\overline{DF} \cong \overline{AB}$. What number is associated with point D? With point F?

The number associated with a point in a number line is the *coördinate of the point*. The number $\frac{3}{2}$, for example, is the coördinate of point C in the number line represented in D1. The number 2 is the coördinate of point D.

C Look again at D1. What is the coördinate of point A? Of point E? Of point G?

D What letters name the endpoints of $\overline{CF}$? Are the endpoints of $\overline{CF}$ also the endpoints of $\overline{FC}$?

E What are the coördinates of the endpoints of $\overline{CF}$?

F Is $1\frac{1}{2}$ the difference of 3 and $\frac{3}{2}$?

The difference of the coördinates of F and C is the *measure* of $\overline{CF}$. The difference of the coördinates of F and C is $1\frac{1}{2}$. Therefore, $1\frac{1}{2}$ is the measure of $\overline{CF}$. The measure of a segment is the difference of the coördinates of the endpoints of the segment. The measure of a segment is the same as the *distance* between the endpoints of the segment.

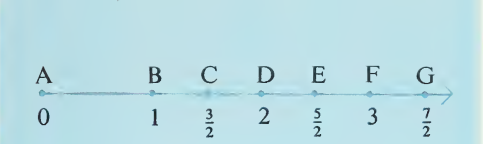
G To find the measure of a segment, you subtract the lesser coördinate from the greater coördinate. Look again at D1. What is the measure of $\overline{DG}$? Of $\overline{BE}$? Of $\overline{AE}$? Of $\overline{AF}$? Tell how you obtained each measure.

H What is the measure of $\overline{FE}$? Name three other segments, each with a measure of $\frac{1}{2}$.

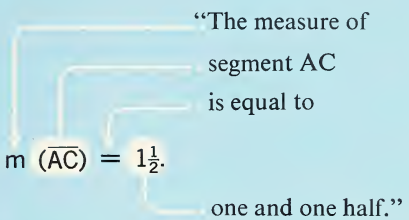
We can use a special symbol to refer to the measure of a segment. D2 shows a way to write and read a sentence about the measure of $\overline{AC}$ in which this special symbol is used.

I Explain why $m(\overline{AC})$ is the same number as $m(\overline{CA})$.

J You know that $\overline{BD} \cong \overline{AB}$ and that $\overline{DF} \cong \overline{AB}$. Is $m(\overline{BD}) = m(\overline{AB})$? Is $m(\overline{DF}) = m(\overline{AB})$? Notice that the symbol for congruence is used in sentences about segments, but the symbol



D1



D2

co ör di nate of a point (kō ôr′də nit). The number associated with a point in a number line. If $\frac{4}{3}$ is associated with point Q, then $\frac{4}{3}$ is the coördinate of point Q.

meas ure of a segment (mezh′ər). The difference of the coördinates of the endpoints of a segment. If the coördinate of S is $\frac{5}{3}$ and the coördinate of R is $\frac{1}{3}$, then the measure of $\overline{RS}$ is $\frac{5}{3} - \frac{1}{3}$, or $\frac{4}{3}$. The measure of $\overline{SR}$ is also $\frac{4}{3}$.

for equality is used in sentences about the measures of the segments.

K What is the measure of $\overline{CE}$? Of $\overline{EG}$? Is $m(\overline{CE}) = m(\overline{EG})$?

Since the measures of segments CE and EG are equal, $\overline{CE} \cong \overline{EG}$.

From exercises J and K you have seen how the measures of congruent segments are related. The properties that relate congruent segments and their measures are expressed below.

If two segments are congruent, then their measures are equal.

If the measures of two segments are equal, then the segments are congruent.

These properties imply, of course, that the same segment has been used to locate the points whose coördinates are 1, 2, 3, and so on.

L Look again at D1. How do you know that $\overline{BF} \cong \overline{CG}$? How do you know that $\overline{CG} \cong \overline{AD}$?

M Suppose that $\overline{KL}$ and $\overline{QR}$ are any two segments. If $\overline{KL} \cong \overline{QR}$, is $m(\overline{KL}) = m(\overline{QR})$? How do you know?

N Now look at the picture of a number line in D3. Is R between Q and S? Decide whether $\overline{QR}$ is less than or greater than $\overline{QS}$.

O What is the measure of $\overline{QR}$? Of $\overline{QS}$?

P Is $m(\overline{QR}) < m(\overline{QS})$?

Q Is T between Q and S? Decide whether $\overline{QT}$ is less than or greater than $\overline{QS}$.

R What is the measure of $\overline{QT}$? Of $\overline{QS}$? Is $m(\overline{QT}) > m(\overline{QS})$?

S What is the measure of $\overline{SU}$? Of $\overline{SV}$? Is $m(\overline{SU}) < m(\overline{SV})$? Is $\overline{SU} < \overline{SV}$?

T What is the measure of $\overline{RU}$? Of $\overline{RT}$? Is $m(\overline{RU}) > m(\overline{RT})$? Is $\overline{RU} > \overline{RT}$?

Now you know how the ideas of "less than" and "greater than" for segments can be related to the ideas of "less than" and "greater than" for measures of segments. These properties are expressed as follows.

unit segment (ū′nit). A segment that has a measure of 1. If the coördinate of D is $6\frac{1}{2}$ and the coördinate of E is $7\frac{1}{2}$, the measure of $\overline{DE}$ is $7\frac{1}{2} - 6\frac{1}{2}$, or 1. $\overline{DE}$ is therefore a unit segment.

For any two segments AB and CD, if $\overline{AB} < \overline{CD}$, then $m(\overline{AB}) < m(\overline{CD})$.

For any two segments AB and CD, if $m(\overline{AB}) < m(\overline{CD})$, then $\overline{AB} < \overline{CD}$.

U Look again at D3. Is $\overline{QS} < \overline{UX}$? How do you know?

So far in this lesson you have learned what is meant by the measure of a segment, and you have used this information to compare segments. In the following exercises you will see how the measure of a given segment can be used to find the measures of other segments.

A Look at the picture of a number line in D4. What is the coördinate of H? What is the coördinate of K? What is the measure of $\overline{HK}$?

B What is $m(\overline{KN})$? Is $\overline{KN} \cong \overline{HK}$? How do you know?

$\overline{HK}$ and $\overline{KN}$ each have a measure of 1. Each segment is a *unit segment*. Any segment that has a measure of 1 is a unit segment.

C How do you know that $\overline{IL}$, represented in D4, is a unit segment? Name two other unit segments represented in D4.

Now look at D5. Suppose that you want to compare the measure of segment AB with the measure of the unit segment MN. You first assign coördinates to points in $\overrightarrow{AB}$ so that each unit segment in $\overrightarrow{AB}$ is congruent to $\overline{MN}$.

D $\overrightarrow{AB}$ and $\overline{MN}$ are represented again in D6. First assign 0 to point A. What is the coördinate of point A? Next, locate point C so that $\overline{AC} \cong \overline{MN}$. What is the coördinate of point C?

E Then locate points D, E, F, and G so that each of the segments CD, DE, EF, and FG is congruent to $\overline{AC}$. For each of these points, name the coördinate.

F How do you know that $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FG}$ are unit segments?

G What is the measure of $\overline{AF}$? What is the measure of $\overline{AG}$?

H Is 4 the coördinate of B? Is 5 the coördinate of B? Explain your answers.

I Is $m(\overline{AB}) > m(\overline{AF})$? Is $m(\overline{AB}) < m(\overline{AG})$? Explain your answers.

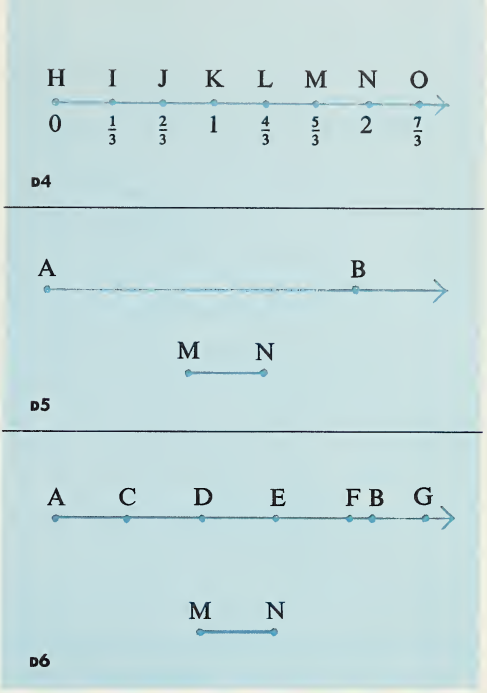
Look again at D6. Notice that segment AG is separated into the congruent segments AC, CD, DE, EF, and FG. Notice also that these segments do not overlap and that their union is the segment AG. We say that segment AG is *subdivided* into five congruent segments. A given segment can be subdivided into any number of congruent segments.

Suppose that, when you subdivide $\overline{FG}$ into ten congruent segments, point B is the right endpoint of the third segment. Since $\overline{FG}$ is a unit segment, .1 is the measure of each of the ten congruent segments. Since point B is the right endpoint of the third segment, .3 is the measure of $\overline{FB}$.

J What is the union of $\overline{AF}$ and $\overline{FB}$? What is the sum of $m(\overline{AF})$ and $m(\overline{FB})$?

K Is $m(\overline{AB}) = m(\overline{AF}) + m(\overline{FB})$? What is $m(\overline{AB})$?

L What is the coördinate of B?



M Now suppose that you subdivide the unit segment FG into eight congruent segments and that B is the right endpoint of the second segment. What is the coördinate of B? What is the measure of $\overline{AB}$?

N Think about exercises J, K, and L. Since B was an endpoint of one of the ten congruent segments included in $\overline{FG}$, you were able to find the coördinate of B. Now suppose that B is contained in one of the ten congruent segments, but is not an endpoint. What would you do to find the coördinate of B?

You have learned how to use a unit segment to help you find coördinates of points in a given segment. In other words, you have learned to find the measure of a given segment by comparing it with another segment. Now you will see how you can obtain different measures for the same segment by selecting different segments to use as unit segments.

O Look at D7. When you use $\overline{ST}$ as a unit segment to find the measure of $\overline{DH}$, the coördinate of D is 0 and the coördinate of H is 3. What is the measure of $\overline{DH}$?

P When you use $\overline{XY}$ as a unit segment, the coördinate of D is 0 and the coördinate of H is 1.8. What is the measure of $\overline{DH}$?

Q Explain why the measure of $\overline{DH}$ that you obtained for exercise O is different from the measure that you obtained for exercise P.

Now you can see that the measure of a given segment depends on the unit segment with which you are comparing the given segment. The measure of a given segment depends upon what unit segment you choose.

On your own

Suppose that the numbers named in each of exercises 1 through 9 are the coördinates of the endpoints of a segment. Find the measure of each segment.

1 $8\frac{1}{3}$, 9	4 $4\frac{1}{2}$, $4\frac{9}{16}$	7 8.93, 4.2
2 $3\frac{2}{3}$, $\frac{1}{2}$	5 2, $7\frac{3}{8}$	8 0, 10
3 7.8, 16	6 $\frac{1}{5}$, 0	9 $2\frac{1}{5}$, $8\frac{5}{10}$

Use the number line represented in D8 to help you answer the questions in exercises 10 through 23.

10 What are the coördinates of the endpoints of $\overline{OS}$? Of $\overline{TU}$?

11 $\overline{TV} \cong \overline{QT}$. What is the coördinate of V?

12 $\overline{MN} \cong \overline{OP}$. What is the coördinate of N?

13 What is $m(\overline{QS})$? What is $m(\overline{UV})$? What is $m(\overline{PT})$?

14 Name two segments included in $\overrightarrow{MV}$, each of which is congruent to $\overline{TU}$.

15 Name three segments included in $\overrightarrow{MV}$, each of which is congruent to $\overline{OP}$.

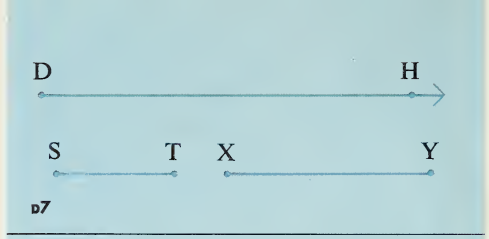
Find the sum expressed in each of exercises 16 through 19.

16 $m(\overline{PQ}) + m(\overline{TV})$

17 $m(\overline{NP}) + m(\overline{ST})$

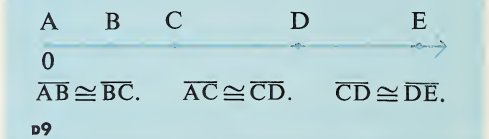
18 $m(\overline{RT}) + m(\overline{UV})$

19 $m(\overline{OP}) + m(\overline{QT})$



D7

D8



$\overline{AB} \cong \overline{BC}$. $\overline{AC} \cong \overline{CD}$. $\overline{CD} \cong \overline{DE}$.

D9

20 Is $m(\overline{PQ}) < m(\overline{RT})$?

21 Which is greater, $m(\overline{OQ})$ or $m(\overline{ST})$?

22 Is $m(\overline{NP}) + m(\overline{QR}) < m(\overline{MN}) + m(\overline{NO})$?

23 Which segment represented in D8 has the same measure as $m(\overline{MQ}) + m(\overline{QU}) + m(\overline{UV})$?

Use the number line represented in D9 and the sentences in the display to help you answer the questions in exercises 24 through 30.

24 If the coördinate of B is 1, what is the coördinate of C? Of D? Of E?

25 If the coördinate of C is 1, what is the measure of $\overline{AD}$? Of $\overline{DE}$? Of $\overline{BD}$?

26 If the coördinate of D is 1, what is the coördinate of C? Of E? Of B?

27 If $m(\overline{AB}) = 2$, what is the measure of $\overline{AC}$? Of $\overline{AE}$? Of $\overline{BD}$?

28 If $m(\overline{AE}) = 1$, what is the coördinate of C? Of D? Of B?

29 If $m(\overline{AC}) + m(\overline{CE}) = 12$, what is the coördinate of C? Of E?

30 If $m(\overline{BC}) = 1$, what is the coördinate of B?

Practical measurement

In the last lesson you used a unit segment to find the measure of a given segment. You learned that you can obtain different measures for a given segment by using different unit segments.

The edge of a board, a taut string, a common pin, a piece of wire—these and many other physical objects suggest segments. You can use what you know about measurement of segments to help you find the measure of a physical object that suggests a segment.

A D1 shows a picture of a springboard. Notice that one edge of the board is shown in red. Now imagine that you are measuring the actual board. To find the measure of the red edge, you must choose a unit segment to compare with the segment represented by the red edge. Suppose that each student in your class were to use a different pencil to represent a unit segment. Do you think that all of the students would obtain the same measure for the edge of the board? Explain your answer.

B If each student were to use one of his shoes to represent a unit segment, would all of the students obtain the same measure for the edge of the board? Explain your answer.

It is clear that the measure of the length of a physical object depends upon the unit segment with which it is compared. If each student in your class used a different pencil as a unit segment to find the measure of the length of a springboard, one student might obtain a measure of 17, one a measure of 21, and so on.

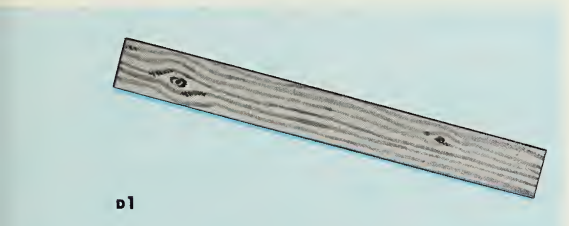
Before the measure of the length of a physical object has much meaning, a definite unit segment must be chosen. For convenience, most people have agreed upon certain units. Such units are known as *standard units*. You are already familiar with many standard units, such as the inch, the foot, the yard, and the mile. In this lesson you will use these familiar units to learn more about the measure of the length of objects.

The *length* of any object that suggests a segment is the measure of the object in terms of a standard unit. If 10 is the measure in feet of the edge of the springboard pictured in D1, then the length of the edge is 10 feet. You can also say, "The edge is 10 feet long."

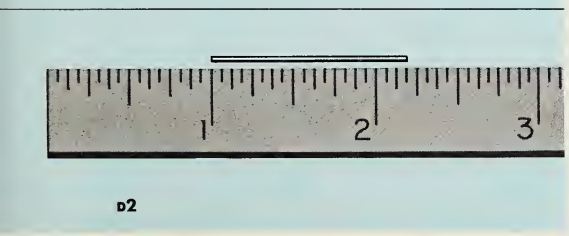
Now you will use a measuring stick, or rule, to find the lengths of objects.

C Look at D2. A piece of wire and part of a rule are pictured in the display. Think of the wire as suggesting a segment. Think of each mark on the rule as suggesting a point in a number line. What point does the upper left corner of the rule suggest?

D The length of the rule between the left edge and the mark labelled "1" is 1 inch. This length



D1



D2

was also used to locate the marks labelled "2" and "3". What is the length of the rule between the mark labelled "1" and the mark labelled "2"?

E Each inch of the rule has been marked off into parts of the same length. Into how many parts has each inch of the rule been marked off? Think of each of these parts as a unit. Then $\frac{1}{16}$ of an inch is the length of the unit.

F How many points are between any two points in a rational-number line? Can you actually mark off an inch into infinitely many parts?

It is impossible to measure a physical object without a certain amount of error. The amount of error of a measurement is controlled by the nature of the object being measured, the measuring instrument used, and the person doing the measuring. Throughout this unit, however, we will treat measurement of physical objects as we treated measurement in lesson 98. We will imagine that it is possible to find the exact measure of an object by comparing the length of the object with a standard unit.

D2

In *Seeing Through Mathematics, Book 2*, you will learn much more about practical measurement. You will learn what precision and accuracy in measurement mean. You will also learn that, practically speaking, you cannot make an exact measurement.

G Look again at D2. The left end of the wire is directly above the mark for 1. The right end of the wire appears to be directly above which mark?

H What is the measure of the piece of wire?

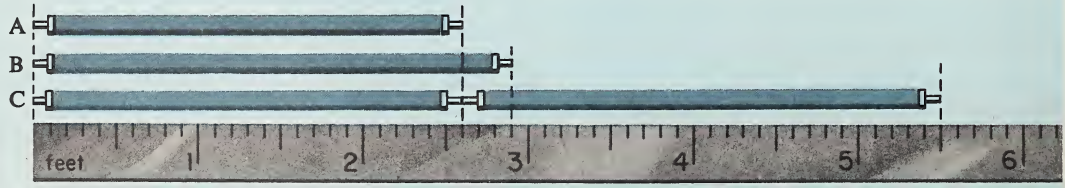
? How do you know?

I What unit did you use to find the measure of the length of the piece of wire? Is $1\frac{3}{16}$ inches the length of the wire?

In the following exercises you will use standard units to help you find sums and differences of measures.

A Look at D3, which shows pictures of some curtain rods and a rule. In the picture of the rule, the length between the left edge of the rule and the mark labelled "1" represents the length of a standard unit of 1 foot. The rule pictured is marked off into lengths of 1 foot by the marks labelled with the numerals 1 through 6.

B Each foot of the rule is marked off into parts of equal length. Into how many parts is each foot marked off? What is the length of the rule between any two consecutive marks? Think of .1 of a foot as the length of the unit.



D3

C How do you know that $2.6 - 0$, or 2.6, is the measure of rod A? What is the length of rod A?

D How do you know that 2.9 is the measure of rod B? What is the length of rod B?

E Now look at picture C in D3. Rods A and B have been placed end to end. Is the right end of the second rod directly above the mark for 5.5 on the rule? What is the total measure of the rods? What is the total length of the two rods?

F You know that the measure of rod A is 2.6 and the measure of rod B is 2.9. What is the sum of these measures? Is this the same as the total measure of both rods?

G Suppose that you cut $1\frac{1}{2}$ yd. of ribbon from a piece of ribbon that has a length of $7\frac{2}{3}$ yd. To find the length of the piece of ribbon that you have left, you will find the difference of two measures. What is the difference of $7\frac{2}{3}$ and $1\frac{1}{2}$? What is the length of the ribbon that is left?

Now you know that you can find the sum or the difference of two measures. When you are asked to find the sum or the difference of two lengths, you find the sum or the difference of the two measures. This sum or difference in terms of a standard unit is the sum or difference of the lengths.

Find the sum of the lengths expressed in each of exercises H through M.

H 7 ft., $4\frac{1}{2}$ ft.

I .4 in., 9.2 in.

J $3\frac{3}{4}$ mi., $8\frac{1}{2}$ mi.

K 2.9 ft., 8.15 ft.

L $6\frac{1}{3}$ yd., $7\frac{5}{8}$ yd.

M 10.75 in., 3 in.

Find the difference of the first length and the second length expressed in each of the exercises below.

N $5\frac{1}{2}$ yd., $2\frac{1}{3}$ yd.

O $14\frac{1}{16}$ in., $5\frac{3}{8}$ in.

P 9 ft., $6\frac{5}{12}$ ft.

Q 98.25 mi., 16.75 mi.

R 20 yd., $14\frac{5}{8}$ yd.

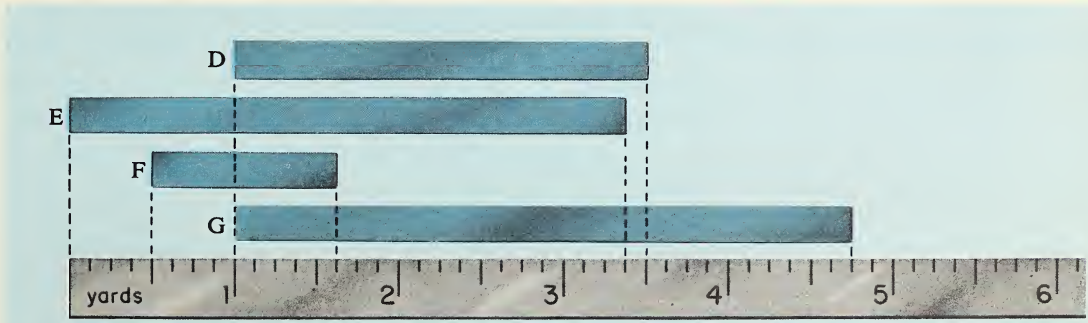
S 9 in., 7.5 in.

In this lesson you have used standard units to find the lengths of objects. You have also learned how to find sums and differences of lengths.

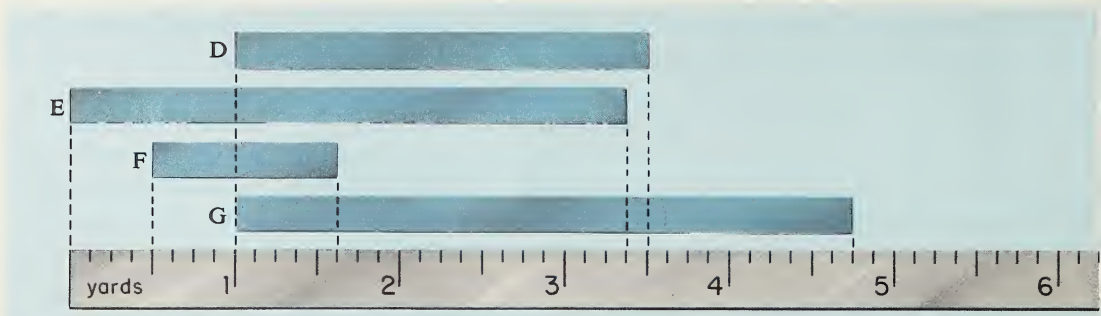
On your own

Use D4 as you answer the questions in exercises 1 through 12. In the picture of the rule, the length between the left edge of the rule and the mark labelled "1" represents the length of a standard unit of 1 yard. The rule pictured is marked off into lengths of 1 yard by the marks labelled with the numerals 1 through 6. Each yard of the rule is marked off into 8 parts of equal length.

1 What is the length of a part of the rule that is between any two consecutive marks?



D4



D4

2 The left end of ribbon D is directly above which mark on the rule? The right end of the ribbon is directly above which mark?

3 What is the measure of ribbon D? How long is ribbon D?

4 How long is ribbon E?

5 How long is ribbon F?

6 How long is ribbon G?

7 Find the sum of the measures of the four ribbons shown in D4. What is the sum of the lengths of these four ribbons?

8 What is the difference of the lengths of ribbon G and ribbon D?

9 What is the difference of the lengths of ribbon G and ribbon E?

10 What is the sum of the lengths of ribbon D and ribbon E?

11 What is the sum of the lengths of ribbon F and ribbon G?

12 What is the difference of the length you found for exercise 10 and the length you found for exercise 11?

Find the sum of the lengths expressed in each of exercises 13 through 17.

13 $3\frac{3}{8}$ in., $5\frac{1}{4}$ in.

14 $7\frac{2}{3}$ yd., $5\frac{5}{12}$ yd., $6\frac{1}{2}$ yd.

15 11.2 mi., 8.5 mi., 2.1 mi.

16 $10\frac{1}{2}$ ft., $6\frac{2}{3}$ ft.

17 $1\frac{1}{3}$ mi., $1\frac{1}{5}$ mi., $9\frac{3}{10}$ mi.

Find the difference of the first length and the second length expressed in each of exercises 18 through 27.

18 $3\frac{1}{2}$ yd., $2\frac{2}{3}$ yd.

19 7.6 ft., 3.0 ft.

20 $5\frac{3}{16}$ in., $4\frac{3}{4}$ in.

21 8 mi., 6.3 mi.

22 $4\frac{1}{4}$ ft., $3\frac{1}{4}$ ft.

23 6 yd., $4\frac{1}{12}$ yd.

24 $1\frac{1}{2}$ rd., $\frac{3}{4}$ rd.

25 17 mi., $8\frac{7}{10}$ mi.

26 18.4 in., 6.9 in.

27 100 ft., $86\frac{2}{3}$ ft.

KEEPING SKILFUL

For each exercise below, find the sum, difference, product, or quotient.

1 $.725 + 9.3 + .54$

2 $78\frac{3}{4} - 59\frac{1}{2}$

3 $34\frac{6}{7} + 49\frac{3}{7}$

4 $9.687 - 3.099$

5 $21.3 \times .54$

6 $4\frac{1}{2} \times 8\frac{2}{3}$

7 $7.21 - 5.396$

8 $41\frac{3}{4} + 62\frac{3}{20}$

9 $101\frac{5}{8} - 85\frac{1}{3}$

10 $\frac{8}{9} \div \frac{3}{4}$

11 $1\frac{3}{17} \times 6\frac{4}{5}$

12 $.163 \times 7.5$

13 $29\frac{1}{2} + 37\frac{3}{5}$

14 $416\frac{1}{7} - 396\frac{2}{3}$

15 $.098 \times 68$

16 $.3 + 76.8 + .706$

17 $5\frac{3}{4} \div \frac{11}{12}$

18 9.36×8.12

19 $3\frac{1}{2} \div 1\frac{3}{4}$

20 $596\frac{5}{6} + 185\frac{4}{9}$

21 $2\frac{5}{8} \div 9\frac{5}{12}$

22 $130\frac{1}{9} - 37\frac{4}{5}$

23 $9.002 - 1.236$

24 $20\frac{1}{3} \times 17\frac{1}{8}$

25 $4.914 \div 6.3$ (Hundredths)

26 $125.72 \div 92$ (Thousandths)

27 $85068 \div 278$ (Ones)

28 $579 \div .136$ (Tenths)

Problems involving standard units of length

In your previous work in arithmetic, you learned how certain standard units of length are related. In this lesson you will review what you know about standard units of length. You will use this knowledge to solve problems that involve more than one standard unit of length.

The table in D1 shows some of the comparisons between standard units of length. Each length named in the first column is the same as the length named opposite it in the second column.

A The table tells you that a length of 12 inches is the same as a length of 1 foot or, as we sometimes say, there are 12 inches in 1 foot. What lengths named in the table are the same as 1 yard? What lengths are the same as 1 rod? What lengths are the same as 1 mile?

Because 12 inches is the same length as 1 foot, we can use the rate pair 12/1 to represent the comparison of the two lengths. Each of the components of 12/1 refers to the same length, but, because the units of measure are different, the numbers are different.

B What rate pair can you use to represent the comparison of the number of inches in a yard with 1 yard?

C What rate pair can you use to represent the comparison of the number of feet in a yard with 1 yard? The comparison of the number of feet in a rod with 1 rod? The comparison of the number of feet in a mile with 1 mile?

LENGTH

12 inches (in.)	1 foot (ft.)
36 inches	1 yard (yd.)
3 feet	1 yard
$16\frac{1}{2}$ feet	1 rod (rd.)
5280 feet	1 mile (mi.)
$5\frac{1}{2}$ yards	1 rod
1760 yards	1 mile

D1

The length of the Lions Gate Bridge at Vancouver is about 516 yd. What is the length of the bridge in feet?

D2

D What rate pair can you use to represent the comparison of the number of yards in a rod with 1 rod? What rate pair can you use to represent the comparison of the number of yards in a mile with 1 mile?

Each rate pair that you have named has a second component of 1. The first component of each rate pair refers to the number of smaller standard units that is the same length as 1 larger standard unit.

Read the problem in D2. U = Z.

E Why will you use 3/1 in the condition for the problem?

F You are told that the length of the Lions Gate Bridge is about 516 yd. You are to find the length of the bridge in feet. Why will you use $n/516$, and not $516/n$, in the condition? To what does n refer?

G Does the sentence at the right express a condition for the problem? $3/1 \sim n/516$

H Tabulate $\{n|3/1 \sim n/516\}$. Give the answer to the problem in D2.

LENGTH

12 inches (in.)	1 foot (ft.)
36 inches	1 yard (yd.)
3 feet	1 yard
$16\frac{1}{2}$ feet	1 rod (rd.)
5280 feet	1 mile (mi.)
$5\frac{1}{2}$ yards	1 rod
1760 yards	1 mile

D1

LENGTH (METRIC)

.394 inch	1 centimetre (cm.) (sen′tə mē′tər)
1.094 yards	1 metre (m.) (mē′ tər)
.621 mile	1 kilometre (km.) (kil′ə mē′tər)

D3

In many countries the standard units of length are not inches, feet, yards, and so on. Many countries use the *metric* (met′rik) *system of measures.* Scientists throughout the world also use the metric system. You will learn about the metric system of measures later. In this lesson you will use a few metric measures.

Look at the table in D3. It shows comparisons between three of the common metric units of length and the units used every day in Canada. Each length named in the first column is approximately the same (to the nearer thousandth) as the length named opposite it in the second column.

A 1 centimetre is the same length as approximately how many inches? 1 metre is the same length as approximately how many yards? 1 kilometre is the same length as approximately how many miles?

B Read the problem in D4. U = Z. Why will you use 1.094/1 in the condition for the problem?

C Why will you use $z/100$, and not $100/z$, in the condition? To what does z refer?

D Is $1.094/1 \sim z/100$ a condition for the problem?

E Tabulate $\{z \mid 1.094/1 \sim z/100\}$. Give the answer to the problem.

F Write a sentence that expresses a condition for the problem below. Then give the answer to the problem. U = Z.

One inch is the same length as approximately how many centimetres? (Hundredths)

In some problems involving standard units of length, it is necessary first to change from one standard unit to a second standard unit and then from the second standard unit to a third. You will use a compound condition to solve this kind of problem.

Read the problem in D5. U = Z × Z.

The table in D1 does not tell you how many rods are in 1 mile. However, the table does tell you how many feet are in 1 rod and how many feet are in 1 mile. It also tells you how many yards are in 1 rod and how many yards are in 1 mile.

You are told that the length of the east boundary of the ranch is 1650 rods. First you can find either the number of feet that is the same length as 1650 rods or the number of yards that is the same length as 1650 rods. We will begin by finding the number of yards that is the same length as 1650 rods.

A Why will you use $5\frac{1}{2}/1$ in one of the simple conditions for the problem? Why will you use $x/1650$, and not $1650/x$, in this condition? For what is x a variable?

B Is $5\frac{1}{2}/1 \sim x/1650$ one of the simple conditions for the problem?

One event for men at the summer Olympic games is the 100-metre run. When a man has completed this race, approximately how many yards has he run?

D4

Mr. Ace owns a ranch in Alberta. The length of the east boundary of his ranch is 1650 rd. What is the length of the east boundary in miles? (Tenths)

D5

You can use $5\frac{1}{2}/1 \sim x/1650$ to find the number of yards that is the same length as 1650 rods. Next you must find the number of miles that is the same length as x yards.

C Does $1760/1$ represent the comparison of 1760 yards with the number of miles that is the same length? Why will you use $1760/1$ in the second condition for the problem?

You want to find a rate pair that is equivalent to $1760/1$ and whose second component is the number of miles that is the same length as x yards. Use x/y to represent this rate pair.

D To what does x refer? To what does y refer?

E Is $1760/1 \sim x/y$ the second simple condition for the problem?

F Does the sentence in red below express a compound condition for the problem in D5?

$$5\tfrac{1}{2}/1 \sim x/1650 \wedge 1760/1 \sim x/y.$$

G What number satisfies $5\frac{1}{2}/1 \sim x/1650$?

H If you replace x in $1760/1 \sim x/y$ by 9075, do you obtain $1760/1 \sim 9075/y$?

I What is the replacement for y rounded off to the nearer tenth?

J What is the length of the east boundary of the ranch in miles?

In this lesson you learned how to use conditions for equivalence to solve problems involving standard units of length.

On your own

In each of exercises 1 through 12, a length is expressed in one unit of measure, and a second unit is named in parentheses. Express the given length in terms of the unit indicated by the word in parentheses.

1 1 in. (feet)
2 1 ft. (yards)
3 20 rd. (yards)
4 52 cm. (inches)
5 18,480 ft. (miles)
6 150 m. (yards)
7 6 mi. (yards)
8 350 km. (miles)
9 1 yd. (metres)
10 1 mi. (kilometres)
11 45 in. (centimetres)
12 415 yd. (metres)

For each of the following problems, first write a sentence that expresses a condition. Then give the answer to the problem. For problems 13 through 23, the universe for the variable is Z. For problems 24 through 27, the universe is $Z \times Z$.

13 The average height of the sequoia is $2\frac{3}{4}$ times the average height of the white oak. The average height of the white oak is 100 ft. What is the average height of the sequoia?

14 Mrs. Mills bought 9 metres of cotton cloth in Mexico City. Approximately how many yards of cloth did she buy? (Tenths)

15 The diameter of one of the wheels of a certain French automobile is 40 centimetres long. The diameter is approximately how many inches long?

16 The surface of the Dead Sea is 1293 ft. below sea level. The surface of the Dead Sea is how many miles below sea level? (Hundredths)

17 The height of Angel Falls in Venezuela is .625 of a mile. What is the height in feet of Angel Falls?

18 One of the largest sharks ever caught was 163 in. long. What was the length of the shark in feet? (Tenths)

19 One of the events for men in the 1908 Olympic Games was a 5-mile run. When a man completed this race, approximately how many kilometres had he run? (Ones)

20 The highest mountain in the world is Mount Everest in the Himalayas. The summit of Mount Everest is 29,028 feet above sea level. How many miles above sea level is the summit of Mount Everest? (Tenths)

21 The average depth of the Pacific Ocean is $2\frac{2}{3}$ miles. What is its average depth in feet?

22 Dave used two kinds of wood, oak and walnut, to make a coffee table. The length of the oak board was 1 ft. 9 in. The length of the walnut board was 2 ft. 5 in. What was the total length of the boards in feet?

THINK How do you know that 1 ft. 9 in. is the same length as $1\frac{3}{4}$ ft.? How do you know that 2 ft. 5 in. is the same length as $2\frac{5}{12}$ ft.?

23 Use your answer to problem 22 to answer the following question: What was the total length of the boards in inches?

24 Jean bought $2\frac{3}{4}$ yards of cotton and $1\frac{1}{8}$ yards of silk. How many inches of material in all did she buy?

25 Mr. Wilson is going to build a road from his barn to his farmhouse. The length of the road will be $\frac{4}{5}$ of a mile. What will be the length of the road in rods?

26 Mr. Johnson had a tract of land surveyed. The surveyor told him that the length of the tract of land was 750 rd. What was the length in miles? (Hundredths)

27 Susan walks $\frac{7}{8}$ mi. to school each day. After mathematics class one day she decided to find out how many inches there are in $\frac{7}{8}$ mi. What number of inches should Susan have obtained for an answer?

APPLYING MATHEMATICS

For each of problems 1 through 15, write a sentence that expresses a condition. Then find the number that satisfies the condition and give the answer to the problem. The universe is given for each problem.

1 Rajah, a Siamese cat, weighs $6\frac{1}{2}$ lb. How many ounces does Rajah weigh? U = Z.

2 Ed sold 35 pk. of apples one day. How many bushels of apples did he sell? U = Z.

3 Mr. Andrews used $4\frac{3}{4}$ gal. of white paint to paint his house. How many quarts of white paint did Mr. Andrews use? U = Z.

4 Nick weighs 110 lb. About how many kilograms does Nick weigh? U = Z.

WEIGHT

16 ounces (oz.)	1 pound (lb.)
2000 pounds	1 ton (T.)
.035 ounce (approx.)	1 gram (g.)
2.2 pounds (approx.)	1 kilogram (kg.)

CAPACITY

8 fluid ounces (fl. oz.)	1 cup (c.)
$2\frac{1}{2}$ cups	1 pint (pt.)
2 liquid pints	1 liquid quart (qt.)
4 liquid quarts	1 gallon (gal.)
8 dry quarts	1 peck (pk.)
4 pecks	1 bushel (bu.)
.88 quart (approx.)	1 litre (l.)

TIME

60 seconds (sec.)	1 minute (min.)
60 minutes	1 hour (hr.)
24 hours	1 day (da.)
7 days	1 week (wk.)
12 months (mo.)	1 year (yr.)
365 days	1 year

Finding solutions of conditions for problems involving standard units of weight, capacity, and time

5 Kay used $2\frac{3}{8}$ cups of milk in a pudding. How many fluid ounces of milk did she use? U = Z.

6 Tom bought 24 oz. of peanuts. He bought how many pounds of peanuts? U = Z.

7 Susan picked 10 qt. of strawberries one morning. She picked how many pecks of strawberries that morning? (Hundredths) U = Z.

8 Mrs. Kane bought 1.8 kilograms of bananas at a market in Mexico City. About how many pounds of bananas did she buy? U = Z.

9 Jim spent $2\frac{1}{4}$ hr. on his homework one evening. How many minutes did Jim spend on his homework that evening? U = Z.

10 Mr. Thompson had 1.5 tons of coal in his basement. He had how many pounds of coal in his basement? U = Z.

11 Some baseball players use bats that weigh 34 oz. About how many grams does a 34-ounce bat weigh? U = Z.

12 5,921,728 tons of steel were produced in Canada in one year. Calculate how many pounds of steel were produced that year in Canada. U = Z.

13 The coffee machine in Mr. Harold's office holds 50 cups of coffee. How many quarts of coffee does the machine hold? U = Z × Z.

14 There are six 45-minute class periods at the Long School. The six class periods is the same period of time as how many hours? U = Z × Z.

15 The gasoline tank in Mr. Ryan's car will hold 56 litres. Approximately how many gallons will the tank hold? U = Z × Z.

For each exercise below, express the given units in the units indicated in parentheses.

16 1 oz. (grams)
17 1 lb. (kilograms)
18 1 qt. (litres)
19 5 da. (hours)
20 5200 lb. (tons)
21 44 lb. (kilograms)
22 3 gal. (quarts)
23 301 da. (weeks)
24 176 oz. (pounds)
25 2.5 hr. (seconds)
26 $3\frac{1}{5}$ yr. (days)
27 30 gal. (pints)

Measure of an angle

You have learned how to find the measure of a segment. In this lesson you will learn how to find the measure of an angle.

A Name the two lines represented in D1.

B In lesson 27 you learned what is meant by a ray, an angle, and a linear pair of angles. Name the rays represented in D1 that have D as an endpoint. Is $\overrightarrow{DE} \cup \overrightarrow{DF} = \angle FDE$?

C Name each angle represented in D1 that has D as a vertex. Name the sides of each angle.

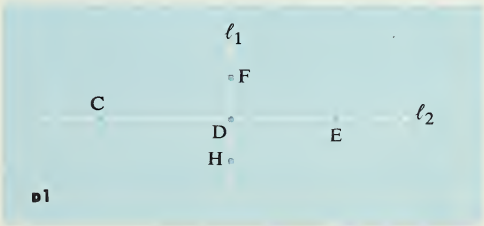
D Which point is the common vertex of $\angle FDE$ and $\angle CDF$? Which ray is the common side of $\angle FDE$ and $\angle CDF$?

E Points C, D, and E are collinear points. How do you know that $\overrightarrow{DE} \cup \overrightarrow{DC} = \ell_2$?

F Are $\angle FDE$ and $\angle CDF$ a linear pair of angles? Is $\angle FDE$ the supplement of $\angle CDF$? Is $\angle CDF$ the supplement of $\angle FDE$?

G $\angle FDE \cong \angle CDF$. Are $\angle FDE$ and $\angle CDF$ right angles? $\angle CDF \cong \angle HDC$. Is $\angle HDC$ a right angle?

H Are $\angle HDC$ and $\angle EDH$ a linear pair of angles? Is $\angle HDC$ the supplement of $\angle EDH$? Is $\angle EDH$ the supplement of $\angle HDC$? How do you know that $\angle EDH$ is a right angle?



In lesson 98 you learned how to compare segments by comparing their measures. You used a unit segment to find the measure of other segments. Now that you have reviewed some ideas about angles, you will learn how to find the measure of an angle.

Look at D2. Several different rays with a common endpoint are represented in the display. $\angle SMR$, $\angle TMS$, $\angle UMT$, $\angle VMU$, and $\angle WMV$ are each congruent to $\angle RMQ$.

A Just as you can associate numbers with endpoints of segments, you can associate numbers with the rays that form angles. The number 0 is associated with $\overrightarrow{MQ}$. What number is associated with $\overrightarrow{MT}$?

B The number 2 is associated with which ray? The number 6 is associated with which ray?

C Name the angle formed by the union of $\overrightarrow{MV}$ and $\overrightarrow{MT}$. What is the difference of the number associated with $\overrightarrow{MV}$ and the number associated with $\overrightarrow{MT}$?

The difference of 5 and 3 is the *measure of the angle* VMT. Thus, $5 - 3$, or 2, is the measure of $\angle VMT$. The measure of an angle is the difference of the numbers associated with the sides of the angle. The difference is obtained by subtracting the lesser number from the greater number.

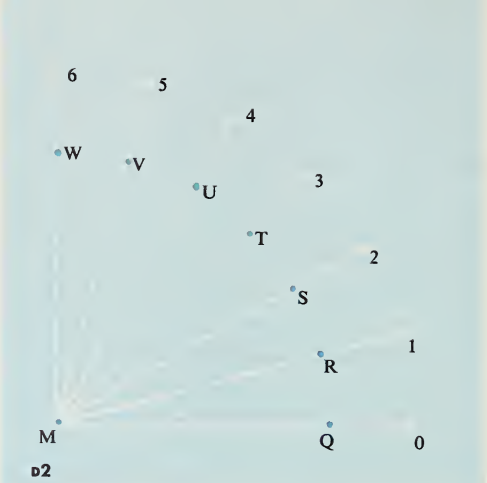
D What number is associated with $\overrightarrow{MQ}$? With $\overrightarrow{MU}$? What is the measure of $\angle QMU$?

E $\angle WMQ$ is a right angle. Is $6 - 0$, or 6, the measure of $\angle WMQ$?

F What is the measure of $\angle RMQ$? Of $\angle UMT$?

Notice that both $\angle RMQ$ and $\angle UMT$ have a measure of 1. Each of these angles is a *unit angle*. An angle that has a measure of 1 is a unit angle.

G What other angles represented in D2 are unit angles?



D2

measure of an angle. The difference of the numbers associated with the sides of the angle. If 9 is associated with ray AC and 3 is associated with ray AB, then $9 - 3$, or 6, is the measure of $\angle CAB$.

unit angle. An angle whose measure is 1.

H What is the measure of $\angle RMS$? Of $\angle SMT$? Of $\angle RMT$? Is the measure of $\angle RMT$ equal to the sum of the measures of $\angle RMS$ and $\angle SMT$?

I What is the measure of $\angle TMV$? Of $\angle VMW$? Of $\angle TMW$? Is the measure of $\angle TMW$ equal to the sum of the measures of $\angle TMV$ and $\angle VMW$?

From the exercises above, you can see how you use a unit angle to determine the measures of other angles. The unit of measurement is the unit angle. The process of measurement compares a given angle with the unit angle.

J Now look at D3. Points M, T, and A are collinear. Name the rays represented in D3. Name the number associated with each of these rays.

K Name the unit angle represented in D3.

L What is the measure of $\angle ATB$? Of $\angle MTK$? Of $\angle HTJ$? How do you know?

M $\angle BTE \cong \angle HTK$. What is the measure of each of these angles?

N What is the measure of $\angle DTF$? Of $\angle FTH$? ? Do you think these two angles are congruent?

$\angle DTF \cong \angle FTH$. The measures of congruent angles are equal just as the measures of congruent segments are equal. The properties that relate congruent angles and their measures are expressed below.

If two angles are congruent, then their measures are equal.

If the measures of two angles are equal, then the angles are congruent.

These properties imply, of course, that the same unit angle has been used to find the measures of the two angles.

O Name four angles represented in D3, each of which is congruent to $\angle DTC$. Name three angles each of which is congruent to $\angle ETI$.

P Is the measure of $\angle DTI$ equal to the sum of the measures of $\angle DTF$ and $\angle FTI$?

Q What is the sum of the measures of $\angle GTH$ and $\angle HTK$? Is this sum equal to the measure of $\angle GTK$?

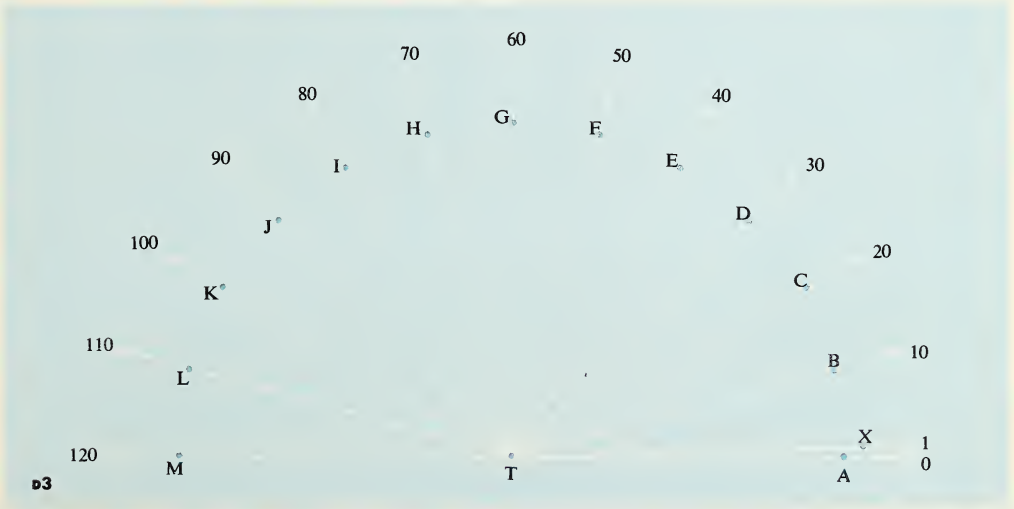
R $\angle ATG$ is a right angle. What is the measure of $\angle ATG$? Name three other angles represented in D3 each of which is congruent to $\angle ATG$.

S Look again at D2. Which angle represented in D2 is a right angle? What is the measure of this angle?

T Do you think that the unit angle you used ? on page 18 is congruent to the unit angle represented in D3? Explain your answer.

The right angle represented in D2 and the right angle represented in D3 have different measures because different unit angles are used. Next you will learn about a standard unit for measuring angles.

As you have learned so far in this lesson, different unit angles can be used in measuring angles. It is customary to choose a unit angle that makes 90 the measure of a right angle. This standard unit of angle measure is the



degree. The measure in degrees of a unit angle is 1. In this book we will use the degree to measure angles. As you continue to study mathematics, you will learn about another unit for measuring angles.

A Look at D4. Points D, C, and A are collinear. $\angle ACB$ is a right angle. What is the measure of $\angle ACB$?

B A unit angle is represented in D4. Name the unit angle.

You know that a right angle is defined as an angle that is congruent to its supplement. You also know that we have chosen a unit angle that makes 90 the measure of a right angle. We have called this unit angle the degree. Thus, the measure of a right angle in degrees is 90.

C What is the measure in degrees of $\angle ACL$? Of $\angle GCQ$? Of $\angle BCR$?

D Look again at D4. What is the measure in degrees of $\angle DCB$? Is $\angle DCB$ a right angle?

E What is the measure in degrees of $\angle LCT$? Is $\angle LCT$ a right angle? What is the measure in degrees of $\angle GCO$? Is $\angle GCO$ a right angle?

F Name three other right angles represented in D4.

G What is the sum of the measures of $\angle ACB$ and $\angle DCB$?

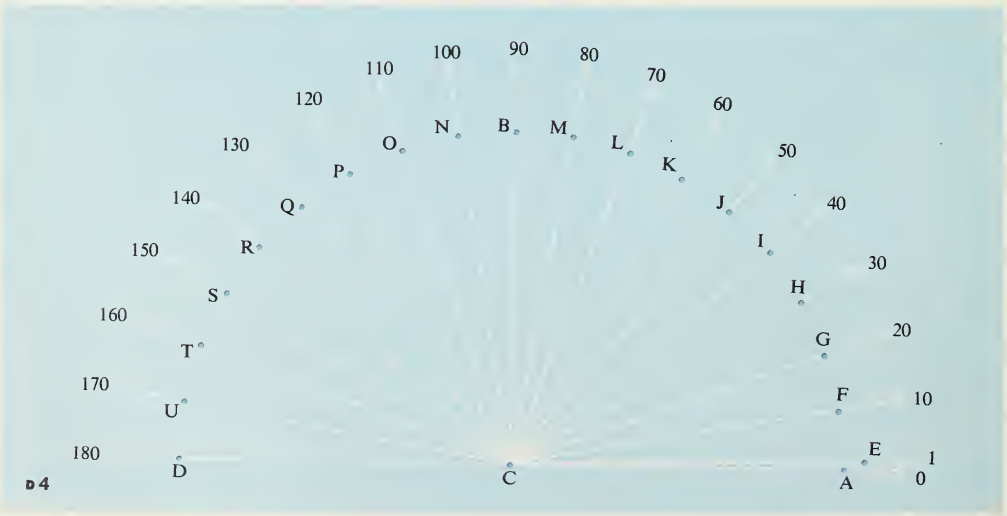
H You know that an angle is the union of two rays that have a common endpoint, but that do not form a line. Is $\overrightarrow{CA} \cup \overrightarrow{CD}$ an angle?

I What is the measure of $\angle ECD$? Of $\angle ACU$? Is the measure of each of these angles less than 180?

J Explain why the measure in degrees of an angle must be less than 180. Explain why the measure in degrees of an angle must be greater than 0.

From now on, when we give the measure in degrees of an angle, you know that the number we use will be a rational number between 0 and 180. The measure in degrees of an angle can be a number like $15\frac{1}{2}$, or $\frac{2}{3}$, or 178. But the measure cannot be 0, or 180, or any number greater than 180.

In the following exercises, you can use what you know about the measure of an angle to



D4

learn how to identify angles according to their measures.

A Study the pictures of angles in D5. Notice that the measure in degrees of each angle is expressed in the display. Which of the angles are right angles?

B What is the measure in degrees of $\angle ABC$? Is the measure of $\angle ABC$ greater than, or less than, 90?

The measure in degrees of $\angle ABC$ is less than 90. $\angle ABC$ is called an *acute angle*. An acute angle is an angle whose measure in degrees is less than 90.

C Name another acute angle represented in D5. What is the measure in degrees of this angle?

D What is the measure in degrees of $\angle FGH$? Is the measure greater than, or less than, 90?

The measure in degrees of $\angle FGH$ is greater than 90. $\angle FGH$ is called an *obtuse angle*. An obtuse angle is an angle whose measure in degrees is greater than 90.

E Name another obtuse angle represented in D5. What is the measure in degrees of this angle?

F Is the measure of an acute angle less than, or greater than, the measure of a right angle? Is the measure of an obtuse angle less than, or greater than, the measure of a right angle?

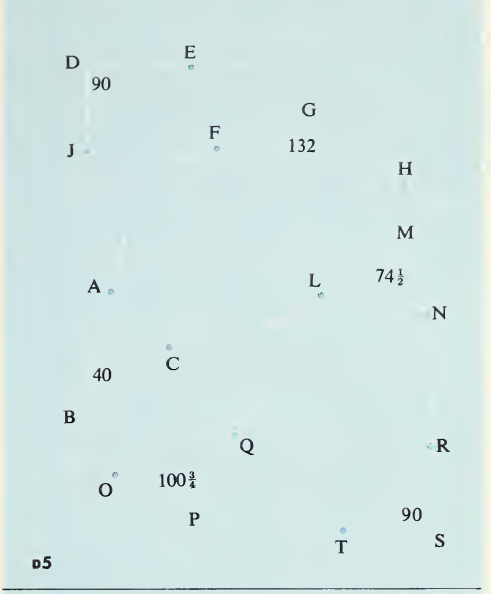
G Is the measure of an acute angle less than, or greater than, the measure of an obtuse angle?

D6 shows how to write and read a sentence that expresses a true statement about the measure of $\angle ABC$, represented in D5. Notice the symbol for degree that is written to the right and above the name of the angle.

H Read the sentence in D6. Then, for each of the other angles represented in D5, write a sentence that expresses a true statement about the measure in degrees of the angle.

acute angle (ə kūt′). An angle whose measure in degrees is less than 90.

obtuse angle (əb tūs′). An angle whose measure in degrees is greater than 90.



D5

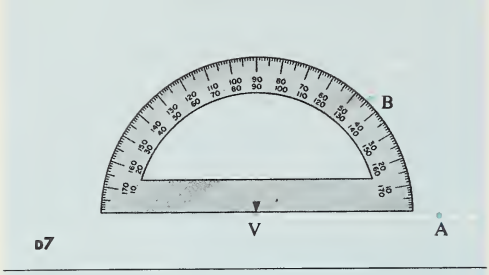
"The measure in degrees of

$\angle ABC^\circ = 40.$

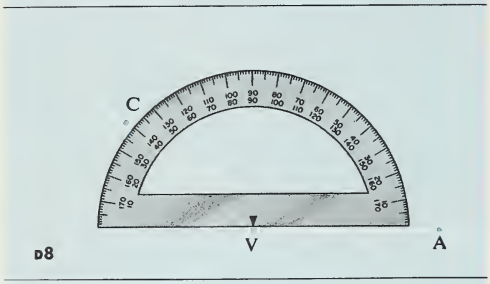
angle ABC is equal to 40."

D6

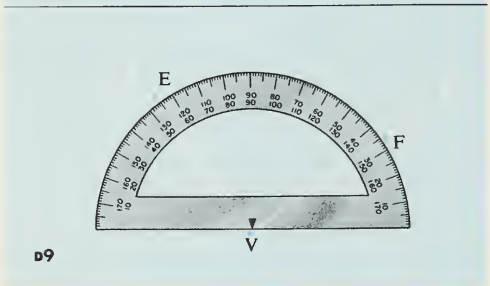
In the preceding work you learned to identify angles according to their measures. When you are working with pictures of angles, you will often want to find measures of these angles in



D7



D8



D9

degrees. An instrument called a *protractor* (prō trak′tər) is a useful device for measuring angles. A picture of such an instrument is shown in D7.

Study D7. Notice that one scale on the protractor begins with the mark that represents 0 on the right. This scale ends with the mark that represents 180 on the left. This is the counterclockwise scale. The other scale, which begins with the mark that represents 0 on the left and ends with the mark that represents 180 on the right, is the clockwise scale. Each mark on the protractor represents a number of degrees. The marks on a protractor thus serve the same kind of purpose as the marks on a yardstick or a thermometer.

A What two rays are represented in D7? How do you know that $\overrightarrow{VA} \cup \overrightarrow{VB} = \angle AVB$?

B The arrowhead on the protractor has been placed on the dot that represents vertex V. Does the picture of $\overrightarrow{VA}$ go under the mark on the protractor that represents 0? This means that the number associated with $\overrightarrow{VA}$ is 0.

C The picture of $\overrightarrow{VB}$ is on top of what mark on the protractor? Why do you use the counterclockwise scale? With what number is $\overrightarrow{VB}$ associated?

D What is the difference of the number associated with $\overrightarrow{VA}$ and the number associated with $\overrightarrow{VB}$?

E What is the measure in degrees of $\angle AVB$? Which kind of angle is $\angle AVB$, an acute, a right, or an obtuse angle?

F Now look at D8. With what number is $\overrightarrow{VC}$ associated? With what number is $\overrightarrow{VA}$ associated? What is the difference of these two numbers? What is the measure in degrees of $\angle CVA$? Which kind of angle is $\angle CVA$, an acute, a right, or an obtuse angle? How do you know?

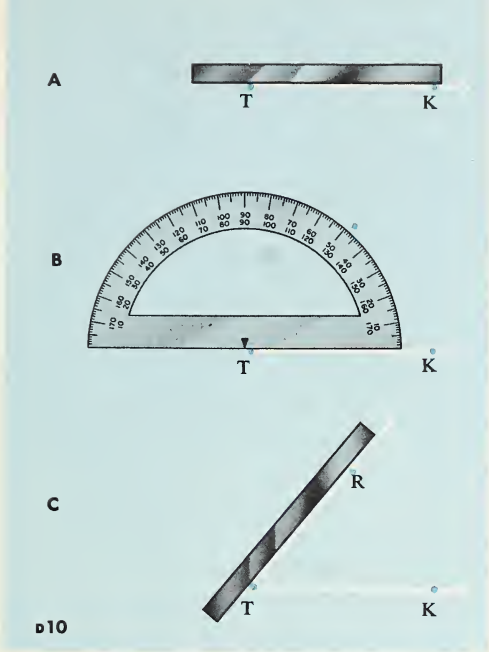
G Study D9. With what number is $\overrightarrow{VE}$ associated? With what number is $\overrightarrow{VF}$ associated? What is the difference of these two numbers?

H What is the measure in degrees of $\angle EVF$? What kind of angle is $\angle EVF$? How do you know?

I You can also use a protractor to make a picture of an angle of a given measure. Suppose, for example, that you want to make a picture of $\angle RTK$ whose measure in degrees is 50. What is the vertex of $\angle RTK$?

J Look at picture A in D10. First use a straightedge to draw a picture of $\overrightarrow{TK}$. What is the endpoint of $\overrightarrow{TK}$?

K Now look at picture B. Where is the arrowhead on the protractor placed? Does the picture of $\overrightarrow{TK}$ go under the bottom edge of the



protractor? Put a dot above the mark that represents 50. Label the dot with the letter R. Finally, use a straightedge to draw a picture of side TR, as shown in picture C.

In this lesson you learned about the measure of angles and that a standard unit of measure for angles is the degree. You also learned how to use a protractor to find the measure in degrees of an angle represented by a picture and how to make a picture of an angle of a given measure.

On your own

1 Think about the angles formed when two lines intersect. Suppose that the measure of one of the angles is 90. What do you know about the measures of the other three angles?

2 Now suppose that one of the angles formed when two lines intersect is an acute angle. Are any of the other angles right angles? Are any of the angles obtuse angles?

3 Name the angle formed by the union of $\overrightarrow{PQ}$ and $\overrightarrow{PL}$. Of $\overrightarrow{KH}$ and $\overrightarrow{KN}$.

4 If $\overrightarrow{AB}$ is associated with $22\frac{1}{2}$ and $\overrightarrow{AC}$ is associated with $66\frac{2}{3}$, what is the measure of $\angle BAC$?

Use the angles represented in D11 in connection with exercises 5 through 8.

5 Is $\angle ABC° = 65.4$? What is $\angle EDF°$? For each of the other angles represented in D11, write a sentence expressing a true statement about the measure of the angle.

6 Which of the angles are acute angles?

7 Which of the angles are obtuse angles?

8 Which of the angles are right angles?

Use a straightedge and a protractor to make pictures of the angles whose measures are expressed below.

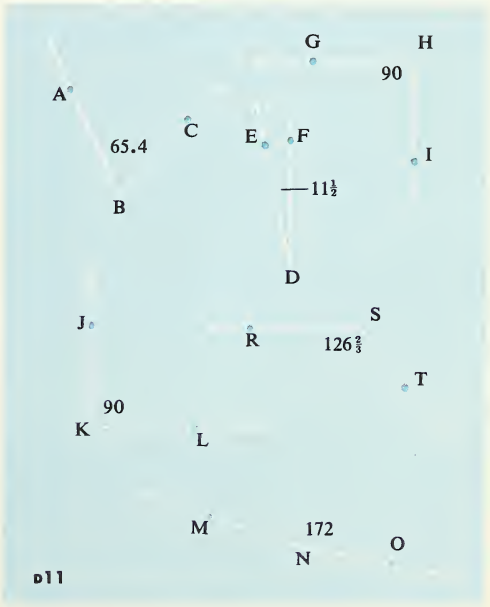
9 $\angle ABC° = 15$.

10 $\angle MNO° = 177$.

11 $\angle DEF° = 35$.

12 $\angle KLM° = 89$.

13 Make pictures of six angles. Use letters to name the angles. Then use a protractor to find the measure of each of the angles.



CHECKING UP

If you have trouble with these two tests, you should refer to lessons 98 through 101.

Test 108

For exercises 1 through 5, use the display below.

1 What is the coördinate of point D?

2 $\overline{EG} \cong \overline{AC}$. Name the coördinate of point G.

3 What is m($\overline{CH}$)?

4 Name two segments that are represented whose measure is $\frac{8}{5}$.

5 Name three segments that are represented whose measure is greater than m($\overline{BI}$).

Test 109

6 What is the measure of a segment whose coördinates are $8\frac{1}{2}$, $4\frac{3}{4}$?

7 Give the sum of the lengths $4\frac{1}{2}$ ft. and $6\frac{2}{3}$ ft.

8 Give the difference of the lengths $12\frac{5}{16}$ in. and $3\frac{3}{4}$ in.

9 If $\angle PQR^\circ = 90$, then $\angle PQR$ is what kind of angle?

10 If $\angle ABC^\circ < 90$, then $\angle ABC$ is what kind of angle?

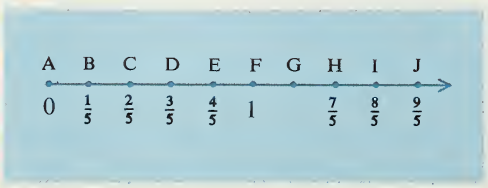
For each problem, write a sentence that expresses a condition, tabulate its solution set, and give the answer to the problem. U = Z.

11 How many feet is the same length as 37 yd.?

12 How many gallons is the same capacity as 50 qt.?

13 1320 yd. is the same length as how many miles? (Hundredths)

14 How many ounces is the same weight as 4.6 lb.?



Angles formed by two coplanar lines

In lesson 101 you learned how to find the measure of an angle. In this lesson you will study the angles formed by two intersecting lines.

A What two lines are represented in D1? Which line is determined by points A and B? Which line is determined by points A and C?

B Are A, B, and C, represented in D1, non-collinear points? How do you know?

C How do you know that ℓ_1 and ℓ_2 are included in the plane determined by A, B, and C? Name this plane.

> REMINDER
> If a line contains two points of a plane, the line is included in the plane.
> *See lesson 9, page 34, Book 1.*

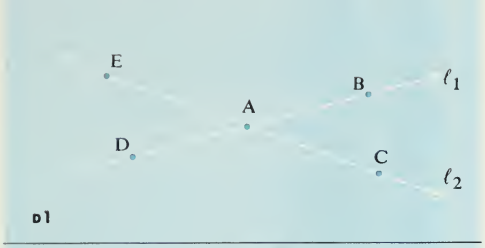
Lines that are included in the same plane are *coplanar lines*. ℓ_1 and ℓ_2, represented in D1, are coplanar lines.

D Look again at D1. Do ℓ_1 and ℓ_2 intersect? Tabulate $\ell_1 \cap \ell_2$.

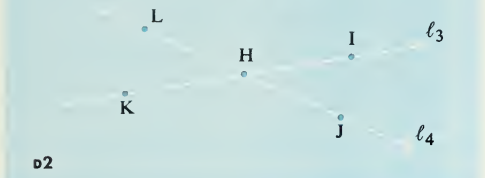
E How do you know that, if two lines intersect, they are coplanar lines?

Lines that intersect are *intersecting lines*. Now you will study the angles formed by the two intersecting lines represented in D1.

F Look again at D1. Point A is the common endpoint of $\overrightarrow{AB}$ and $\overrightarrow{AD}$. Point A is the common endpoint of which other rays represented?



D1



D2

co pla nar lines (kō plā′nər). Lines that are included in the same plane.

op po site rays (op′ə zit). Two rays that have the same endpoint and that are subsets of the same line. If A is between B and C, then ray AB and ray AC are opposite rays.

opposite angles. Two angles whose sides form two pairs of opposite rays.

G Describe $\overrightarrow{AB} \cup \overrightarrow{AD}$. Describe $\overrightarrow{AE} \cup \overrightarrow{AC}$.

H Is $\angle BAC$ the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$? Which side of $\angle BAC$ is included in ℓ_1? Which side of $\angle BAC$ is included in ℓ_2?

I Name the angle formed by $\overrightarrow{AB} \cup \overrightarrow{AE}$. By $\overrightarrow{AE} \cup \overrightarrow{AD}$. By $\overrightarrow{AD} \cup \overrightarrow{AC}$.

J How many angles are formed by ℓ_1 and ℓ_2?

K What lines are represented in D2? Are points H, I, and J collinear points? How do you know?

L Are ℓ_3 and ℓ_4 intersecting lines? Tabulate $\ell_3 \cap \ell_4$.

M How do you know that ℓ_3 and ℓ_4 are coplanar lines?

N Name the common endpoint of $\overrightarrow{HI}$ and $\overrightarrow{HK}$. Tabulate $\overrightarrow{HI} \cap \overrightarrow{HK}$.

O Is H between I and K? Are $\overrightarrow{HI}$ and $\overrightarrow{HK}$ subsets of the same line?

Two rays that have the same endpoint and that are subsets of the same line are *opposite rays*. Remember that, when we say "two rays," we mean "two different rays." $\overrightarrow{HI}$ and $\overrightarrow{HK}$, represented in D2, are opposite rays. $\overrightarrow{HI}$ is the opposite ray of $\overrightarrow{HK}$. $\overrightarrow{HK}$ is the opposite ray of $\overrightarrow{HI}$.

P Look again at D2. Point H is the common endpoint of which two rays besides $\overrightarrow{HI}$ and $\overrightarrow{HK}$? Tabulate the intersection of these two rays.

Q Describe $\overrightarrow{HJ} \cup \overrightarrow{HL}$. Are $\overrightarrow{HJ}$ and $\overrightarrow{HL}$ opposite rays? How do you know?

R Is $\overrightarrow{HK} \cup \overrightarrow{HL} = \angle KHL$? Is $\overrightarrow{HI} \cup \overrightarrow{HJ} = \angle IHJ$? Now think about the sides of the angles KHL and IHJ. Is side HK the opposite ray of side HI? Is side HL the opposite ray of side HJ?

S Is each side of angle IHJ the opposite ray of a side of angle KHL? Explain your answer.

$\angle KHL$ and $\angle IHJ$ are *opposite angles*. If two angles have sides that form two pairs of opposite rays, the two angles are opposite angles.

T $\angle LHI$ is the union of which rays represented in D2? $\angle JHK$ is the union of which rays?

U Is each side of $\angle LHI$ the opposite ray of a side of $\angle JHK$? Is each side of $\angle JHK$ the opposite ray of a side of $\angle LHI$? Explain your answer.

V How do you know that $\angle LHI$ and $\angle JHK$ are opposite angles?

Sometimes opposite angles are called *vertical angles*. In this book, we will use the term "opposite angles." However, remember that the term "vertical angles" also refers to two angles that have sides that form two pairs of opposite rays.

Now you will apply what you know about the measure of an angle to the measures of pairs of angles that are formed by two intersecting lines.

A Look at the pictures of ℓ_5 and ℓ_6 in D3. Tabulate $\ell_5 \cap \ell_6$. Name the angles formed by the intersecting lines ℓ_5 and ℓ_6.

B Are Q, M, and N collinear points? How do you know that $\angle OMN$ and $\angle QMO$ are a linear pair of angles?

C What is the measure in degrees of $\angle OMN$? Of $\angle QMO$? What is the sum of $\angle OMN°$ and $\angle QMO°$?

If the sum of the measures in degrees of two angles is 180, the angles are *supplementary angles*. $\angle OMN$ and $\angle QMO$ are supplementary angles. $\angle OMN$ is a *supplement of* $\angle QMO$. $\angle QMO$ is a supplement of $\angle OMN$.

D Look again at D3. Are P, M, and O collinear points?

E How do you know that $\angle QMO$ and $\angle PMQ$ are a linear pair of angles?

F What is $\angle QMO°$? What is $\angle PMQ°$? What is the sum of $\angle QMO°$ and $\angle PMQ°$?

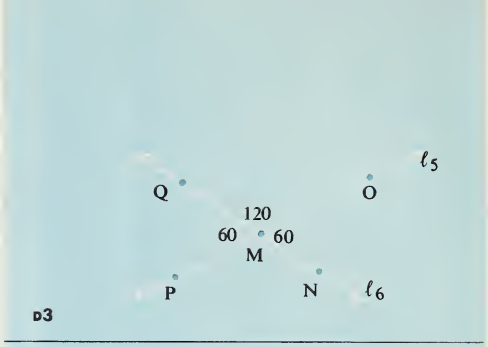
G Are $\angle QMO$ and $\angle PMQ$ supplementary angles? How do you know?

Exercises B through G have helped you to understand an important property concerning the sum of the measures of two angles that form a linear pair. This property is expressed below.

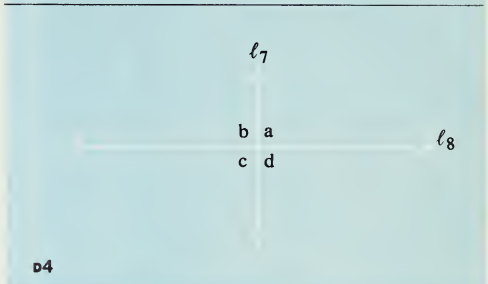
If two angles form a linear pair, then they are supplementary angles.

H Look again at D3. How do you know that $\angle PMQ$ and $\angle NMP$ are a linear pair of angles?

I Are $\angle PMQ$ and $\angle NMP$ supplementary angles? What angle is a supplement of $\angle PMQ$? Of $\angle NMP$? What is the sum of $\angle PMQ°$ and $\angle NMP°$?



D3



D4

sup ple men ta ry angles (sup′lə men′tə ri). A pair of angles, the sum of whose measures in degrees is 180. Each angle of a pair of supplementary angles is a **supplement** of the other angle.

You can use your knowledge of how addition is related to subtraction to find $\angle NMP°$.

J You know that $\angle PMQ° + \angle NMP° = 180$. Is $\angle NMP°$ the difference of 180 and $\angle PMQ°$? What is $\angle PMQ°$?

K Is 120 the difference of 180 and 60? What is $\angle NMP°$?

L Name the opposite rays represented in D3 that are subsets of ℓ_5. Name the opposite rays that are subsets of ℓ_6. Name the two pairs of opposite angles represented in D3.

M Is $\angle OMN° = \angle PMQ°$? How do you know that $\angle OMN \cong \angle PMQ$?

N Is $\angle QMO° = \angle NMP°$? Is $\angle QMO \cong \angle NMP$? Explain your answers.

Now you know that, in each pair of the opposite angles formed by ℓ_5 and ℓ_6, the angles

are congruent. An important property of opposite angles is expressed below.

Opposite angles are congruent.

Now look at the pictures of ℓ_7 and ℓ_8 in D4. Notice that we have used the letters a, b, c, d as names for angles. It often is convenient to name angles this way.

O How do you know that $\angle a$ and $\angle b$ are a linear pair of angles?

P $\angle a \cong \angle b$. Is $\angle a$ a right angle? Is $\angle b$ a right angle? Explain your answers.

Q What is the measure of $\angle a$? Of $\angle b$?

R Name two pairs of opposite angles represented in D4. How do you know that $\angle c$ and $\angle d$ are right angles?

S If two angles are opposite angles, how do you know that their measures are equal?

T Is $\angle b^\circ = 90$? Is $\angle d^\circ = 90$? Are $\angle b$ and $\angle d$ a pair of supplementary angles? Are $\angle b$ and $\angle d$ a pair of linear angles?

U Think about the four angles formed by two intersecting lines. What is the sum of the measures of the four angles? Explain your answer.

V Think of ℓ_9 and ℓ_{10} as any two coplanar lines that have no points in common. Tabulate $\ell_9 \cap \ell_{10}$.

W How do you know that ℓ_9 and ℓ_{10} are parallel? How many angles are formed by any two parallel lines?

REMINDER

Parallel lines are two coplanar lines that do not have any points in common.

See lesson 26, page 101, Book 1.

Now you will use your knowledge of angles and their measures to solve problems concerning angles. Read the problem in D5. The universe is the set of rational numbers between 0 and 180.

Angles a, b, c, and d are formed by two intersecting lines. $\angle a$ is a supplement of $\angle b$. $\angle a$ is also a supplement of $\angle d$. $\angle a^\circ$ is 98. What is the measure of each of angles b, c, and d?

D5

A You know that $\angle a$ and $\angle b$ are supplementary angles. What is the sum of the measures of $\angle a$ and $\angle b$?

B Do you know $\angle b^\circ$? Use x as a variable for $\angle b^\circ$. Why is the universe for x the set of rational numbers between 0 and 180?

C Is $98 + x = 180$ a simple condition for the problem? What number satisfies $98 + x = 180$?

D How do you know that $\angle b^\circ$ is 82?

You can use the measures of $\angle a$ and $\angle b$ to find the measures of the other two angles.

E Is $\angle d \cong \angle b$? Is $\angle d^\circ = \angle b^\circ$? Explain your answers.

F What is the measure of $\angle d$?

G Is $\angle c \cong \angle a$? Is $\angle c^\circ = \angle a^\circ$? Explain your answers.

H What is the measure of $\angle c$?

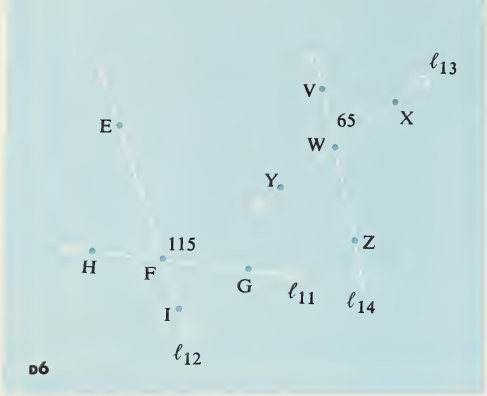
You can use the numbers that you obtained for exercises C, F, and H to get the answer to the problem in D5.

The measure of $\angle b$ is 82; the measure of $\angle c$ is 98; and the measure of $\angle d$ is 82.

In this lesson you learned that opposite and supplementary angles are formed by two intersecting lines.

On your own

1 Draw a picture of two intersecting lines. Then use a protractor to find the measure of each angle formed by the two lines.



D6

2 What is the sum of the measures that you found for exercise 1?

3 Draw a picture of two lines that intersect to form two angles, one of which has a measure of 127. Use your protractor to help you.

4 What is the measure of each of the other angles represented in the picture you drew for exercise 3?

Use D6 to help you answer the questions below. ℓ_{11} and ℓ_{12} are intersecting lines. ℓ_{13} and ℓ_{14} also are intersecting lines.

5 Name the angles formed by ℓ_{11} and ℓ_{12}.

6 Name the angles formed by ℓ_{13} and ℓ_{14}.

7 Name four linear pairs of angles formed by ℓ_{11} and ℓ_{12}.

8 Name four linear pairs of angles formed by ℓ_{13} and ℓ_{14}.

9 Name four pairs of opposite angles represented in D6.

10 What is $\angle GFI°$?

11 Name three other angles that have the same measure as $\angle GFI$.

12 What is $\angle XWZ°$?

13 Name three other angles that have the same measure as $\angle XWZ$.

14 Name two angles formed by ℓ_{13} and ℓ_{14} that are each a supplement of $\angle EFH$.

15 Name two angles formed by ℓ_{11} and ℓ_{12} that are each a supplement of $\angle YWZ$.

Angles formed by three coplanar lines

In the last lesson you studied angles formed by two coplanar lines. In this lesson you will investigate angles formed by three coplanar lines.

A Lines ℓ_1, ℓ_2, and ℓ_3, which are represented in D1, are included in the same plane. Are ℓ_1, ℓ_2, and ℓ_3 coplanar? How do you know?

B What point do ℓ_1, ℓ_2, and ℓ_3 have in common?

C Point R is the endpoint of six rays represented in D1. Name these rays.

D Is $\overrightarrow{RA} \cup \overrightarrow{RB} = \angle ARB$? Name the angle formed by the union of $\overrightarrow{RB}$ and $\overrightarrow{RC}$.

E What is the common vertex of $\angle ARB$ and $\angle BRC$? What is their common side?

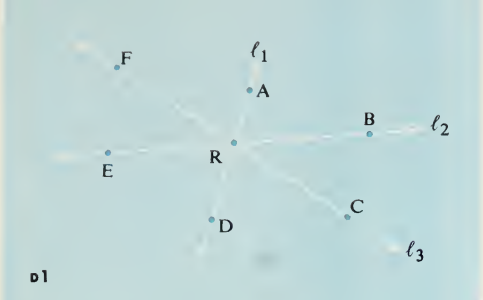
F Are any points that are contained in the interior of $\angle ARB$ also contained in the interior of $\angle BRC$? Tabulate the intersection of the interior of $\angle ARB$ and the interior of $\angle BRC$.

G How do you know that the interiors of $\angle ARB$ and $\angle BRC$ are disjoint sets?

$\angle ARB$ and $\angle BRC$ have a common vertex and a common side. The interiors of these angles are disjoint sets. $\angle ARB$ and $\angle BRC$ are *adjacent angles*. Any two angles that have a common vertex and a common side and whose interiors are disjoint sets are adjacent angles.

ad ja cent angles (ə jā′sənt). Two angles that have a common vertex and a common side and whose interiors are disjoint sets.

Angles formed by three intersecting lines; transversal of two coplanar lines; corresponding angles; converse of a statement



D1

D2

H Look again at D1. What is the common vertex of $\angle CRD$ and $\angle ERF$? Are the interiors of $\angle CRD$ and $\angle ERF$ disjoint sets? Do these two angles have a common side?

I Are $\angle CRD$ and $\angle ERF$ adjacent angles? Explain your answer.

J How do you know that $\angle BRC$ and $\angle BRD$ are not adjacent angles?

K You know that $\angle ARB$ and $\angle BRC$ are adjacent angles. Name three other pairs of adjacent angles represented in D1.

L Are $\angle FRA$ and $\angle ARC$ adjacent angles?

M Are $\angle FRA$ and $\angle ARC$ a linear pair of angles?

N Is $\angle FRA^\circ + \angle ARC^\circ = 180$?

O Name another linear pair of angles represented in D1. Are these angles adjacent?

P Is each linear pair of angles also a pair of adjacent angles? Is each pair of adjacent angles also a linear pair? Explain your answers.

Q If two angles form a linear pair, are they supplementary angles? If two angles are supplementary angles, do they necessarily form a linear pair? Explain your answers.

R If two angles are adjacent angles, are they necessarily supplementary angles? If two angles are supplementary angles, are they necessarily adjacent? Explain your answers.

S Are $\angle ARB$ and $\angle ERD$, represented in D1, adjacent angles? Are they opposite angles? Explain your answers.

T How do you know that a pair of opposite angles cannot be a pair of adjacent angles?

In the preceding exercises, you studied the kinds of angles that are formed when three coplanar lines intersect in one point. Next you will learn about the angles that are formed when three coplanar lines do not all intersect in the same point.

A Three coplanar lines are represented in D2. Name these lines. Do all three lines have one point in common?

B Tabulate $\ell_6 \cap \ell_4$. Tabulate $\ell_6 \cap \ell_5$.

The point in which ℓ_6 intersects ℓ_5 is not the point in which ℓ_6 intersects ℓ_4. A line that intersects one of two coplanar lines in a point and also intersects the other line in a different point is a *transversal* of those two lines. ℓ_6 is a transversal of ℓ_5 and ℓ_4.

C Is the transversal of two coplanar lines always included in the same plane as the two lines? Explain your answer.

D ℓ_4 is a transversal of which lines represented in D2? Name the line that is a transversal of ℓ_4 and ℓ_6.

transversal of two coplanar lines (transvèr'səl). A line that intersects two coplanar lines in such a way that the point in which it intersects one line is different from the point in which it intersects the other line.

E Look at D3. Name the three coplanar lines represented in the display. Is ℓ_7 a transversal of ℓ_8 and ℓ_9? How do you know?

F Think of a point Q contained in the same plane as ℓ_8 and ℓ_9. If $\ell_8 \cap \ell_9 = \{Q\}$, are ℓ_8 and ℓ_9 parallel lines? How do you know?

G Notice that the letters a, b, c, and so on are used to name the angles represented in D3. Name each pair of adjacent angles represented.

H Name each pair of opposite angles.

I Does the transversal, ℓ_7, separate the plane into two half-planes?

J Is the interior of $\angle b$ a subset of one of the half-planes bounded by the transversal, ℓ_7? Is the interior of $\angle f$ a subset of the same half-plane?

K Is the side of $\angle b$ that is included in ℓ_7 a proper subset of the side of $\angle f$ that is included in ℓ_7?

REMINDER

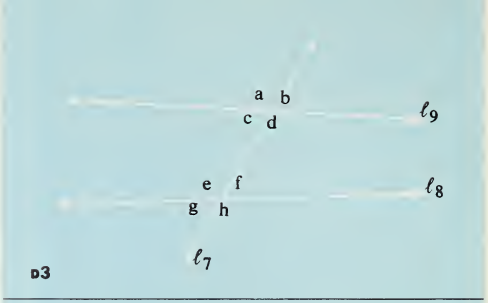
A proper subset of a given set does not contain all the members of the given set.

See lesson 59, page 253, Book 1.

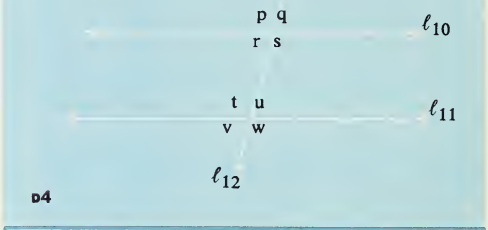
$\angle b$ and $\angle f$ are *corresponding angles*. Corresponding angles are formed by two coplanar lines and a transversal. The interiors of both angles are subsets of the same half-plane bounded by the transversal, and a side of one angle is a proper subset of a side of the other angle.

L Are the interiors of $\angle d$ and $\angle f$ subsets of the same half-plane bounded by ℓ_7? Is one side of $\angle d$ a proper subset of a side of $\angle f$? Is one side of $\angle f$ a proper subset of a side of $\angle d$? Are $\angle d$ and $\angle f$ corresponding angles?

M How do you know that $\angle e$ and $\angle b$ are not a pair of corresponding angles?



D3



D4

cor re spond ing angles (kôr′ə spon′ding). A pair of angles formed by two coplanar lines and a transversal so that the following is true: The interiors of both angles are subsets of the same half-plane bounded by the transversal, and a side of one angle is a proper subset of a side of the other angle.

N How do you know that $\angle b$ and $\angle h$ are not a pair of corresponding angles?

O Are $\angle c$ and $\angle g$ corresponding angles? Explain your answer.

P Name another pair of corresponding angles represented in D3.

You have learned what a transversal of two coplanar lines is and what we mean by pairs of corresponding angles. In the following exercises, you will learn about corresponding angles that are congruent.

A Lines ℓ_{10} and ℓ_{11} represented in D4 are parallel. Are ℓ_{10} and ℓ_{11} included in the same plane? Tabulate $\ell_{10} \cap \ell_{11}$.

B Is ℓ_{12} a transversal of ℓ_{10} and ℓ_{11}?

C Is $\angle p \cong \angle s$? Is $\angle r \cong \angle q$? How do you know? Name two other pairs of opposite angles represented in D4.

D Are $\angle q$ and $\angle u$ corresponding angles? How do you know?

E $\angle q \cong \angle u$. How do you know that $\angle r \cong \angle v$? Are $\angle r$ and $\angle v$ corresponding angles?

F Is $\angle p$ a supplement of $\angle q$? Is $\angle t$ a supplement of $\angle u$? How do you know that $\angle p \cong \angle t$? Are $\angle p$ and $\angle t$ corresponding angles?

G Is $\angle s$ a supplement of $\angle q$? Is $\angle w$ a supplement of $\angle u$? How do you know that $\angle s \cong \angle w$? Are $\angle s$ and $\angle w$ corresponding angles?

Look again at D4. The angles of each pair of corresponding angles formed by the parallel lines and the transversal are congruent. This property of two coplanar lines and a transversal is expressed below.

If two lines are parallel, then the angles of each pair of corresponding angles formed by the two lines and a transversal are congruent.

H Suppose that $\angle p^\circ = 105$. Name three other angles represented in D4 whose measure is also 105.

I If $\angle p^\circ = 105$, what is the measure of $\angle q$? Name three other angles represented in D4 that have the same measure as $\angle q$.

J Now look at D5. The three lines represented in D5 are coplanar. Is ℓ_{14} the transversal of ℓ_{13} and ℓ_{15}?

K Name the pairs of corresponding angles represented in D5.

L $\angle h \cong \angle n$. Name two other angles represented in D5 that are congruent to $\angle n$.

M $\angle h$ and $\angle n$ are corresponding angles. These angles are also congruent. How do you know that each of the other pairs of corresponding angles represented in D5 is a pair of congruent angles?

D5

Since the corresponding angles formed by ℓ_{13} and ℓ_{15} and their transversal are congruent, ℓ_{13} and ℓ_{15} are parallel lines. This property of two coplanar lines and a transversal is expressed below.

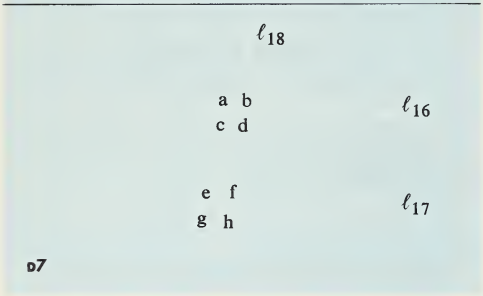
If any two corresponding angles that are formed by two coplanar lines and a transversal are congruent, then the two lines are parallel.

Compare the property expressed in the paragraph preceding exercise H with the property expressed above. From the first property, you know that, if two lines are parallel, then any two corresponding angles are congruent. From the second property, you know that, if any two corresponding angles are congruent, then the lines are parallel. Notice that the second property may be obtained from the first property by interchanging the ideas expressed in the "if" clause and the "then" clause. The second property is the *converse* (kon′vėrs) of the first property, and the first property is the converse of the second property.

N Think about the true statement expressed by the following sentence: "If two angles are a linear pair of angles, then the angles are adjacent." What idea is expressed in the "if" clause? What idea is expressed in the "then" clause?

O If you interchange the ideas expressed in the two clauses, you obtain the following converse: "If two angles are adjacent, then the angles are a linear pair of angles." Is this converse true?

D6



D7

P ? Is the converse of a statement necessarily true? Is the converse of a statement necessarily false? Give examples to explain your answers.

In this lesson you studied the different kinds of angles formed by three coplanar lines. You also learned what a transversal is.

On your own

Use D6 to help you with exercises 1 through 9. The three lines represented are coplanar.

1 Name two angles that are adjacent to $\angle VTW$.

2 Explain why $\angle ZTY$ is not adjacent to $\angle ZTX$.

3 $\angle UTV^\circ = 25$. $\angle UTW^\circ = 90$. Name two angles each of which has a measure of 65.

4 Name two linear pairs of angles.

5 Name two pairs of opposite angles.

6 Is $\angle WTX \cong \angle UTZ$? How do you know?

7 Use the information given in exercise 3 to find the measure of $\angle YTX$. Of $\angle WTX$. Of $\angle UTZ$.

8 Which angles represented in D6 are right angles?

9 Name three pairs of congruent angles represented in D6.

Use D7 to help you with exercises 10 through 15. The three lines represented in D7 are coplanar. ℓ_{16} and ℓ_{17} are parallel lines.

10 Name the transversal.

11 Name four linear pairs of angles.

12 Name four pairs of adjacent angles. Choose pairs of angles different from the pairs you named for exercise 11.

13 Name four pairs of opposite angles.

14 Name four pairs of corresponding angles.

15 $\angle a^\circ = 110$. Give the measure of each of the other angles.

16 What is the converse of the statement expressed by the following sentence? "If a line intersects one of two coplanar lines in a point and also intersects the other line in a different point, then the line is a transversal of the two coplanar lines."

17 Is the statement expressed in exercise 16 true? Is the converse of the statement true?

18 What is the converse of the statement expressed by the following sentence? "If each of two angles is a right angle, then the angles are congruent."

19 Is the statement expressed in exercise 18 true? Is the converse of the statement true?

KEEPING SKILFUL

For each exercise below, tabulate the solution set of the condition. U = Z.

1 $.25/2 \sim x/14$.

2 $3/4 \sim \frac{1}{2}/x$.

3 $x + 4\frac{1}{3} = 7\frac{1}{4}$.

4 $12\frac{1}{2}/100 \sim x/1$.

5 $\frac{1}{4}/\frac{1}{8} \sim x/1$.

6 $17/23 \sim 8/x$.

7 $56\frac{1}{2}/100 \sim x/1$.

8 $x - 17\frac{6}{11} = 4\frac{1}{2}$.

9 $8/7 \sim 9/x$.

10 $1.3/6 \sim .39/x$.

11 $6\frac{1}{7} + 7\frac{2}{3} = x$.

12 $4\frac{6}{11}/100 \sim x/1$.

Measures associated with triangles

In this lesson you will use what you know about measures of segments and angles to learn more about triangles. You remember, from lesson 28, *Book I*, that a triangle is the union of the three segments that are determined by any three noncollinear points.

In lesson 28 you also learned to identify triangles according to the number of congruent sides each triangle has. You learned that a scalene triangle has no two of its sides congruent, that an isosceles triangle has two of its sides congruent, and that an equilateral triangle has three sides congruent. Now you are ready to identify triangles according to the measures of their sides.

A Look at D1. Name the triangles represented in the display.

The sentences in D1 express true statements about the sides of each of the triangles represented.

B How do you know that $m(\overline{DF}) > m(\overline{FE})$? Is $m(\overline{FE}) > m(\overline{DE})$? How do you know that $m(\overline{DF}) > m(\overline{DE})$?

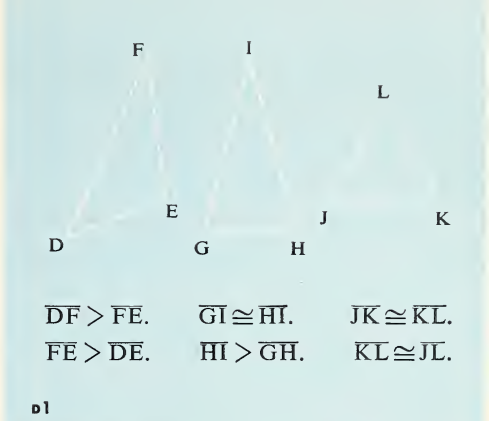
C Is a scalene triangle a triangle in which no two sides have the same measure? Explain your answer.

D Which triangle represented in D1 has no two sides with the same measure? Is this triangle scalene?

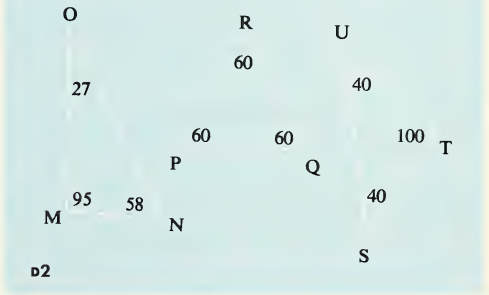
E How do you know that $m(\overline{GI}) = m(\overline{HI})$?

F Is $m(\overline{HI}) > m(\overline{GH})$?

G Is $m(\overline{GI}) > m(\overline{GH})$?



D1



D2

H Is a triangle in which two sides have the same measure an isosceles triangle? Explain your answer.

I Which triangles represented in D1 each have two sides with the same measure? Are these triangles isosceles?

J Is $m(\overline{JK}) = m(\overline{KL})$? Is $m(\overline{KL}) = m(\overline{JL})$? How do you know that $m(\overline{JK}) = m(\overline{JL})$?

K Is an equilateral triangle a triangle in which all three sides have the same measure?

L Which triangle represented in D1 has three sides with the same measure? Is this triangle equilateral?

In the following exercises, you will study some important ideas concerning the measures of the angles of a triangle.

A Name the triangles represented in D2.

You may name an angle of a triangle by using only the letter of the vertex, if it is clear from this letter which angle you mean. For example, you may use the letter M to name ∠OMN represented in D2.

B In △MNO, what is the measure of ∠M in degrees? Of ∠N in degrees? Of ∠O in degrees?

C Is $95 + 58 + 27 = 180$? What is the sum of the measures of the angles of △MNO?

D What is the measure of each angle of △PQR? What is the sum of ∠P°, ∠Q°, and ∠R°?

E What is the measure of each angle of △STU? What is the sum of ∠S°, ∠T°, and ∠U°?

Now you will see if, in any triangle, the sum of the measures of the angles in degrees is 180. You can use the picture in D3 and your knowledge of opposite and corresponding angles to help you. The lines that are represented in D3 are coplanar. ℓ_7 and ℓ_4 are parallel lines.

F Are R, S, and T noncollinear points? Name the segments that are determined by points R, S, and T. Is $\overline{RS} \cup \overline{ST} \cup \overline{TR} = \triangle RST$?

To find the sum of ∠d°, ∠e°, and ∠f°, the first step is to find three angles that have the same measures as the angles of △RST.

G Is ℓ_6 a transversal of ℓ_7 and ℓ_4? How do you know that ∠d and ∠c are corresponding angles?

H Is ∠d° = ∠c°? How do you know? Remember that ℓ_7 and ℓ_4 are parallel.

I Is ℓ_5 a transversal of ℓ_7 and ℓ_4? Are ∠e and ∠a corresponding angles? Is ∠e° = ∠a°?

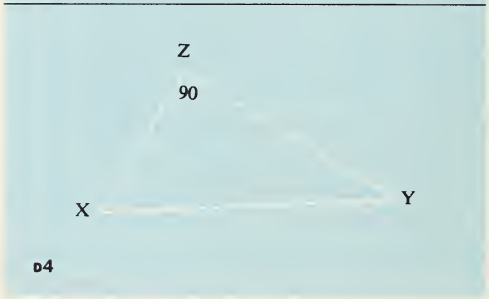
J How do you know that ∠f and ∠b are opposite angles?

K Is ∠f° = ∠b°? How do you know?

Notice that in △RST, ∠d° is the same as ∠c°, ∠e° is the same as ∠a°, and ∠f° is the same as ∠b°.

D2

D3



D4

L Is ∠d° + ∠e° + ∠f° = ∠c° + ∠a° + ∠b°? Explain your answer.

M How do you know that ∠a and ∠b are adjacent angles? Is ∠XTV° = ∠a° + ∠b°?

N Are ∠XTV and ∠c a linear pair of angles? How do you know?

O How do you know that ∠c° + ∠XTV° = 180? Is ∠c° + (∠a° + ∠b°) = 180?

P Is ∠d° + ∠e° + ∠f° = 180? Explain your answer.

Now you know that, for any triangle, the sum of the measures of the angles in degrees

is 180. This property of the measures of the angles of a triangle is expressed below.

For any triangle, the sum of the measures of the angles in degrees is 180.

Look at D4. The triangle represented in D4 is a right triangle. You know that any right triangle has one right angle.

Q Does a right triangle have one angle whose measure in degrees is equal to 90? Which angle represented in D4 has a measure in degrees of 90?

R Is $\angle X^\circ + \angle Y^\circ + \angle Z^\circ = 180$? How do you know? What is the sum of the measures of $\angle X$ and $\angle Y$?

S Is $\angle X^\circ < 90$? Is $\angle Y^\circ < 90$? Explain your answers. Are both $\angle X$ and $\angle Y$ acute angles?

T How do you know that a triangle cannot have more than one angle with a measure in degrees of 90?

A triangle that has one obtuse angle is an *obtuse triangle*. An obtuse triangle has one angle whose measure in degrees is more than 90.

U How do you know that a triangle cannot have more than one obtuse angle?

V You have seen that a right triangle has one right angle and two acute angles. Can a triangle have three acute angles?

A triangle with three acute angles is an *acute triangle*. An acute triangle has three angles, each of whose measures in degrees is less than 90.

W Is it possible for a triangle to have only one acute angle? Explain your answer.

obtuse triangle. A triangle in which one angle is an obtuse angle.

acute triangle. A triangle that has three acute angles.

Now you know how to identify triangles according to the measures of their sides and according to the measures of their angles. You also know that, in any triangle, the sum of the measures of the angles in degrees is 180.

On your own

For exercises 1 through 13, use the picture in D5 to help you. Four coplanar lines are represented in D5. ℓ_4 and ℓ_1 are parallel lines.

1 Tabulate $\ell_1 \cap \ell_4$.

2 Tabulate $\ell_2 \cap \ell_3 \cap \ell_4$.

3 Tabulate $\ell_2 \cap \ell_3$.

4 Tabulate $\ell_1 \cap \ell_2$.

5 Tabulate $\ell_1 \cap \ell_3$.

6 Name the sides of $\triangle ABC$.

7 Name the angles of $\triangle ABC$.

8 Which angle is the opposite angle of $\angle ACB$?

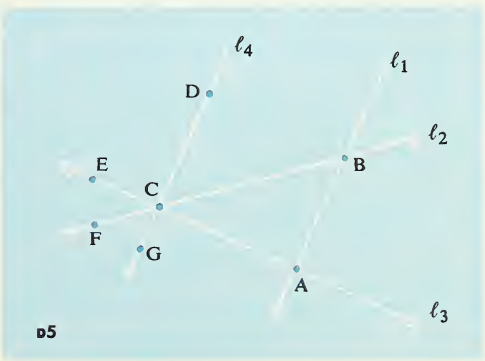
9 If $\angle ECF^\circ = 40$, what is the measure of $\angle ACB$ in degrees? How do you know?

10 If $\angle FCG^\circ = 50$, what is the measure of $\angle CBA$ in degrees? How do you know?

11 Using the information given in exercises 9 and 10, give the measure of $\angle ECD$ in degrees. Of $\angle BAC$ in degrees.

12 What is the measure of $\angle GCA$ in degrees?

13 Identify $\triangle ABC$ according to the measures of its angles.



D5

For exercises 14 through 32, use the sentences and the pictures in D6 to help you.

14 In $\triangle ABC$, what is the measure of $\angle B$ in degrees?

15 In $\triangle ABC$, is $m(\overline{AB}) = m(\overline{AC})$? When you identify $\triangle ABC$ according to the measures of its sides, you say it is equilateral. What do you say when you identify $\triangle ABC$ according to the measures of its angles?

16 In $\triangle DEF$, what is the measure of $\angle D$ in degrees?

17 Is $m(\overline{DF}) < m(\overline{EF})$? Identify $\triangle DEF$ according to the measures of its sides. According to the measures of its angles.

18 In $\triangle GHI$, what is the measure of $\angle G$ in degrees?

19 Identify $\triangle GHI$ according to the measures of its sides. According to the measures of its angles.

20 In $\triangle JKL$ what is the measure of $\angle K$ in degrees?

21 Is $m(\overline{JL}) < m(\overline{JK})$? Identify $\triangle JKL$ according to the measures of its sides. Identify $\triangle JKL$ according to the measures of its angles.

22 Which triangles represented in D6 have two or more acute angles?

23 Which triangle has three acute angles?

24 Which triangles have an obtuse angle?

25 Which triangle has three angles with the same measure? Is this triangle an isosceles triangle? Is this triangle also an equilateral triangle?

26 Which triangle has exactly two angles with the same measure? Is this triangle an isosceles triangle?

27 Is it possible for an acute triangle also to be an isosceles triangle? To be a scalene triangle? To be an equilateral triangle?

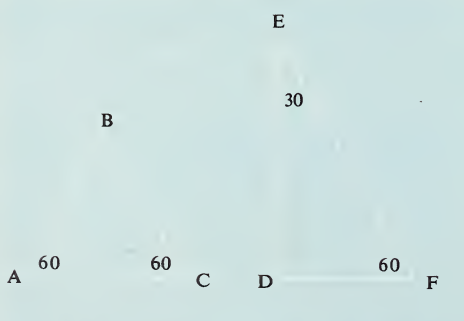
28 Is it possible for an obtuse triangle also to be an isosceles triangle?

29 Which triangle represented in D6 is both obtuse and isosceles?

30 Is it possible for an obtuse triangle also to be a scalene triangle?

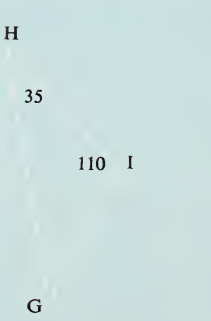
31 Which triangle represented in D6 is both obtuse and scalene?

32 Is it possible for a right triangle also to be a scalene triangle? How do you know?

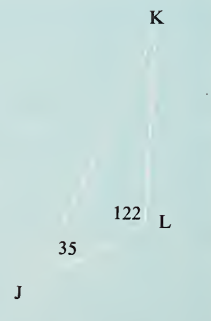


$m(\overline{AB}) = m(\overline{BC})$.
$m(\overline{BC}) = m(\overline{AC})$.

$m(\overline{DF}) < m(\overline{ED})$.
$m(\overline{ED}) < m(\overline{EF})$.



$m(\overline{HI}) = m(\overline{IG})$.
$m(\overline{IG}) < m(\overline{HG})$.



$m(\overline{JL}) < m(\overline{KL})$.
$m(\overline{KL}) < m(\overline{JK})$.

D6

Congruent triangles

In *Book 1* you learned that two congruent triangles have three pairs of congruent angles and three pairs of congruent sides. In this lesson you will learn more about congruent triangles.

First you will learn how the sets of vertices of two triangles may be put in one-to-one correspondence. Look at D1. You can use the triangles represented in the display to determine the ways in which the sets of vertices of the two triangles can be put in one-to-one correspondence.

REMINDER

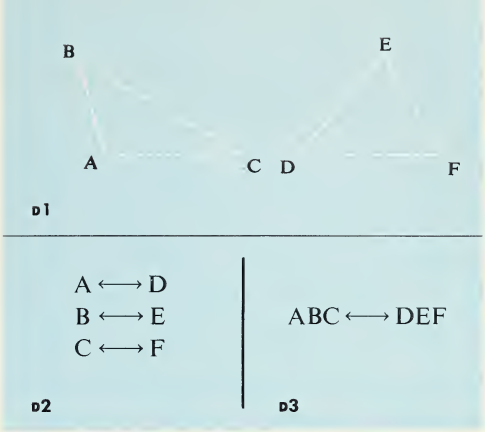
A one-to-one correspondence is two one-to-one mappings that map a first set onto a second set and the second set onto the first set.

See lesson 57, page 246, Book 1.

A Is {A, B, C} the set of vertices of △ABC? Tabulate the set of vertices of △DEF.

B Suppose that vertex A of △ABC and vertex D of △DEF are mapped onto each other. Suppose also that vertex B and vertex E are mapped onto each other. To complete a one-to-one correspondence, what other two vertices will be mapped onto each other?

The diagram in D2 represents a one-to-one correspondence between the sets of vertices of △ABC and △DEF. Compare the diagram in D2 with the diagram in D3. Both diagrams



represent the same one-to-one correspondence, but the one represented in D3 is shorter. Notice that, in D3, the order of the letters in the names of the two triangles indicates which vertices are mapped onto each other.

C How can you tell from the diagram in D3 that vertices A and D are mapped onto each other? That vertices B and E are mapped onto each other? That vertices C and F are mapped onto each other?

From now on, when we speak of a one-to-one correspondence between the vertices of two triangles, we will mean a one-to-one correspondence between the two sets of vertices.

D Next, think about another way in which the vertices of the two triangles represented in D1 can be put in one-to-one correspondence. If vertices A and D are mapped onto each other, and vertices B and F are mapped onto each other, onto what vertex will vertex C be mapped? Onto what vertex will vertex E be mapped? Make a diagram like the one in D3 to represent this way of putting the vertices of the two triangles in one-to-one correspondence.

You have seen two ways in which the vertices of △ABC and △DEF can be put in one-to-one correspondence. The vertices of

△ABC and △DEF can be put in one-to-one correspondence in six ways. These six ways are represented in D4.

$ABC \longleftrightarrow DEF$ $ABC \longleftrightarrow DFE$
$ABC \longleftrightarrow EDF$ $ABC \longleftrightarrow EFD$
$ABC \longleftrightarrow FDE$ $ABC \longleftrightarrow FED$

D4

You can use the idea of one-to-one correspondence between the vertices of two triangles to understand the ways in which the sides and angles of the two triangles may correspond.

E Look at D5. Name the angles and the sides of each triangle represented in the display.

F Diagram A in D5 represents a one-to-one correspondence between the vertices of the two triangles. Which vertices are mapped onto each other by this correspondence?

G M and N are the endpoints of side MN of △MNP. R and S are the endpoints of which side? Are the endpoints of $\overline{MN}$ mapped onto the endpoints of $\overline{RS}$? Are the endpoints of $\overline{RS}$ mapped onto the endpoints of $\overline{MN}$?

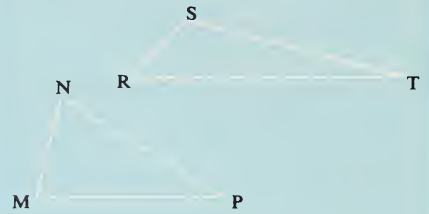
Since the endpoints of $\overline{MN}$ and $\overline{RS}$ are mapped onto each other, side MN and side RS are *corresponding sides* of the two triangles.

H How do you know that side NP and side ST are corresponding sides? How do you know that side MP and side RT are corresponding sides?

Since the vertices of ∠M and ∠R are mapped onto each other, ∠M and ∠R are *corresponding angles* of the two triangles.

I Are ∠N and ∠S corresponding angles? Are ∠P and ∠T corresponding angles? Explain your answers.

Notice what we have done. We have established a one-to-one correspondence between the vertices of two triangles. This enables us to determine pairs of corresponding sides and pairs of corresponding angles of the two triangles. Diagrams A, B, and C in D5 summarize the correspondences that we have established.



A $MNP \longleftrightarrow RST$

B $\overline{MN} \longleftrightarrow \overline{RS}$
$\overline{NP} \longleftrightarrow \overline{ST}$
$\overline{MP} \longleftrightarrow \overline{RT}$

C $\angle M \longleftrightarrow \angle R$
$\angle N \longleftrightarrow \angle S$
$\angle P \longleftrightarrow \angle T$

D5

J The diagram at the right represents another way of putting the vertices of △MNP and △RST in one-to-one correspondence. Use this correspondence between the vertices to name the corresponding sides and the corresponding angles of the two triangles.

$PMN \longleftrightarrow RST$

K You know that there are six ways of putting the vertices of △MNP and △RST in one-to-one correspondence. In how many ways can the sides of △MNP be put in one-to-one correspondence with the sides of △RST? In how many ways can the angles of △MNP be put in one-to-one correspondence with the angles of △RST?

So far you have seen that each of the six ways in which the vertices of two triangles can be mapped onto each other determines a different correspondence between the corresponding sides and a different correspondence between the corresponding angles of the two triangles. Next, you will use what you know about con-

gruent segments, congruent angles, and correspondence to determine when two triangles are congruent.

A Two triangles are represented in D6. Use the given diagram for a correspondence between their vertices to name the corresponding sides and the corresponding angles.

B Each sentence in D6 expresses a true statement. Are the corresponding sides of $\triangle ABC$ and $\triangle JKL$ congruent? Are the corresponding angles of these two triangles congruent?

There is a correspondence between the vertices of $\triangle ABC$ and $\triangle JKL$ in which the corresponding sides are congruent, and the corresponding angles are congruent. We say that this correspondence between the vertices of the two triangles is a *congruence correspondence*. If there is a congruence correspondence between the vertices of two triangles, then the triangles are congruent. If two triangles are congruent, then there is a congruence correspondence between the vertices of the triangles.

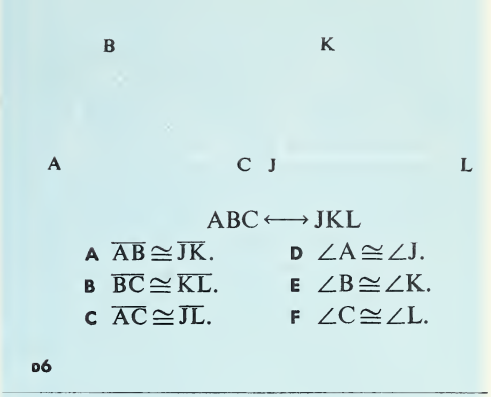
C Look again at D6. Is $\triangle ABC \cong \triangle JKL$? Explain your answer.

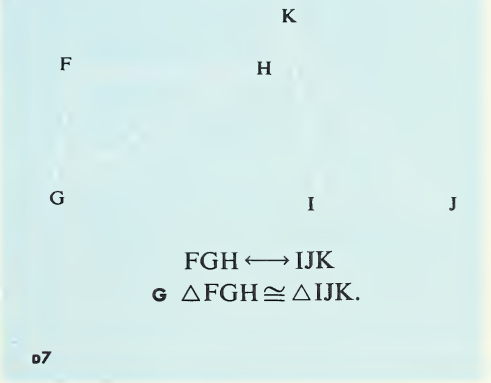
In the sentence $\triangle PQR \cong \triangle STU$, the order in which the vertices are named indicates the congruence correspondence between the vertices. When we speak of corresponding sides or corresponding angles of congruent triangles, we will mean the corresponding parts determined by a congruence correspondence between the vertices.

D $\triangle PQR \cong \triangle STU$. Name the corresponding sides of the two triangles. Are the corresponding sides congruent? How do you know?

E Name the corresponding angles of $\triangle PQR$ and $\triangle STU$. How do you know that the corresponding angles are also congruent?

F Is a correspondence between the vertices of any pair of triangles always a congruence correspondence?





G There are six correspondences between the vertices of any pair of congruent triangles. Is each correspondence always a congruence correspondence? Explain your answer.

Now you know that, if a correspondence between the vertices of two triangles is a congruence correspondence, the corresponding sides and the corresponding angles are congruent, and the two triangles are congruent. Next, you will learn how you can decide if two triangles are congruent.

H Use the diagram for correspondence in D7 to name the corresponding sides and the corresponding angles of the two triangles represented in the display.

I Look at D7. Sentence G expresses a true statement about the triangles that are represented. How do you know that $\overline{FG} \cong \overline{IJ}$?

J Name the other two pairs of corresponding sides. Are these corresponding sides also congruent?

K What is the converse of the following true statement? "If two triangles are congruent, then the corresponding sides of the two triangles are congruent." Do you think that the converse is also true?

It is true that, if the corresponding sides of two triangles are congruent, then the two triangles are congruent. This property of two triangles is expressed below.

For any pair of triangles, ABC and DEF, if $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, *and* $\overline{BC} \cong \overline{EF}$, *then* $\triangle ABC \cong \triangle DEF$.

L Look again at D7. How do you know that $\angle F \cong \angle I$? That $\angle G \cong \angle J$? That $\angle H \cong \angle K$?

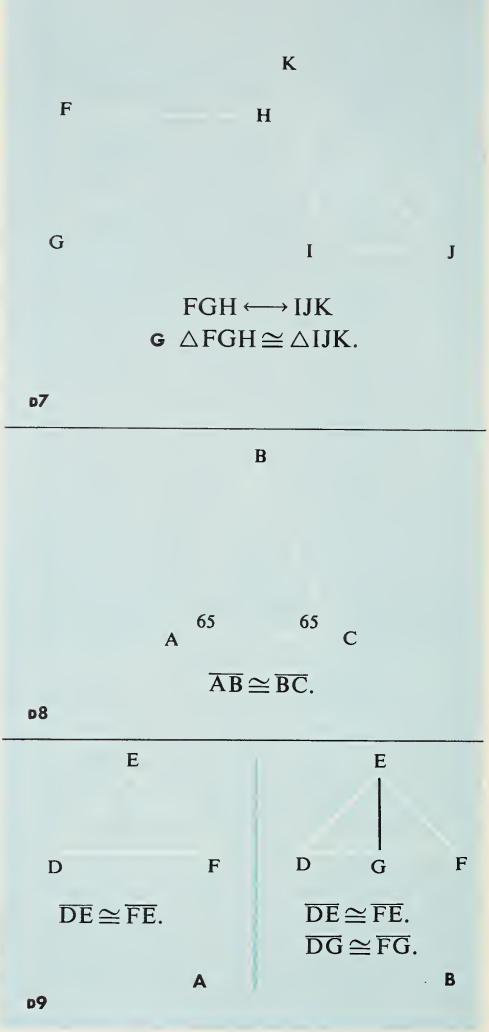
M What is the converse of the following true statement? "If two triangles are congruent, then the corresponding angles of the two triangles are congruent." Do you think that the converse is also true? Make a sketch to help explain your answer.

If the corresponding sides of two triangles are congruent, then the two triangles are congruent. You can use this information to learn more about isosceles triangles.

A Look at D8. How do you know that the triangle represented in this display is an isosceles triangle?

B Does a side of $\angle C$ intersect $\overline{AB}$ in point A? In which point does the other side of $\angle C$ intersect $\overline{AB}$?

Since the intersection of $\angle C$ and $\overline{AB}$ contains exactly two points, $\angle C$ is the angle *opposite* side AB of $\triangle ABC$. Side AB is the side opposite $\angle C$ of $\triangle ABC$.



C Which angle is opposite side BC? How do you know? Is $\angle A \cong \angle C$?

You can show that, for any isosceles triangle, the angles opposite the congruent sides of the triangle are congruent.

D Look at D9. How do you know that $\triangle DEF$, represented in picture A, is an isosceles triangle? Name the angles opposite the congruent sides of $\triangle DEF$.

Now look at picture B. Point G has been chosen so that $\overline{DG} \cong \overline{FG}$.

E Do point G and vertex E of $\angle DEF$ determine a segment? Is this segment represented in picture B? Name this segment.

F Think about triangles DEG and FEG. Is $\overline{EG} \cong \overline{EG}$? Are three sides of $\triangle DEG$ congruent to the corresponding sides of $\triangle FEG$? Is $\triangle DEG \cong \triangle FEG$?

G Are $\angle D$ and $\angle F$ corresponding angles of $\triangle DEG$ and $\triangle FEG$? Is $\angle D \cong \angle F$?

We have shown that the angles opposite the congruent sides of an isosceles triangle are congruent. This property of the angles of an isosceles triangle is expressed below.

For any triangle XYZ, if $\overline{XY} \cong \overline{ZY}$, *then* $\angle X \cong \angle Z$.

H Is each angle of any equilateral triangle congruent to each of the other angles of the triangle? Explain your answer.

Now you know what is meant by a congruence correspondence between two triangles. You also know that, for any isosceles triangle, the angles opposite the congruent sides are congruent.

On your own

Use the triangles represented in D10 in connection with exercises 1 through 5.

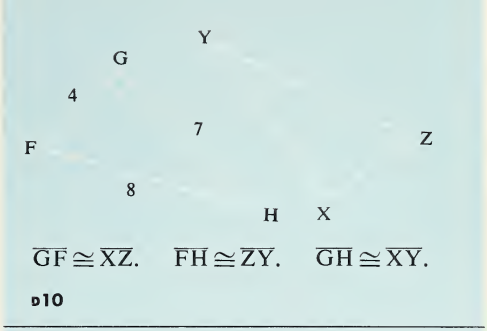
1 Are the two triangles represented in D10 congruent? Explain your answer. Make a diagram to represent the congruence correspondence between the vertices of the two triangles.

2 What is the measure of $\overline{XY}$? Of $\overline{ZY}$? Of $\overline{XZ}$?

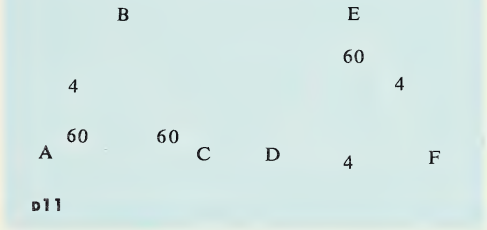
3 Are the corresponding angles of $\triangle GFH$ and $\triangle XZY$ congruent?

4 Suppose that you put the sets of vertices in one-to-one correspondence by mapping vertices F and X onto each other, vertices H and Z onto each other, and vertices G and Y onto each other. For this correspondence between vertices, name the corresponding sides and the corresponding angles of the two triangles.

F ↔ X, H ↔ Z, G ↔ Y



D10



D11

5 Are the corresponding sides you named for exercise 4 congruent? Is there a congruence correspondence between the vertices?

Use the triangles represented in D11 to help you with exercises 6 through 12.

6 What is the measure of $\angle B$? Of each of the angles of $\triangle DEF$?

7 Does $\triangle ABC$ have three congruent sides? Does $\triangle DEF$ have three congruent sides?

8 Is $\triangle ABC$ equilateral?

9 Is each side of $\triangle ABC$ congruent to each side of $\triangle DEF$?

10 Make diagrams to show the six ways of putting the vertices of the two triangles in one-to-one correspondence.

11 How many of the correspondences that you named for exercise 10 are congruence correspondences? All - equilateral triangles

12 Explain why each of the six correspondences between the vertices of the two equilateral triangles is a congruence correspondence.

The quadrilateral

In lesson 29, *Book I*, you learned that a simple polygon is a special kind of simple closed curve. In this lesson you will study simple polygons that have four sides.

A Look at D1. Points A, B, C, and D are coplanar points. How do you know that ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are coplanar lines?

B Name the four segments represented in D1 that are determined by points A, B, C, and D. Each of the segments that you named is in which line?

C Is the geometric figure formed by the union of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ a simple closed curve? Is figure ABCD a simple polygon?

D Name the sides and the vertices of polygon ABCD. How many sides does polygon ABCD have?

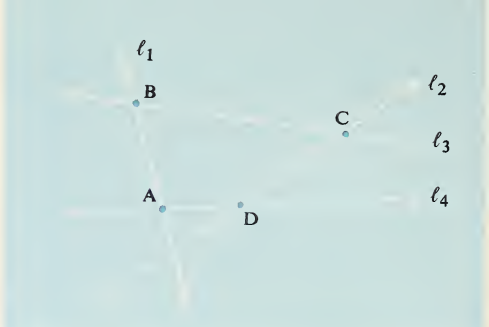
Polygon ABCD is a *quadrilateral*. A quadrilateral is a simple polygon that has four sides.

E Look again at D1. Do $\overline{AB}$ and $\overline{CD}$ intersect? Describe $\overline{AB} \cap \overline{CD}$.

$\overline{AB}$ and $\overline{CD}$ are *opposite sides* of the quadrilateral ABCD. Opposite sides of a quadrilateral are two sides that do not intersect.

F Tabulate $\overline{BC} \cap \overline{DA}$. Why are $\overline{BC}$ and $\overline{DA}$ opposite sides of quadrilateral ABCD?

You know that the sides of triangles and quadrilaterals are segments and that the sides of angles are rays. Triangles and quadrilaterals, therefore, do not include angles. When we speak of an angle of a triangle or a quadrilateral, we mean an angle each of whose sides includes a side of the triangle or quadrilateral.



D1

quad ri lat er al (kwod′rə lat′ər əl). A simple polygon that has four sides.

G Look again at D1. Name the four angles of quadrilateral ABCD. Name the two sides of each of these angles.

H What rays form $\angle A$ of the quadrilateral? $\angle C$ of the quadrilateral?

I Tabulate the intersection of side AB of $\angle A$ and side CB of $\angle C$. Tabulate $\overrightarrow{AD} \cap \overrightarrow{CD}$.

J How many points are in the intersection of $\angle A$ and $\angle C$? Name these points.

We call $\angle A$ and $\angle C$ *opposite angles* of quadrilateral ABCD. Opposite angles of a quadrilateral are two angles of the quadrilateral that intersect in exactly two points. Remember that the opposite angles of a quadrilateral are different from the opposite angles formed by two intersecting lines.

K Look at D1 again. Name the vertex of $\angle A$. Name the vertex of $\angle C$.

Points A and C are *opposite vertices* of quadrilateral ABCD. Opposite vertices of a quadrilateral are the vertices of two opposite angles.

L Tabulate $\angle B \cap \angle D$. Why are $\angle B$ and $\angle D$ opposite angles of quadrilateral ABCD?

Next you will study a special kind of quadrilateral and some of its properties.

A Quadrilateral EFGH is represented in D2. Name the two pairs of opposite sides of quadrilateral EFGH.

B $\overline{EF}$ is a subset of which line? $\overline{GH}$ is a subset of which line? $\overline{FG}$ is a subset of which line? $\overline{HE}$ is a subset of which line?

C $\ell_5 \cap \ell_6 = \{\ \}$. How do you know that ℓ_5 and ℓ_6 are parallel lines?

Notice that $\overline{EF}$ is included in ℓ_5 and $\overline{GH}$ is included in ℓ_6. Since ℓ_5 and ℓ_6 are parallel, $\overline{EF}$ and $\overline{GH}$ are *parallel segments*. Parallel segments are segments that are included in parallel lines.

D ℓ_7 and ℓ_8 are also parallel. What do you know about the opposite sides FG and HE?

Notice that the opposite sides of quadrilateral EFGH are parallel. Quadrilateral EFGH is a *parallelogram*. A parallelogram is a quadrilateral in which the opposite sides are parallel.

E Now look at D3. Figure IJKL is a parallelogram. Name the two pairs of opposite sides of parallelogram IJKL. Name the two pairs of opposite angles.

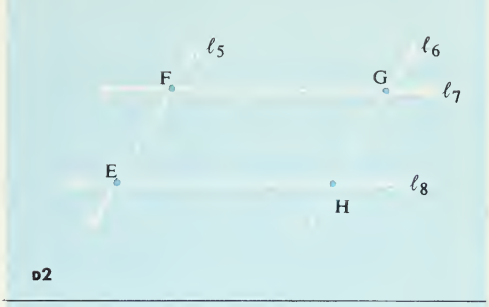
F Name the two pairs of opposite vertices of parallelogram IJKL.

G What segment is determined by points I and K?

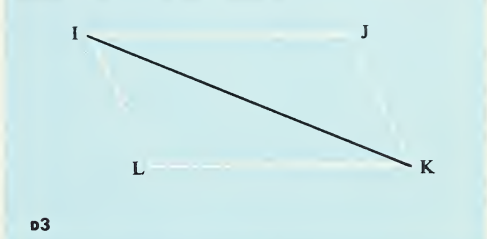
$\overline{IK}$ is a *diagonal* of parallelogram IJKL. A diagonal of a parallelogram is a segment determined by two opposite vertices of the parallelogram.

H Look again at D3. $\triangle IJK$ is the union of what three segments? $\triangle KLI$ is the union of what three segments?

Notice that $\triangle IJK$ is formed by the union of two sides of parallelogram IJKL and diagonal IK. Notice also that $\triangle KLI$ is formed by the union of the other two sides of parallelogram IJKL and diagonal IK. We say, therefore, that a parallelogram and one of its diagonals form two triangles.



D2



D3

par al lel o gram (par′ə lel′ə gram). A quadrilateral in which the opposite sides are parallel.

di ag o nal of a parallelogram (dī ag′ə nəl). A segment determined by two opposite vertices of a parallelogram.

I Now think about $\overline{JL}$. How do you know that $\overline{JL}$ is a diagonal of parallelogram IJKL?

J Name the two triangles formed by parallelogram IJKL and diagonal JL.

In this book we will accept without proof the property that is expressed below.

A parallelogram and either of its diagonals form two congruent triangles.

K Parallelogram IJKL and diagonal IK form two triangles so that $\triangle IJK \cong \triangle KLI$. Name the corresponding sides and the corresponding angles of $\triangle IJK$ and $\triangle KLI$.

L Is $\overline{IJ} \cong \overline{KL}$? Is $\overline{JK} \cong \overline{LI}$? Is $\angle IJK \cong \angle KLI$? Explain your answers.

M The parallelogram and diagonal JL form ? two triangles so that $\triangle JKL \cong \triangle LIJ$. Is $\angle JKL \cong \angle LIJ$? How do you know?

Now you know that the opposite sides of parallelogram IJKL are congruent. You know also that the opposite angles are congruent.

You have learned two important properties that concern the sides and angles of a parallelogram. These properties are expressed below.

The opposite sides of a parallelogram are congruent.

The opposite angles of a parallelogram are congruent.

In the following exercises, you will study special kinds of parallelograms.

A Look at D4. Polygon NOPQ is a parallelogram. The measure of ∠QNO is expressed in the display. Is ∠QNO a right angle?

Now you are going to decide whether each of the other angles of parallelogram NOPQ is a right angle, also.

B Are ∠QNO and ∠RNQ a linear pair of angles? How do you know that ∠RNQ is a right angle?

C Are ∠RNQ and ∠NOP corresponding angles? How do you know that ∠NOP is a right angle?

D How do you know that ∠QNO ≅ ∠OPQ? How do you know that ∠NOP ≅ ∠PQN?

E Which angles of parallelogram NOPQ are right angles?

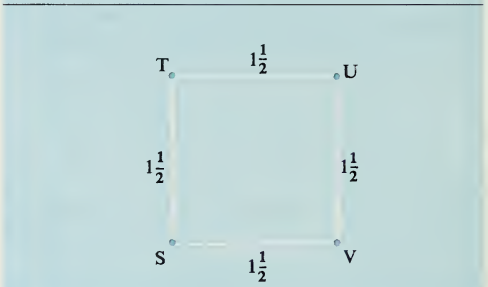
You have developed a property concerning parallelograms. The property is expressed below.

If one angle of a parallelogram is a right angle, then each of the angles of the parallelogram is a right angle.

Each angle of parallelogram NOPQ is a right angle. Parallelogram NOPQ is a *rectangle*. A rectangle is a parallelogram in which each angle is a right angle.

F Suppose that ∠M of parallelogram KLMN is a right angle. What do you know about each of the angles of parallelogram KLMN? How do you know that KLMN is a rectangle?

D4



D5

rectangle (rek′tang′gəl). A parallelogram in which each angle is a right angle.

square (skwār). A rectangle in which each of the sides is congruent to each of the other sides.

Another property of parallelograms is expressed below.

If one angle of a parallelogram is a right angle, the parallelogram is a rectangle.

G Look at D5. Polygon STUV is a rectangle. The measure of each side of rectangle STUV is expressed in the display. How do you know that each side of rectangle STUV is congruent to each of the other sides?

If each side of a rectangle is congruent to each of the other sides, the rectangle is a *square*.

H Is rectangle STUV a square?

Any two sides of a quadrilateral that intersect are called *adjacent sides*. In square STUV, represented in D5, side ST and side TU are adjacent.

I How do you know that, if two adjacent sides of a rectangle are congruent, the rectangle is a square?

J Is every parallelogram a quadrilateral? Is every quadrilateral a parallelogram?

K Is every rectangle a quadrilateral? Is every rectangle a parallelogram? Is every parallelogram a rectangle?

In this lesson you studied different kinds of quadrilaterals. You learned that parallelograms are quadrilaterals whose opposite sides are parallel. You also developed several properties of parallelograms and learned what we mean by a rectangle and a square.

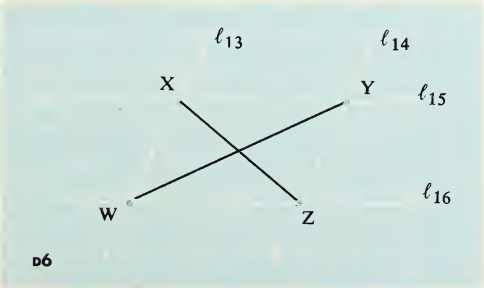
On your own

For each of exercises 1 through 11, use D6 to help you answer the questions.

1 Each of points W, X, Y, and Z is in plane WXY. Name a segment, represented in D6, that is a subset of ℓ_{13}. A subset of ℓ_{14}. A subset of ℓ_{15}. A subset of ℓ_{16}.

2 Is figure WXYZ a triangle? A quadrilateral?

3 Name the two pairs of opposite sides of figure WXYZ.



D6

4 $\ell_{13} \cap \ell_{14} = \{\ \}$. $\ell_{15} \cap \ell_{16} = \{\ \}$. What kind of lines are ℓ_{13} and ℓ_{14}? What kind of lines are ℓ_{15} and ℓ_{16}?

5 Name a segment, represented in D6, that is parallel to $\overline{WX}$. That is parallel to $\overline{XY}$.

6 How do you know that figure WXYZ is a parallelogram?

7 Name a side of parallelogram WXYZ that is congruent to $\overline{WX}$. To $\overline{XY}$.

8 Name the two pairs of opposite angles of parallelogram WXYZ.

9 Name an angle of parallelogram WXYZ that is congruent to $\angle W$. To $\angle X$.

10 Name the triangles formed by diagonal XZ and parallelogram WXYZ.

11 How do you know that triangles WXY and YZW are congruent?

12 Suppose that figure BCDE is a parallelogram in which $\angle B$ is a right angle. What is the measure of each angle of figure BCDE?

KEEPING SKILFUL

For each exercise below, find the sum, difference, product, or quotient.

1 $.76 + 8.2 + .14$

2 $1\frac{3}{4} \times \frac{5}{7}$

3 $31.6 - 9.86$

4 $150\frac{2}{3} + 698\frac{11}{12}$

5 3.02×1.8

6 $90\frac{3}{4} - 72\frac{2}{5}$

7 $\frac{5}{8} \div \frac{3}{11}$

8 $3\frac{1}{3} \times 4\frac{1}{8}$

9 $29\frac{1}{7} + 64\frac{5}{8}$

10 $16 - 3.9076$

11 $.17 \times 8.92$

12 $298\frac{1}{2} - 179\frac{6}{7}$

13 $\frac{11}{12} \div 7\frac{1}{3}$

14 $.075 \times .96$

15 $.186 + 106 + 9.84$

16 $5\frac{3}{8} \div 6\frac{2}{5}$

17 $901\frac{7}{9} + 898\frac{4}{5}$

18 $100\frac{1}{8} - 72\frac{11}{13}$

19 $9\frac{1}{11} \div 8\frac{4}{7}$

20 $.7623 \times 100$

21 $62.68 \div 6.8$ (Hundredths)

22 $1035 \div 13.8$ (Ones)

23 $.0987 \div .138$ (Hundredths)

24 $24.581 \div 1.32$ (Tenths)

25 $.3834 \div 7.4$ (Thousandths)

APPLYING MATHEMATICS

For each problem, write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

In these problems, when we say "rectangular," "triangular," "the shape is square," and so on, we mean that the boundary should be thought of as a rectangle, a triangle, a square, and so on. The lengths of two adjacent sides of a rectangle are usually referred to as "length" and "width."

1 Pat is going to put a fence around a rectangular flower garden. The length of the rectangle is $8\frac{1}{2}$ ft. The width of the rectangle is 4 ft. How many feet of fencing does Pat need for enclosing the garden? U = Z.

THINK The number of feet of fencing that Pat needs is the same number as the sum of the measures of the sides of the rectangle. Is $8\frac{1}{2} + 4 + 8\frac{1}{2} + 4 = x$ the condition for the problem?

For problem 1, you found the sum of the measures of the sides of a rectangle. The sum of the measures of the sides of a simple polygon is the *perimeter* of the polygon.

2 Mr. Long made a triangular patio with sides of 6 yd., 12 yd., and 15 yd. What is the perimeter of the patio in yards? U = Z.

3 The shape of a baseball diamond is square. The length of a side is 90 ft. What is the perimeter of a baseball diamond in feet? U = Z.

4 Use your answer to problem 3 to find the answer to the following problem: What is the perimeter of a baseball diamond in yards? U = Z.

THINK You will use the answer for problem 3. Is $3/1 \sim 360/x$ a condition for the problem?

5 Ken has a piece of plywood in the shape of a parallelogram with sides of 9 ft. and $17\frac{1}{2}$ ft. What is the perimeter of the piece of plywood in inches? U = Z × Z.

6 Sue is going to put lace edging on a rectangular tablecloth that is 70 in. long and 54 in. wide. If she allows 4 in. for waste, how many inches of edging does she need in all? U = Z.

7 Use your answer to problem 6 to find the answer to the following problem: The edging that Sue is going to buy costs $.35 a yard. How much in all will the edging for the tablecloth cost? U = Z × Z.

8 The length of each side of a triangular window is $35\frac{1}{2}$ in. What is the perimeter of this window in inches? U = Z.

9 The lengths of two sides of a parallelogram are $11\frac{1}{3}$ ft. and $9\frac{2}{3}$ ft. What is the perimeter of the parallelogram in yards? U = Z × Z.

10 The length of a rectangle is $3\frac{1}{2}$ ft. The width of the rectangle is 64 in. What is the perimeter of the rectangle in inches? U = Z × Z.

For problems 11 through 13, the universe is the set of rational numbers between 0 and 180.

11 $\angle m$ and $\angle n$ are supplementary angles. $\angle m^\circ = 127$. What is the measure of $\angle n$?

12 $\angle g^\circ = 108$. The sum of $\angle g^\circ$ and $\angle h^\circ$ is 253. What is the measure of $\angle h$?

13 The measure of each of two angles of an isosceles triangle is 11.5. What is the measure of the third angle of the triangle?

For problems 14 and 15, the universe is the Cartesian set formed from the set of rational numbers between 0 and 180.

14 $\angle k^\circ = 82$. $\angle h$ and $\angle k$ are supplementary angles. The sum of $\angle h^\circ$ and $\angle m^\circ$ is 277. What is the measure of $\angle h$? Of $\angle m$?

15 $\angle b^\circ = 60$. The sum of $\angle b^\circ$ and $\angle c^\circ$ is 110. The sum of $\angle c^\circ$ and $\angle d^\circ$ is 215. What is the measure of $\angle c$? Of $\angle d$?

Area of the rectangle

In lesson 98 you learned to assign a number to a segment. This number is the measure of the segment. In lesson 101 you learned to assign a number to an angle. This number is the measure of the angle. In this lesson you will learn to assign a number to a region consisting of a rectangle and its interior. This number will be the area of the rectangle.

A Which geometric figures represented in D1 are simple closed curves? Which are simple polygons?

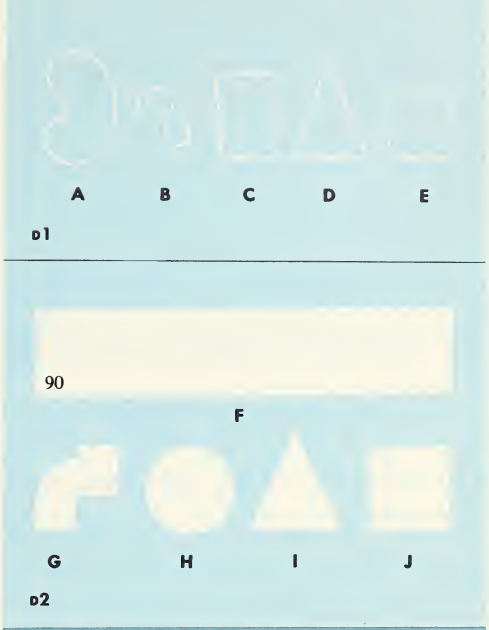
In lesson 29, *Book I*, you learned that a simple closed curve separates a plane into two regions. One of the regions is the interior of the simple closed curve, and the other region is the exterior. The simple closed curve is the boundary of each of the regions and is not included in either region.

B Tabulate the intersection of a simple closed curve and its interior. Are a simple closed curve and its interior disjoint sets? What points are in the union of a simple closed curve and its interior?

The union of a simple closed curve and its interior is called a *closed region.*

C Look again at D1. You can think of picture A as representing a closed region. What other pictures in D1 can represent closed regions? Explain why picture B cannot represent a closed region.

Now suppose that you want to find the measure of a closed region. First you must have a unit closed region, just as you had a unit segment and a unit angle. In this book, we will call a unit closed region a *unit region.*



closed region (rē′jən). The union of a simple closed curve and its interior.

area of a closed region (ãr′ēə). The number of unit regions that do not overlap and that can be included in the closed region.

The number of unit regions that do not overlap and that can be included in a closed region is the *area* of the closed region. The area of a unit region is 1.

D Now look at D2. Picture F represents the union of a rectangle and its interior. Is the union of the rectangle and its interior a closed region? How do you know?

A closed region formed by the union of a rectangle and its interior is called a *rectangular region.* First you will find the area of a rectangular region like the one represented in picture F.

Concept of area of a closed region; finding the area of a rectangular region

E Each of the closed regions represented in pictures G, H, I, and J is a unit region. What is the area of each of these closed regions?

F Which of the unit regions represented in D2 do you think is the most convenient to use in finding the area of a rectangular region?

The unit region represented in picture J is a square unit region, which we will call a *square unit*. A square unit is the union of a square and its interior. The measure of each side of the square is 1.

You will use a square unit for finding areas. The union of the square units into which a closed region can be subdivided is the given closed region.

G Look again at D2. Suppose that 5 square units like the one represented in picture J are included without overlapping in the closed region represented in picture F. What is the area of the rectangular region in square units?

Now you will use a condition for equivalence to find the area of the rectangular region represented in D3. In the following exercises, we will say "area of a rectangle" to mean "area of a rectangular region."

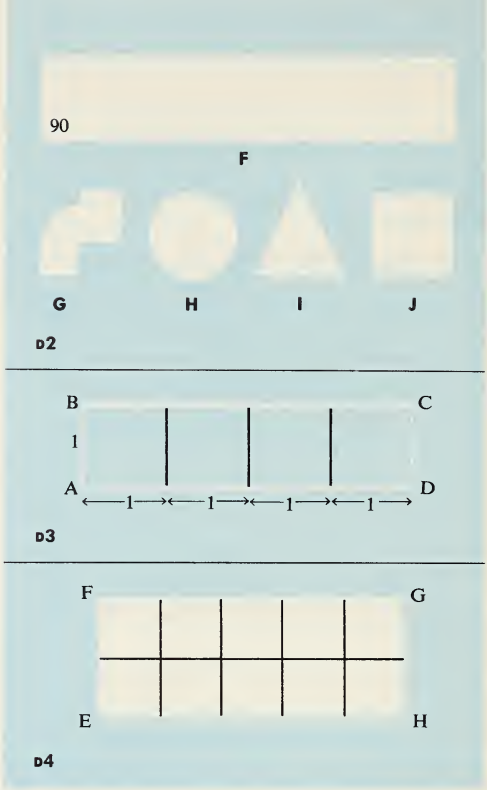
Look at D3. The rectangular region formed by rectangle ABCD and its interior is subdivided into square units.

H How many square units are included in the rectangular region ABCD? Do these square units overlap? What is the union of these square units?

I What is the area of rectangle ABCD?

Since 4 square units are included in the rectangular region ABCD, the area of rectangle ABCD is 4. You can also say that the area of rectangle ABCD is 4 square units.

Notice that you can obtain the area by finding the sum $1 + 1 + 1 + 1$. In doing this, you use the property of area that is expressed in the next column.



If a rectangular region is the union of square units that do not overlap, then the area of the rectangular region is the sum of the areas of the square units.

J The picture in D4 represents rectangular region EFGH subdivided into square units. What is the measure of each side of rectangle EFGH?

K How many square units are included in the bottom row represented in D4? You can use the rate pair 5/1 to represent the fact that there are 5 square units in 1 row.

L How many rows of square units are included in the rectangular region represented in D4?

To find the number of square units in the 2 rows that are included in the rectangular re-

gion EFGH, you can find the first component of a rate pair that is equivalent to 5/1 and that has 2 as a second component. In $x/2$, x is a variable for the number of square units in the 2 rows. U = Z.

M Is the area of rectangle EFGH the same number as the number of square units in the 2 rows? Is x also a variable for the area of rectangle EFGH?

N Can you use the condition expressed below at the right to find the number of square units in the rectangular region EFGH?

$5/1 \sim x/2$

O Tabulate $\{x|5/1 \sim x/2\}$. How many square units are there in the rectangular region EFGH?

P What is the area of rectangle EFGH?

Next look at D5. Notice that the rectangular region JKLM is subdivided into square units and parts of square units.

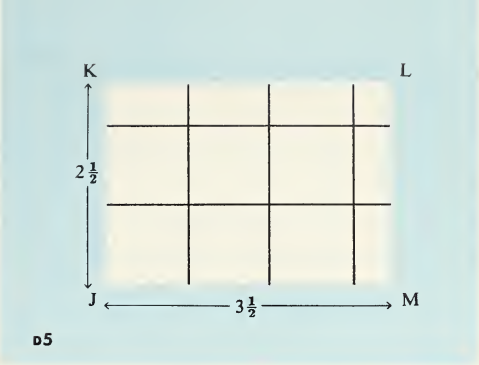
Q In rectangle JKLM, represented in D5, what is the measure of side JM? What is the measure of side KJ?

R Are more than 3 square units included in the bottom row of the rectangular region JKLM? Are fewer than 4 square units included in this row? The measure of side JM tells you how many square units are in the bottom row of the rectangular region JKLM. Are $3\frac{1}{2}$ square units included in this row of the rectangular region?

S The measure of side KJ tells you how many rows of square units there are in the rectangular region JKLM. How many rows are in the rectangular region?

T You can find the area of rectangle JKLM in the same way you found the area of the rectangle represented in D4. What does the rate pair $3\frac{1}{2}/1$ represent?

U Use m as a variable for the area of rectangle JKLM. Can you use the rate pair $m/2\frac{1}{2}$ to



D5

represent the fact that there are m square units in $2\frac{1}{2}$ rows? U = Z.

V Does the sentence at the right express a condition for finding the area of rectangle JKLM in square units?

$3/1 \sim m/2\frac{1}{2}$

W What is the area of rectangle JKLM in square units?

X The measure of side WX of rectangle WXYZ is 2.2, and the measure of side XY is 8.3. What condition can you use to find the area of rectangle WXYZ in square units? U = Z.

Y What is the area of rectangle WXYZ in square units?

In lessons 99 and 100 you worked with standard units of length, such as the inch, the foot, and so on. The square units you have been using so far in this lesson are not standard units. When you are finding areas, you will work with standard square units, such as the *square inch*, the *square foot*, the *square yard*, the *square rod*, and the *square mile*.

A A square inch is a square unit, each of whose sides is 1 inch long. A square foot is a square unit, each of whose sides is 1 foot long. Describe a square yard. Describe a square mile. Describe a square rod.

Now read problem A in D6. To answer the question in problem A, you must find the area of the floor in square feet. In the following exercises, you will develop a condition for the problem. U = Z.

B Why can you use the rate pair expressed at the right in a condition for the problem? $15/1$

C The other rate pair in the condition is expressed at the right. For what is x a variable? $x/12\frac{1}{2}$

D Does the sentence at the right express a condition for the problem? $15/1 \sim x/12\frac{1}{2}.$

E Tabulate $\{x \mid 15/1 \sim x/12\frac{1}{2}\}$.

Now you can give the answer to problem A.

$187\frac{1}{2}$ square feet of carpeting are needed to cover the floor from wall to wall.

F Is $187\frac{1}{2}$ square feet of carpeting the same as $187\frac{1}{2}$ square inches of carpeting? As $187\frac{1}{2}$ square yards of carpeting?

G Explain why, when you use different standard square units to find an area, you obtain different areas for the same rectangle.

Read problem B in D6. You will develop a compound condition for this problem. U = Z × Z.

To answer the question in problem B, you can first find the area of the floor of the hallway in square yards.

H If you use 6/1 in the first condition, why will you also use $x/1\frac{1}{3}$, and not $1\frac{1}{3}/x$? For what is x a variable?

I Does the sentence at the right express one simple condition for the problem? $6/1 \sim x/1\frac{1}{3}.$

Next find the cost of x square yards of carpeting.

J Can you use the rate pair 8.77/1 to represent the rate of the price of the carpeting per square yard?

A The length of a rectangular floor is 15 ft. The width of the floor is $12\frac{1}{2}$ ft. How many square feet of carpet are needed to cover the floor from wall to wall?

B Mrs. Ames wants to buy wall-to-wall carpeting for a rectangular hallway that is 6 yd. long and $1\frac{1}{3}$ yd. wide. She has chosen carpeting that is priced at $8.77 per square yard. How much will she have to pay for enough carpeting to cover the hallway floor?

D6

K Use y as a variable for the number of dollars that is the total cost of the carpeting. You know that x is a variable for the number of square yards in the area of the floor. If you use 8.77/1 in the other simple condition for the problem, why will you use y/x, and not x/y, in this condition? For what is y a variable?

L Does the sentence at the right express another simple condition for the problem? $8.77/1 \sim y/x.$

M Does the sentence in red below express a compound condition for problem B?

$$6/1 \sim x/1\tfrac{1}{3} \wedge 8.77/1 \sim y/x.$$

N What replacement for x satisfies $6/1 \sim x/1\frac{1}{3}$?

O If you replace x by 8 in the condition $8.77/1 \sim y/x$, what condition do you obtain?

P What replacement for y satisfies $8.77/1 \sim y/8$?

Q Give the answer to problem B in D6.

In this lesson you learned what a closed region is. You learned that area is a number associated with a closed region. You also learned how to find the area of a rectangle.

On your own

In each of exercises 1 through 14, the length and the width of a rectangle are expressed in standard units. Find the area of the rectangle in standard square units. U = Z.

	Length	Width		Length	Width
1	13 ft.	8 ft.	8	$4\frac{1}{2}$ yd.	$4\frac{1}{2}$ yd.
2	24 in.	5 in.	9	90 ft.	90 ft.
3	$6\frac{1}{2}$ in.	$2\frac{3}{4}$ in.	10	0.5 in.	1.6 in.
4	7 ft.	8 ft.	11	3 mi.	2 mi.
5	1.4 yd.	.9 yd.	12	$17\frac{2}{3}$ in.	3 in.
6	17 mi.	$9\frac{1}{2}$ mi.	13	8.5 yd.	4.6 yd.
7	2.8 in.	3.6 in.	14	10.7 ft.	6.4 ft.

For each of the following problems, write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

15 Clay Township is in the shape of a rectangle 13.9 mi. long and 11.6 mi. wide. What is its area in square miles? U = Z.

16 Mr. Wilson is going to paint a rectangular basement floor. The floor is $12\frac{1}{2}$ ft. wide and $15\frac{1}{3}$ ft. long. What is the area of the floor in square feet? U = Z.

17 Use your answer to problem 16 to find the answer to the following problem: One gallon of paint will cover about 600 square feet of floor. About how many gallons of paint will Mr. Wilson need to give the floor one coat? U = Z.

18 A rectangle is $7\frac{1}{3}$ yd. long and $6\frac{1}{2}$ yd. wide. What is the area of this rectangle in square yards? U = Z.

19 The area of a rectangle is 240 square inches. The length of this rectangle is 20 in. What is the width of the rectangle? U = Z.

20 Mrs. Carpenter bought a mirror for a closet door. The mirror is 18 in. wide and 4 ft. long. What is the area of the mirror in square inches? U = Z × Z.

21 Mr. Gordon is reseeding a rectangular lawn 60 ft. long and 50 ft. wide. He uses 1 lb. of grass seed for 500 square feet of lawn. How many pounds of grass seed will he use in all? U = Z × Z.

22 It cost Mr. Humphries $125 to have the floor of his kitchen covered with tile. If the rectangular floor is 14 ft. long and $11\frac{1}{2}$ ft. wide, what was the cost per square foot to the nearer cent? U = Z × Z.

23 Al and Bob were measuring the top of a bench that they planned to cover with plastic material. Al found that the bench top was $3\frac{1}{2}$ ft. long. Bob found that the bench top was 15 in. wide. What was the area of the bench top in square feet? U = Z × Z.

KEEPING SKILFUL

Tabulate the solution set of each condition expressed below. U = Z.

1	$17\frac{1}{2} + x = 36\frac{3}{4}$.	11	$.86(.3 + .9) = m$.
2	$\frac{5}{6}/1 \sim n/100$.	12	$11\frac{1}{5}/100 \sim s/1$.
3	$.24/.48 \sim r/1$.	13	$\frac{1}{3}/1 \sim \frac{6}{11}/x$.
4	$m + 11\frac{2}{5} = 40\frac{1}{5}$.	14	$\frac{1}{2} \div (\frac{7}{8} \times \frac{3}{14}) = r$.
5	$\frac{1}{4}/\frac{3}{4} \sim n/1$.	15	$\frac{6}{7}/1 \sim x/100$.
6	$12/16 \sim s/24$.	16	$1\frac{3}{4}(1\frac{2}{7} + 2\frac{1}{2}) = n$.
7	$20 - r = 9\frac{9}{11}$.	17	$\frac{9}{11}/1 \sim r/100$.
8	$\frac{1}{2}/9 \sim \frac{1}{4}/x$.	18	$28 \times 69 \div 14 = x$.
9	$.7/8 \sim x/5.6$	19	$18/17 \sim 31\frac{1}{2}/n$.
10	$n - .698 = 1.504$.	20	$9\frac{6}{7} + 6\frac{3}{4} - 8\frac{1}{2} = s$.

For each exercise below, find the sum, difference, product, or quotient.

21	$7\frac{1}{2} \times \frac{4}{5}$	29	$16\frac{3}{4} - 9\frac{7}{8}$
22	$.67 + 8.6 + .5$	30	$.013 \times 86$
23	$1\frac{7}{8} \div 1\frac{1}{4}$	31	$246\frac{4}{7} + 389\frac{3}{8}$
24	4.9×5.6	32	$8.8937 - 6.936$
25	$9\frac{1}{2} - 6\frac{3}{5}$	33	$8\frac{3}{4} \times 4\frac{4}{5}$
26	890×703	34	$164356 - 99805$
27	$8\frac{1}{2} \div 1\frac{5}{16}$	35	$23 \div 7\frac{1}{2}$
28	$6745 + 605 + 917$	36	$.876 \times .038$

Area of the parallelogram and area of the triangle

In the last lesson you learned that area is a number assigned to a closed region. You also learned how to find the area of a parallelogram that is a rectangle. In this lesson you will learn how to find the area of two other important figures—the area of a parallelogram that is not a rectangle and the area of a triangle.

When we speak of the area of a parallelogram, a triangle, or any other simple polygon, we mean the area of the closed region that is the union of the polygon and its interior.

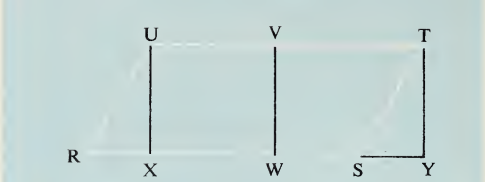
First, you will learn more about the measures of the segments associated with a parallelogram.

A A parallelogram that is not a rectangle is represented in D1. Name the pairs of opposite sides of parallelogram RSTU.

B Tabulate $\overline{VW} \cap \overline{RS}$. Tabulate $\overline{VW} \cap \overline{TU}$.

C Is one endpoint of $\overline{VW}$ contained in the line that includes a side of parallelogram RSTU? Is the other endpoint of $\overline{VW}$ contained in the line that includes the opposite side of parallelogram RSTU?

The line that includes $\overline{VW}$ is perpendicular to both the line that includes $\overline{RS}$ and the line that includes $\overline{TU}$. $\overline{VW}$ is perpendicular to $\overline{TU}$ and to $\overline{RS}$. Two segments, like $\overline{VW}$ and $\overline{RS}$, are perpendicular if they intersect and if they are subsets of lines that are perpendicular. A segment and a given line are perpendicular if they intersect and if the segment is a subset of a line that is perpendicular to the given line.



D1

al ti tude of a parallelogram (al′tə tüd). A segment that is perpendicular to the lines that include a pair of opposite sides of the parallelogram. The endpoints of the segment are contained in the lines that include the opposite sides. Each of the opposite sides is a **base** of the parallelogram.

$\overline{VW}$ is an *altitude* of parallelogram RSTU. An altitude of a parallelogram is a segment that is perpendicular to both of the lines that include a pair of opposite sides of the parallelogram. The endpoints of the altitude are contained in the lines that include the opposite sides. Each of the opposite sides is called a *base* of the parallelogram.

D Look again at D1. $\angle UXS$ and $\angle XUT$ are right angles. How do you know that $\overline{UX}$ is an altitude of parallelogram RSTU?

E $\overline{TY}$ is perpendicular to the lines that include $\overline{TU}$ and $\overline{RS}$. How do you know that $\overline{TY}$ is also an altitude of parallelogram RSTU?

F Explain why $\overline{UR}$ is not an altitude of parallelogram RSTU. Explain why $\overline{RS}$ is not an altitude of the parallelogram.

G $\overline{VW}$ is an altitude of parallelogram RSTU. $\overline{RS}$ is one of the bases associated with this altitude. Which of the other sides is a base associated with altitude VW?

H Look again at D1. Polygon XWVU is a parallelogram. Is polygon XWVU a rectangle? How do you know?

I Is $m(\overline{UX}) = m(\overline{VW})$? How do you know?

J Suppose that polygon DEFG is any parallelogram. Think about the altitudes whose

endpoints are contained in the lines that include one pair of opposite sides of parallelogram DEFG. Use what you know about the opposite sides of a rectangle to explain why each of these altitudes has the same measure.

K Why can any one of the sides of a rectangle be thought of as an altitude of the rectangle? As a base of the rectangle?

Once an altitude is selected for a parallelogram, we will refer to either of the sides included in the lines perpendicular to this altitude as *the* base of the parallelogram.

Now you will learn how to find the area of a parallelogram. To do this, you will use what you know about finding the area of a rectangle.

Look at D2. Polygon ABCD represented in the display is a parallelogram that is not a rectangle. Polygon EFCD is a rectangle.

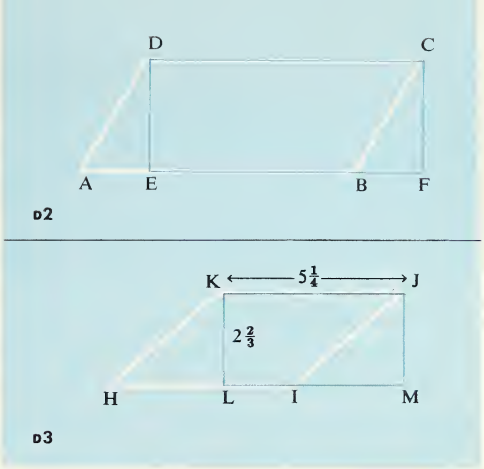
A Is the rectangular region EFCD the union of the closed regions EBCD and BFC? Is the area of rectangle EFCD the sum of the areas of the closed regions EBCD and BFC?

B Is the closed region ABCD the union of the closed regions AED and EBCD? Is the area of the closed region ABCD the sum of the areas of the closed regions EBCD and AED?

We will accept the fact that $\triangle AED \cong \triangle BFC$, and also the fact that two congruent triangles have the same area. Thus, the area of $\triangle AED$ is the same as the area of $\triangle BFC$.

C You know that the area of parallelogram ABCD is the sum of the areas of $\triangle AED$ and polygon EBCD. You also know that the area of rectangle EFCD is the sum of the areas of $\triangle BFC$ and polygon EBCD. How do you know that the area of parallelogram ABCD is the same as the area of rectangle EFCD?

A property that relates the area of a rectangle and the area of a parallelogram that is not a rectangle is expressed in the next column.



If a parallelogram that is a rectangle and a parallelogram that is not a rectangle have a common base and a common altitude, then the two parallelograms have the same area.

D Look again at D2. $\overline{DE}$ is perpendicular to $\overline{AB}$ and $\overline{DC}$. How do you know that $\overline{DE}$ is an altitude of parallelogram ABCD? An altitude of rectangle EFCD?

E Which segment represented in D2 is a common side of parallelogram ABCD and rectangle EFCD? Is this side the common base of these polygons?

F Suppose $m(\overline{DE}) = 3$ and $m(\overline{DC}) = 5$. You can use the condition $5/1 \sim n/3$ to find the area in square units of rectangle EFCD. What is the area of rectangle EFCD? U = Z.

G Look again at D2. Is 15 square units the area of parallelogram ABCD? How do you know?

Look at D3. Polygon HIJK is a parallelogram that is not a rectangle. Polygon LMJK is a rectangle.

H What two altitudes that are common to polygons HIJK and LMJK are represented in D3?

I What is the common base of HIJK and LMJK?

J Write a sentence to express a condition that you can use to find the area of rectangle LMJK.

K Can you use the condition you used for exercise J to find the area of parallelogram HIJK? Explain your answer.

L What is the area in square units of parallelogram HIJK? U = Z.

M Suppose that you know the measure of an altitude and the measure of the base of a parallelogram. Explain how you can use these measures to make a condition for equivalence for finding the area of the parallelogram.

N The length of an altitude of a parallelogram is 5.2 in. The length of the base of the parallelogram is 4.6 in. Write a sentence to express a condition you can use to find the area of the parallelogram in square inches.

O What is the area in square inches of the parallelogram described in exercise N? U = Z.

In the following exercises, you will learn how to find the area of a triangle. You can use what you know about the area of a parallelogram to help you. First, you will learn about the altitude and base of a triangle.

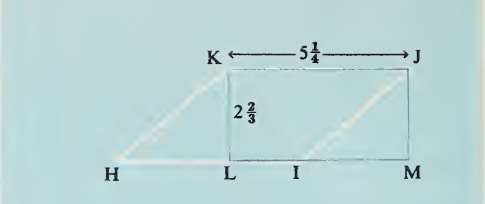
A Look at D4. Name the sides and the angles of △ABC.

B Does $\overline{CD}$ contain vertex C? Tabulate $\overline{CD} \cap \overline{AB}$.

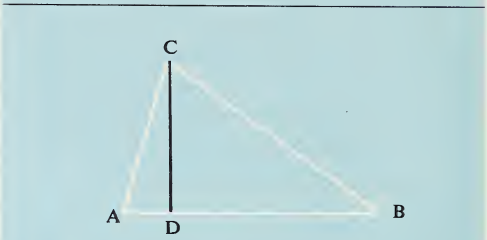
C $\overline{CD}$ is perpendicular to $\overline{AB}$. Is $\overline{CD}$ perpendicular to the side opposite ∠C?

$\overline{CD}$ is an *altitude* of △ABC. An altitude of a triangle is a segment determined by a vertex of an angle of the triangle and a point in the line that includes the side opposite the angle. The altitude is perpendicular to the line that includes the opposite side. The side opposite the angle is called a *base* of the triangle.

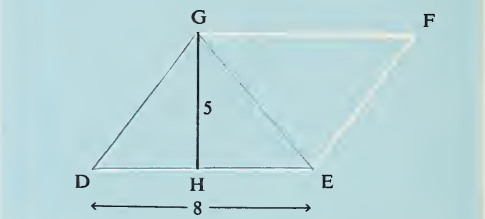
D How many altitudes can a triangle have? How many bases can a triangle have?



D3



D4



D5

altitude of a triangle. A segment determined by a vertex of an angle of the triangle and a point in the line that includes the side opposite the angle. The segment is perpendicular to the line that includes the side opposite the angle. The side opposite the angle is a **base** of the triangle.

E Look at D5. How do you know that $\overline{GE}$ is a diagonal of parallelogram DEFG?

F Name the two triangles formed by parallelogram DEFG and diagonal GE.

G How do you know that the area of △DEG is the same as the area of △GEF?

H Is the area of the parallelogram represented in D5 the sum of the areas of △DEG and △GEF?

I Is the area of parallelogram DEFG twice the area of △DEG? Is the area of parallelogram DEFG twice the area of △GEF?

A property that relates the area of a parallelogram and the area of a triangle is expressed below.

If a parallelogram and a triangle have a common base and a common altitude, then the area of the parallelogram is 2 times the area of the triangle.

J Look again at D5. $\overline{GH}$ is perpendicular to $\overline{DE}$ and to $\overline{GF}$. Is $\overline{GH}$ an altitude of parallelogram DEFG? Of △DEG?

K Is $\overline{DE}$ a base of parallelogram DEFG? Is $\overline{DE}$ also a base of △DEG?

L Can you use the condition $8/1 \sim m/5$ to find the area in square units of parallelogram DEFG? What is the area of the parallelogram? U = Z.

M How can you use the number you found for exercise L to find the area of △DEG?

You know that you can use a condition for equivalence to find the area of a parallelogram. Next you will learn how to use this condition to find the area of a triangle.

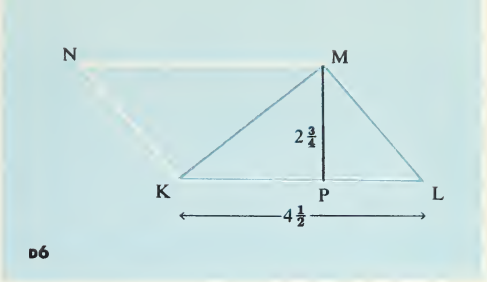
A Look at D6. $\overline{MP}$ is a common altitude of parallelogram KLMN and △KLM. Name the common base of KLMN and △KLM.

B How do you know that the area of KLMN is 2 times the area of △KLM?

C What is the measure of base KL? What is the measure of altitude MP?

Now suppose that x is a variable for the area of △KLM. The universe for x is Z. Since the area of KLMN is 2 times the area of △KLM, you can use $2x$ to refer to the area of KLMN. The condition expressed at the right can be used to find the area of △KLM.

$4\frac{1}{2}/1 \sim 2x/2\frac{3}{4}$.



D6

D How do you obtain $4\frac{1}{2}(2\frac{3}{4}) = 1(2x)$ from $4\frac{1}{2}/1 \sim 2x/2\frac{3}{4}$?

E How do you obtain $4\frac{1}{2}(2\frac{3}{4}) = 2x$ from $4\frac{1}{2}(2\frac{3}{4}) = 1(2x)$?

F How do you obtain $\frac{99}{8} = 2x$ from $4\frac{1}{2}(2\frac{3}{4}) = 2x$? How do you find the number that satisfies $\frac{99}{8} = 2x$? Is $\frac{99}{16}$ the replacement for x that satisfies $\frac{99}{8} = 2x$?

G What is the area in square units of △KLM? Use a mixed numeral to express your answer.

H Suppose that n is a variable for the area of △EFG. $\overline{EN}$ is an altitude of △EFG. The measure of $\overline{EN}$ is $3\frac{1}{3}$. The measure of base FG is 7. Why will you use $2n$ in a condition for finding the area of △EFG?

I Can you use the condition $7/1 \sim 2n/3\frac{1}{3}$ to find the area of △EFG?

J Tabulate $\{n \mid 7/1 \sim 2n/3\frac{1}{3}\}$. U = Z. What is the area of △EFG?

K The length of an altitude of a triangle is 3.2 yd. The length of the base of the triangle is 2.5 yd. What is the area of the triangle in square yards? U = Z.

Now you know how to use conditions for equivalence to find the area of a parallelogram that is not a rectangle. You also know how to find the area of a triangle.

On your own

In each of exercises 1 through 6, the length of an altitude and the length of the base of a parallelogram are expressed. Find the area of the parallelogram in square units. U = Z.

	Altitude	Base		Altitude	Base
1	6 in.	12 in.	4	11.3 ft.	6.5 ft.
2	$12\frac{1}{2}$ ft.	3 ft.	5	$10\frac{3}{4}$ in.	$10\frac{3}{4}$ in.
3	4 yd.	$2\frac{1}{2}$ yd.	6	1.2 mi.	3.765 mi.

In each of exercises 7 through 12, the length of an altitude and the length of the base of a triangle are expressed. Find the area of the triangle in square units. U = Z.

	Altitude	Base		Altitude	Base
7	5 ft.	10 ft.	10	7.2 ft.	7.5 ft.
8	$1\frac{1}{2}$ yd.	4 yd.	11	12 in.	$9\frac{1}{2}$ in.
9	3.6 in.	8.5 in.	12	0.8 yd.	2.5 yd.

The universe is given for each of the following problems. Write a sentence that expresses a condition, tabulate the solution set, and give the answer for each problem.

13 For use in his machine shop, Ray buys metal pieces shaped like a parallelogram. The length of an altitude of each piece is $8\frac{1}{2}$ ft. The length of the base is $5\frac{1}{2}$ ft. What is the area of each piece in square feet? U = Z.

14 Don made a table top out of a triangular piece of wood. The length of an altitude of the piece was 4.2 ft. The length of the base was 5 ft. What was the area of the table top in square feet? U = Z.

15 Joan cut out a piece of felt in the shape of a parallelogram. The length of an altitude of the piece was 8 in. The length of the base was 2 ft. What was the area of the piece of felt in square inches? U = Z × Z.

16 Mrs. Harris plans to put colored glass in a window that is shaped like a parallelogram. The length of an altitude of the window is $2\frac{2}{3}$ ft. The length of the base is 3 ft. If the colored glass costs $1.50 per square foot, how much will the glass for the window cost? U = Z × Z.

17 The Kings have a triangular rose garden. The length of an altitude of the garden is 5 yd. The length of the base is 8 ft. What is the area of the rose garden in square yards? U = Z × Z.

KEEPING SKILFUL

For each exercise below, simplify the expression.

1	$9(62 - 9) + 48 \div 6$	6	$8.67 - 1.2 \div 4$
2	$17 \times 96 \div 32$	7	$36 \times .35 \div .6$
3	$1.5 \times 9.2 + .142$	8	$(16\frac{1}{8} - 8\frac{5}{6}) \times 1\frac{3}{4}$
4	$(4\frac{1}{2} \times 6\frac{2}{5}) \div \frac{15}{4}$	9	$27 \times 6\frac{1}{3} \div 4\frac{2}{3}$
5	$7\frac{1}{2} + 9\frac{2}{3} - 11\frac{3}{4}$	10	$27 \times (6\frac{1}{3} \div 4\frac{2}{3})$

Name the basic fraction that indicates the reciprocal of each number named below.

11	$4\frac{1}{8}$	13	1.6	15	.075	17	8.875	19	$3\frac{8}{9}$
12	.25	14	$7\frac{2}{3}$	16	$9\frac{3}{11}$	18	.036	20	5.13

Express each quotient named below by a mixed numeral.

21	$\frac{17}{8}$	23	$\frac{109}{4}$	25	$\frac{413}{11}$	27	$\frac{698}{8}$	29	$\frac{1000}{17}$
22	$\frac{64}{3}$	24	$\frac{216}{15}$	26	$\frac{100}{7}$	28	$\frac{65}{35}$	30	$\frac{610}{3}$

Tabulate the solution set of each condition expressed below. U = Z.

31	$\frac{3}{4}/1 \sim c/100.$	39	$17\frac{3}{7} + x = 36\frac{1}{6}.$
32	$7/6 \sim 175/x.$	40	$\frac{1}{8}/r \sim 8/1.$
33	$\frac{1}{2}(7\frac{2}{3} \div \frac{7}{8}) = n.$	41	$1.89(7.5) = m.$
34	$12\frac{1}{2}/100 \sim s/1.$	42	$\frac{1}{7}/1 \sim x/100.$
35	$46\frac{1}{3} - d = 23\frac{5}{9}.$	43	$.903 + s = 1.006.$
36	$\frac{1}{3}/\frac{1}{4} \sim x/1.$	44	$.25/1 \sim r/100.$
37	$\frac{5}{7} \div (1\frac{1}{2} \times 8) = m.$	45	$\frac{6}{7}(4\frac{1}{8} + 6\frac{2}{3}) = x.$
38	$66\frac{2}{3}/100 \sim s/1.$	46	$4\frac{6}{11}/100 \sim m/1.$

For each exercise below, find the sum, difference, product, or quotient.

47	$946 + 876 + 145$	55	$.369 \times 1000$
48	3.08×7.05	56	$18.609 - 3.712$
49	$4\frac{2}{3} + 6\frac{5}{8}$	57	$4\frac{3}{8} \times 5\frac{3}{4}$
50	$193456 - 183765$	58	863×109
51	$7.0621 - .9486$	59	14.6×3.96
52	$37\frac{1}{3} - 15\frac{4}{5}$	60	$16\frac{1}{7} - 5\frac{3}{8}$
53	$.46 + .871 + 1.45$	61	809×530
54	$11\frac{1}{5} \times 9\frac{1}{7}$	62	$3\frac{5}{9} \div 7\frac{2}{3}$
63	$9 \div 7$ (Hundredths)		
64	$60564 \div 196$ (Ones)		
65	$90.56 \div .86$ (Tenths)		
66	$204.369 \div 3.12$ (Thousandths)		

APPLYING MATHEMATICS

For each problem, write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 Mr. Jones has a triangular field that is bounded by three highways. The length of each side of the field is 167 rd. What is the perimeter of this field in rods? U = Z.

2 A pilot flew an airplane 2510 mi. in 4 hr. He flew at an average speed of how many miles per hour? U = Z.

3 Carol had 4 cups of milk. She used $\frac{5}{8}$ of the milk to make custard. How many cups of milk did she use to make the custard? U = Z.

4 Henry has painted $\frac{2}{3}$ of the fence around his garden. He has painted $12\frac{1}{2}$ ft. of the fence. How long is the fence around his garden? U = Z.

5 The rectangular floor of a certain convention room is $113\frac{1}{4}$ ft. long and $80\frac{1}{4}$ ft. wide. What is the area of the floor in square feet? U = Z.

6 Marilyn bought a rectangular piece of material that was 54 in. wide and 72 in. long. How many square inches of material did she buy? U = Z.

7 Use your answer to problem 6 to find the answer to the following problem: How many square feet of material did Marilyn buy? U = Z.

THINK 1 square foot is the same area as how many square inches? Can you use $144/1 \sim 3888/x$ as a condition for the problem? For what is x a variable?

8 Mr. Hansen is going to pave his rectangular driveway. The driveway is $50\frac{1}{2}$ ft. long and 6 ft. wide. What is the area of the driveway in square yards? U = Z × Z.

AREA	
144 square inches (sq. in.)	1 square foot (sq. ft.)
9 square feet	1 square yard (sq. yd.)
160 square rods (sq. rd.)	1 acre (A.)
640 acres	1 square mile (sq. mi.)

9 In a certain year the population of Belleville, Ontario, was about $\frac{1}{12}$ the population of Hamilton, Ontario. The population of Belleville was about 20,000. What was the approximate population of Hamilton? U = Z.

10 The Johnson Realty Company bought a rectangular piece of land that was $\frac{1}{2}$ mi. wide and $\frac{1}{4}$ mi. long. How many acres did the company buy? U = Z × Z.

11 Harold has a piece of glass in the shape of a parallelogram. The length of an altitude is $9\frac{1}{4}$ in. The length of the base is 14 in. What is the area of this piece of glass in square inches? U = Z.

12 The shape of a field is square. The length of a side is 60 ft. What is the perimeter of this field in yards? U = Z × Z.

13 In all, Bill walked less than 11 mi. on Friday and Saturday. He walked 5 mi. farther on Saturday than on Friday. How many miles could he have walked each day? U = C × C.

14 Mrs. Harms made a triangular flag. The length of an altitude was $2\frac{1}{2}$ ft. The length of the base was 4 ft. What was the area of the flag in square feet? U = Z.

15 Lake Superior is 383 mi. long. Lake St. Clair is 26 mi. long. Lake Superior is how many times as long as Lake St. Clair? (Hundredths) U = Z.

16 Roger had a board 6 ft. long. He used $3\frac{3}{4}$ ft. of this board to repair the front steps. The length of board Roger used was what fraction of the total length? U = Z.

17 For a bazaar, Mrs. Hayes ordered 13 qt. of coffee ice cream, 11 gal. of vanilla ice cream, and $8\frac{1}{2}$ gal. of chocolate ice cream. In all, she ordered how many gallons of ice cream? U = Z × Z.

18 The areas of three lots are 18,750 sq. ft., 12,750 sq. ft., and 18,000 sq. ft. What is the average area per lot? U = Z × Z.

19 The area of the rectangular floor of a room in a municipal building is 12,788 sq. ft. The width of the floor is 92 ft. What is the length of the floor? U = Z.

20 Mrs. Smith bought a rectangular rug that was 12 ft. long and 8 ft. wide. What was the area of the rug in square yards? U = Z × Z.

21 Use your answer to problem 20 to find the answer to the following problem: If the rug cost $13.50 per square yard, what was the total cost of the rug? U = Z.

22 Mr. Ames sold 420 pecks of potatoes in September and 442 pecks of potatoes in October. What was the per cent of increase in the number of pecks sold? (Tenths) U = Z × Z.

23 Mrs. Jackson's kitchen is square. The length of a side is 8 ft. What is the area of the kitchen floor in square yards? U = Z × Z.

24 Jim built a triangular patio. The length of an altitude was $16\frac{1}{4}$ ft. The length of the base was 26 ft. What was the area of the patio in square feet? U = Z.

25 The area of Mr. Jay's lot is 3150 sq. ft. The area of Mr. Hunt's lot is 10,800 sq. ft. The area of Mr. Jay's lot is what fraction of the area of Mr. Hunt's lot? (Hundredths) U = Z.

26 Mr. Horton's field is a parallelogram in shape. The length of an altitude is 160 rd. The length of the base is 320.5 rd. How many acres are included in Mr. Horton's field? U = Z × Z.

27 Carolyn received a triangular piece of material from Paris. The length of an altitude was 1.75 metres. The length of the base was 2.50 metres. What was the area of the material in square metres? U = Z.

28 The length of a rectangular field is 45 rods. The width of the field is 50 rods. How many acres are in this field? (Tenths) U = Z × Z.

29 The length of a rectangle is $7\frac{1}{2}$ ft. The width of the rectangle is $3\frac{2}{3}$ ft. What is the area of the rectangle in square inches? U = Z × Z.

30 The length of each side of a triangle is $4\frac{1}{3}$ ft. What is the perimeter of this triangle in yards? U = Z × Z.

31 The area of a parallelogram is 4992 sq. rd. The length of a base of the parallelogram is 64 rd. What is the length of the altitude of the parallelogram? U = Z.

32 The width of a rectangle is 16 yd. The length of this rectangle is 59 ft. What is the perimeter of the rectangle in feet? U = Z × Z.

33 The area of a triangle is 23.12 sq. in. The length of an altitude is 3.4 in. What is the length of the base of the triangle? U = Z.

34 The length of an altitude of a parallelogram is 24 in. The length of the base of the parallelogram is $36\frac{1}{2}$ in. What is the area of the parallelogram in square feet? U = Z × Z.

35 The lengths of the sides of a triangle are $17\frac{1}{2}$ yd., 45 ft., and $16\frac{1}{2}$ yd. What is the perimeter of the triangle in yards? U = Z × Z.

36 Use your answer to problem 35 to find the answer to the following problem: What is the perimeter of the triangle in inches? U = Z.

37 The area of a rectangle is 46.24 sq. yd. The length is 3.4 yd. What is the width of the rectangle in yards? U = Z.

38 The length of an altitude of a triangle is 7.6 metres. The length of the base of the triangle is 3.8 metres. What is the area of the triangle in square metres? U = Z.

39 The area of a parallelogram is 86 sq. in. The length of an altitude is 24 in. What is the length of the base in feet? U = Z × Z.

APPLYING MATHEMATICS

For each problem, write a sentence that expresses a condition. If possible, tabulate the solution set. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The universe is given for each problem.

1 Phil had \$5.17. He earned more money, but still had less than \$7.00. How much money did Phil earn? $U = Z$.

2 One gallon of gasoline costs 38.5¢. At this rate, how much did Mr. Carter have to pay for 14 gal. of gasoline? $U = Z$.

3 Last week Mr. Smith drove .5 as many miles as Mr. Hayes. Mr. Hayes drove 128.5 mi. last week. How many miles did Mr. Smith drive? $U = Z$.

4 Richard saves $12\frac{1}{2}\%$ of his monthly take-home pay. He saves \$17.75 each month. How much is Richard's monthly take-home pay? $U = Z$.

5 There are 3.65 times as many provincial parks in British Columbia as there are in Alberta. There are 40 provincial parks in Alberta. How many provincial parks are there in British Columbia? $U = Z$.

6 In a trophy race one summer, the speed of the second-place boat was 102.935 mph (miles per hour). This speed was 5.151 mph less than the speed of the winning boat. What was the speed of the winning boat? $U = Z$.

7 Ellen needs $1\frac{1}{4}$ lb. of salted nuts for a party. The nuts she wants are sold in $\frac{3}{8}$ lb. cans. How many cans of these nuts does Ellen have to buy? $U = Z$.

8 Mr. Keniston had $15\frac{1}{2}$ bushels of apples. He sold some, but not all, of these apples and then had less than $7\frac{1}{8}$ bushels left. How many bushels of apples did Mr. Keniston sell? $U = Z$.

9 In one province, the sales tax on automobiles was 1.5% of the selling price. At this rate, how much sales tax did Mr. Morris have to pay on an automobile priced at \$2400? $U = Z$.

10 Babe Ruth had 128 hits during the 1922 baseball season. He had 205 hits during the 1923 season. What was the per cent of increase in the number of hits? (Hundredths) $U = Z \times Z$.

11 Mrs. Hanson bought $1\frac{3}{4}$ times as much hamburger as pork sausage. She bought $3\frac{1}{2}$ lb. of hamburger. How much pork sausage did she buy? $U = Z$.

12 In 1960, there were 68 weekly newspapers in Manitoba, 87 in British Columbia, 14 in New Brunswick, and 30 in Nova Scotia. What was the average number of weekly newspapers per province in 1960? (Tenths) $U = Z \times Z$.

13 In 1958, 3,000,000 bu. of peaches were harvested in Canada. That same year about 16,000,000 bu. of blueberries were harvested. How many times more bushels of blueberries than bushels of peaches were harvested? $U = Z$.

14 Martha bought $4\frac{1}{2}$ yd. of material to make a dress. After she had made the dress, she had 12 in. of material left. How much material did Martha use to make the dress? $U = Z \times Z$.

15 In 1950, there were about 16,000 motor vehicles registered in Newfoundland. Seven years later, there were about 48,000 motor vehicles registered. What was the per cent of increase? $U = Z \times Z$.

16 The heights of five players on a basketball team are 74 in., 71 in., 70 in., 72 in., and 75 in. What is the average height of these five players? (Tenths) $U = Z \times Z$.

17 At the time of the 1951 census, the population of Brantford, Ontario, was about 36,000. In the census of 1956, the population of Brantford was about 51,000. What was the per cent of increase in population? $U = Z \times Z$.

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you need it.

Test 110

What word, words, or symbols best complete exercises 1 through 8?

1 The difference of the coördinates of two points is the ~~~ of the segment determined by the two points. (5)

2 The measure of a unit segment is ~~~. (6)

3 The sum of the lengths $7\frac{3}{4}$ ft. and $2\frac{2}{3}$ ft. is ~~~ ft. (11)

4 The union of a simple closed curve and its ~~~ is a closed region. (47)

5 1 mile is the same distance as ~~~ feet. (13)

6 The set of quadrilaterals is a subset of the set of ~~~. (42)

7 If $\angle KCN° < 90$, then $\angle KCN$ is an ~~~ angle. (21)

8 The number of unit regions that do not overlap and that are included in a closed region is the ~~~ of the closed region. (47)

Test 111

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

9 If $\overline{AB} < \overline{RS}$, then $m(\overline{RS}) < m(\overline{AB})$. (6)

10 To find the number of ounces in 5 lb., you can use the solution set of the condition $16/1 \sim 5/x$. U = Z. (16)

11 The measure in degrees of any angle is greater than 0 and less than 180. (20)

12 The union of opposite rays is a line. (25)

13 The measure of a unit angle in degrees is 90. (18)

14 If $\angle JPN° > 90$, then $\angle JPN$ is an obtuse angle. (21)

15 Each pair of adjacent angles is also a linear pair of angles. (29)

16 The converse of a true statement may be a false statement. (32)

17 A right triangle cannot have an obtuse angle. (35)

Test 112

From the list below each exercise, choose the word, words, or symbols that correctly complete the exercise.

18 The number associated with a point in a number line is the ~~~ of the point. (5)

a measure
b opposite
c supplement
d coördinate

19 One cup is the same capacity as ~~~ fl. oz. (16)

a 2
b 4
c 8
d 16

20 If 22 is associated with $\overrightarrow{MN}$, and 68 is associated with $\overrightarrow{MO}$, then ~~~ is the measure of $\angle OMN$. (18)

a 22
b 46
c 68
d 90

21 An acute triangle has exactly ~~~ acute angles. (35)

a 0
b 1
c 2
d 3

22 If $\angle a \cong \angle b$, then $\angle a° \sim\sim\sim \angle b°$. (19)

a $>$
b $\cong$
c $<$
d $=$

23 A pair of opposite angles *cannot* be a pair of ~~~ angles. (29)

a congruent
b supplementary
c adjacent
d obtuse

24 In any triangle, the sum of the measure of the angles in degrees is ~~~. (35)

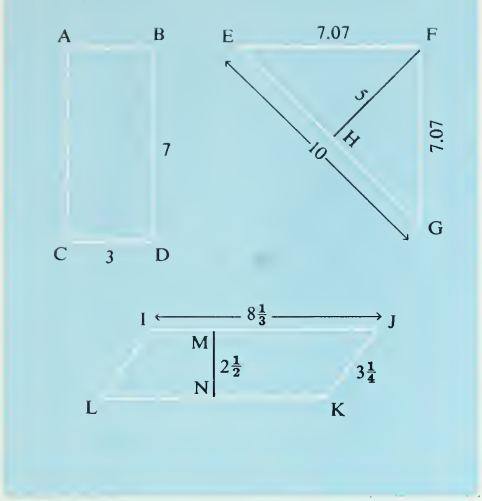
a 60
b 120
c 180
d 360

25 In a right triangle, the sum of the measures of the two acute angles in degrees is ~~~. (35)

a 90
b 180
c greater than 90
d less than 90

26 A right triangle can be ~~~. (35)

a equilateral
b acute
c obtuse
d scalene



Test 113a

Use the display above for each exercise.

27 Identify △EFG according to the measures of its sides. (33)

For exercises 28 and 29, △EFH ≅ △GFH.

28 Side EH and which side of △GFH are mapped onto each other? (39)

29 ∠FHE and which angle of △GFH are mapped onto each other? (39)

For each of exercises 30 through 35, write a sentence that expresses a condition, tabulate its solution set, and give the answer to the exercise. The measures are given in feet.

30 What is the perimeter of △EFG in feet? U = Z. (46)

31 What is the perimeter of rectangle ABDC in feet? U = Z. (46)

32 What is the perimeter of parallelogram IJKL in yards? U = Z × Z. (46)

33 What is the area of rectangle ABDC in square feet? U = Z. (50)

34 What is the area of △EFG in square feet? $\overline{FH}$ is perpendicular to $\overline{EG}$. U = Z. (55)

35 What is the area of parallelogram IJKL in square yards? $\overline{MN}$ is perpendicular to $\overline{KL}$ and to $\overline{IJ}$. U = Z × Z. (53, 54)

Test 113b

What words or symbols best complete exercises 36 through 42?

36 The coördinate of point A is 7.5. The coördinate of point B is 4.7. The measure of segment AB is ~~~. (5)

37 Lines CD and JK intersect in point N. ~~~ are a linear pair of angles. (17)

a ∠JND and ∠KND **c** ∠KNC and ∠DNJ
b ∠CNJ and ∠DNK **d** ∠KND and ∠JNC

38 If two angles form a linear pair, then they are ~~~ angles. (26)

a opposite **c** vertical
b right **d** supplementary

39 Ray ~~~ is the common side of ∠KNM and ∠MNL. (28)

40 A ~~~ triangle is a triangle in which no two sides have the same measure. (33)

a right **c** scalene
b isosceles **d** equilateral

41 In △TUV, side VT is the side opposite ~~~. (40)

42 If each side of a rectangle is congruent to each of the other sides, the rectangle is a ~~~. (44)

Test 113c

What words or symbols best complete exercises 43 through 49?

43 If the measures of two segments are equal, then the segments are ~~~. (6)

44 Two given lines intersect each other. Therefore, the two given lines ~~~. (24)

a are opposite sides of a parallelogram
b are disjoint sets
c are included in the same plane
d form two similar triangles

45 Any pair of ~~~ angles are congruent. (27)

a vertical **c** obtuse
b supplementary **d** linear

46 If any two corresponding angles that are

formed by two coplanar lines and a transversal are congruent, then the two lines are ~~~. (31)

47 In $\triangle RST$, $\angle R^\circ = 42$ and $\angle S^\circ = 35$. Therefore, $\triangle RST$ is ~~~ triangle. (35)

a an acute **c** a right
b an obtuse **d** an isosceles

48 A quadrilateral is a simple polygon that has ~~~ sides. (42)

49 The length of an altitude of a parallelogram is $3\frac{2}{3}$ ft. The length of the base is $5\frac{1}{4}$ ft. The area of the parallelogram is ~~~ sq. ft. (54)

Test 113d

What words or symbols best complete exercises 50 through 56?

50 For any two segments KL and MN, if ~~~, then $\overline{KL} > \overline{MN}$. (6)

a $m(\overline{KL}) = 5.36$ and $m(\overline{MN}) = 3.56$
b $m(\overline{MN}) = 3.65$ and $m(\overline{KL}) = 3.56$
c $m(\overline{KL}) = 5.63$ and $m(\overline{MN}) = 6.53$
d $m(\overline{MN}) = 6.35$ and $m(\overline{KL}) = 5.36$

51 The difference of $7\frac{5}{8}$ ft. and $4\frac{3}{4}$ ft. is ~~~.

52 $\angle ABC^\circ = 50$. $\angle DEF$ is the supplement of $\angle ABC$. $\angle DEF^\circ =$ ~~~. (26)

a 310 **b** 130 **c** 90 **d** 40

53 If two lines are parallel, then the angles of each pair of corresponding angles formed by the two lines and a transversal are ~~~. (31)

a supplementary **c** congruent
b right angles **d** opposite angles

54 There are ~~~ ways of putting the vertices of two triangles in one-to-one correspondence.

55 A ~~~ of a parallelogram is a segment determined by two opposite vertices of the parallelogram. (43)

56 The length of a rectangle is 8 ft. and the width is 5 ft. If x represents the area of the rectangle, then ~~~. (50)

a $5/8 \sim 1/x$ **c** $x/8 \sim 1/5$
b $1/8 \sim x/5$ **d** $8/1 \sim x/5$

Unit 9

Sets of numbers and sets of points

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Compound conditions that include the connective "and"

As you studied the lessons in *Seeing Through Mathematics, Book 1*, you discovered that many ideas—set, condition, and variable, among others—were used again and again. These ideas will continue to be important in the mathematics presented in this book. Therefore, before you examine any new ideas, you will find it worth while to take another look at some of the basic ideas that were introduced in *Book 1*.

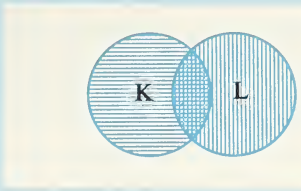
A Sets K, L, and M are tabulated in D1. Which set is the set of natural numbers from 4 through 8? Which set is the set of odd numbers less than 9? Which set is the set of odd numbers greater than 3 and less than 9?

B Is each member of M a member of K? Is each member of M a member of L? Are there any members of M that are not members of both K and L?

As you know, the *intersection* of two sets is the set that contains those objects, and only those objects, that belong to both sets. You can express the intersection of K and L by the symbol $K \cap L$.

$K = \{1, 3, 5, 7\}$.
$L = \{4, 5, 6, 7, 8\}$.
$M = \{5, 7\}$.

D1



D2

C Is $M = K \cap L$? How do you know?

Now look at the Venn diagram in D2. The picture of the rectangle and its interior represents a universe. The pictures of the circles and their interiors represent the sets tabulated in D1.

D The universe is the set of natural numbers. How do you know that each of the three sets tabulated in D1 is a *subset* of the set of natural numbers?

REMINDER

If each member of one set is also a member of a second set, then the first set is a subset of the second set.

See lesson 2, page 11, Book 1.

Certain exercises in this book are labelled with a large question mark. Exercise E below is such an exercise. Try to answer such exercises without help. When you think you know what response to make, turn to the response section in the back of the book. The responses in this section will show you whether or not your thinking is correct. Do not turn to this section until you have tried to respond to the exercise without help.

E ? What numbers are common members of K and L? Which portion of the Venn diagram in D2 represents the intersection of set K and set L?

Look at the Venn diagram in D3. The universe is the set of all geometric figures. A is the set of all triangles. B is the set of all rectangles.

F Is A a subset of the universe? Is B a subset of the universe?

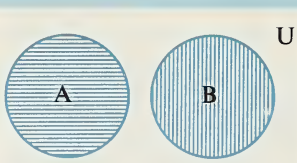
G Do A and B have any common members?

? Are A and B disjoint sets?

H Is the intersection of A and B the empty

? set? Tabulate $A \cap B$.

I How does the diagram in D3 show that $A \cap B$ is the empty set?



D3

Next you will review how to find the solution sets of conditions.

A What conditions are expressed in D4? Notice that, in each condition, the universe for the variable x is R_a. Remember that the symbol R_a is a name of the set of rational numbers of arithmetic.

$U = R_a$.

A $x - \frac{2}{3} > \frac{1}{2}$.

B $\frac{1}{4} + x < \frac{15}{8}$.

C $x + 7.2 > 8.6$.

D $6.5 - x > 1.8$.

D4

C Tabulate the solution set of $x - \frac{2}{3} = \frac{1}{2}$. What computation did you use to obtain the member of the solution set?

REMINDER

A rational number of arithmetic is a set of equivalent fractions.

$\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots\}$ is a rational number.

Any member of $\{\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \ldots\}$ may be used to indicate the rational number.

See lesson 72, page 316, Book 1.

The symbol R_a names the set of all rational numbers of arithmetic.

See lesson 73, page 318, Book 1.

REMINDER

The solution set of a condition consists of all members of the universe that satisfy the condition.

See lesson 4, page 18, Book 1.

Each member of the solution set of a condition is a solution of the condition.

See lesson 6, page 24, Book 1.

First you will find the solution set of condition A. When we say "condition A," we mean "the condition expressed by the open sentence A."

B You know that you can use a condition for equality, $x - \frac{2}{3} = \frac{1}{2}$, to help you find the solutions of condition A. If you know that the difference of x and $\frac{2}{3}$ is $\frac{1}{2}$, do you also know that x is the sum of $\frac{2}{3}$ and $\frac{1}{2}$?

D Is each rational number that is greater than $\frac{7}{6}$ a solution of $x - \frac{2}{3} > \frac{1}{2}$?

E Think about the set whose members are all the rational numbers that satisfy the condition $x > \frac{7}{6}$, or, as we sometimes say, all x that satisfy the condition $x > \frac{7}{6}$. Is this set the solution set of $x - \frac{2}{3} > \frac{1}{2}$?

The symbols $\{x \mid x - \frac{2}{3} > \frac{1}{2}\}$ and $\{x \mid x > \frac{7}{6}\}$ are both standard descriptions of the solution set of $x - \frac{2}{3} > \frac{1}{2}$. However, when you are asked to

give a standard description, always give the simpler description (for example, $\{x|x>\frac{7}{6}\}$).

F Is $\{x|x>\frac{7}{6}\}$ an infinite set? Why can you not tabulate this set?

G Look again at D4. What condition for equality can you use to help you find the solutions of condition B?

H To find the solution of $\frac{1}{4}+x=\frac{15}{8}$, you can again use the relationship between addition and subtraction. If you know that the sum of $\frac{1}{4}$ and x is $\frac{15}{8}$, do you also know that x is the difference of $\frac{15}{8}$ and $\frac{1}{4}$?

I Tabulate the solution set of $\frac{1}{4}+x=\frac{15}{8}$. What computation did you use to obtain the member of the solution set?

J Is each member of R_a that is less than $\frac{13}{8}$ a solution of $\frac{1}{4}+x<\frac{15}{8}$? Is the solution set of the condition an infinite set, or is it a finite set?

K Give a standard description of the solution set of $\frac{1}{4}+x<\frac{15}{8}$.

L What condition for equality can you use to help you find the solutions of condition C?

M If you know that the sum of x and 7.2 is 8.6, do you also know that x is the difference of 8.6 and 7.2?

N Tabulate the solution set of $x+7.2=8.6$.

O Is each rational number that is greater than 1.4 a solution of $x+7.2>8.6$? Give a standard description of the solution set of $x+7.2>8.6$.

P Now look again at D4. What condition for equality can you use to help you find the solutions of condition D?

Q If the difference of 6.5 and x is 1.8, is 6.5 the sum of x and 1.8? If 6.5 is the sum of x and 1.8, is x the difference of 6.5 and 1.8?

R Tabulate the solution set of $6.5-x=1.8$.

S Is each member of R_a that is less than 4.7 a solution of $6.5-x>1.8$? Give a standard description of the solution set of $6.5-x>1.8$.

$U=R_a$.
$\frac{1}{8}+x<\frac{11}{8}\wedge\frac{4}{3}-x>\frac{2}{3}$.

D5

You can use what you know about finding solutions of simple conditions in one variable to find the solutions of compound conditions formed from two simple conditions and the connective "and."

A What two simple conditions make up the compound condition expressed in D5?

B What is the universe for x in each simple condition?

REMINDER

A compound condition is a condition that is made up of two or more simple conditions and one or more connectives.

The connective "and" ($\wedge$) in a compound condition that is made up of two simple conditions tells you that each solution must satisfy both simple conditions.

See lesson 24, pages 90 and 91, Book 1.

C What tells you that each solution of the compound condition must satisfy not only $\frac{1}{8}+x<\frac{11}{8}$ but also $\frac{4}{3}-x>\frac{2}{3}$?

D $\{x|x<\frac{5}{4}\}$ is the solution set of which simple condition expressed in D5?

E Give a standard description of the solution set of $\frac{4}{3}-x>\frac{2}{3}$.

F Why is $\{x|x<\frac{5}{4}\wedge x<\frac{2}{3}\}$ the solution set of the compound condition expressed in D5?

G Does $\{x|x<\frac{5}{4}\wedge x<\frac{2}{3}\}$ contain all the members of R_a that satisfy both simple conditions?

$U = R_a$.

$\frac{1}{8} + x < \frac{11}{8} \wedge \frac{4}{3} - x > \frac{2}{3}$.

D5

H Is the intersection of $\{x|x < \frac{5}{4}\}$ and $\{x|x < \frac{2}{3}\}$ the same set as $\{x|x < \frac{5}{4} \wedge x < \frac{2}{3}\}$? Explain your answer.

I Are $\{x|x < \frac{2}{3}\}$ and $\{x|x < \frac{5}{4} \wedge x < \frac{2}{3}\}$ the same set? How do you know?

Now you will use pictures of rational-number lines to make graphs of the solution set of each simple condition and of the compound condition expressed in D5.

REMINDER

A rational-number line is a set of points that are associated with rational numbers.

See lesson 73, page 319, Book 1.

A bar above or below the picture of a rational-number line can be used to indicate the graph of a set of rational numbers.

See lesson 90, pages 393 and 394, Book 1.

J Study graph E in D6. Remember that the curved stroke tells you that the dot for point $\frac{5}{4}$ is not contained in the graph of $\{x|x < \frac{5}{4}\}$. The vertical stroke through the dot for point 0 tells you that this dot is contained in the graph of $\{x|x < \frac{5}{4}\}$. Would the dots indicated by the bar form the graph of $\{x|x < \frac{5}{4}\}$?

K Look at graph F. How are the dots in the graph of $\{x|x < \frac{2}{3}\}$ indicated? Is 0 a member of $\{x|x < \frac{2}{3}\}$? Is $\frac{2}{3}$ a member? Explain your answers.

L Look at graph G. The portion of the graph of R_a that has a bar above it as well as below it is the graph of the solution set of the compound condition. Is the dot for point 0 in the graph of $\{x|x < \frac{5}{4} \wedge x < \frac{2}{3}\}$? Is the dot for point $\frac{2}{3}$ in this graph? Does this graph contain all the dots for points associated with members of R_a that are less than $\frac{2}{3}$?

M Why is the graph of $\{x|x < \frac{5}{4} \wedge x < \frac{2}{3}\}$ the same as the graph of $\{x|x < \frac{2}{3}\}$?

N Now look at D7. What is the universe for y? Remember that the letter Z is a name of the set of non-zero rational numbers of arithmetic.

$U = R_a$.

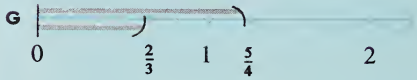
E

$\{x|x < \frac{5}{4}\}$

F

$\{x|x < \frac{2}{3}\}$

G



$\{x|x < \frac{5}{4} \wedge x < \frac{2}{3}\}$

D6

$U = Z$.

H $y > \frac{11}{4}$.

I $y < \frac{17}{4}$.

J $y > \frac{11}{4} \wedge y < \frac{17}{4}$.

K $\frac{11}{4} < y \wedge y < \frac{17}{4}$.

L $\frac{11}{4} < y < \frac{17}{4}$.

D7

O Give a standard description of the solution set of condition H. Of condition I.

P Is condition J formed from conditions H and I and the connective "and"?

Q Is the symbol $\{y|y > \frac{11}{4} \wedge y < \frac{17}{4}\}$ a standard description of the solution set of condition J?

R Does $\frac{11}{4} < y$ have the same solution set as $y > \frac{11}{4}$? Explain your answer.

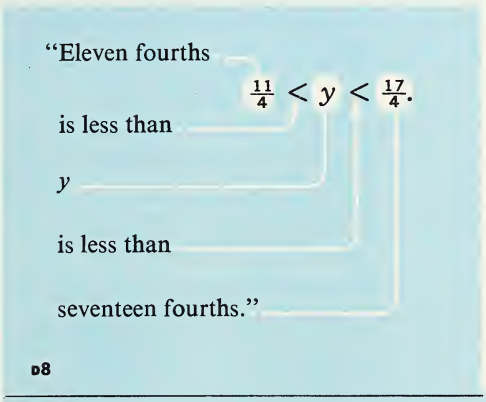
S Does condition K have the same solution set as condition J? Explain your answer.

T Is 3 a member of $\{y|\frac{11}{4} < y \wedge y < \frac{17}{4}\}$?

Since $\frac{11}{4}$ is less than 3 and 3 is less than $\frac{17}{4}$, we say that 3 is *between* $\frac{11}{4}$ and $\frac{17}{4}$.

U Name two other members of the solution set of $\frac{11}{4} < y \wedge y < \frac{17}{4}$. Is each of these members a number between $\frac{11}{4}$ and $\frac{17}{4}$?

V Is every member of $\{y|\frac{11}{4} < y \wedge y < \frac{17}{4}\}$ between $\frac{11}{4}$ and $\frac{17}{4}$?



D8

Look again at sentence K. The compound condition requires that $\frac{11}{4}$ be less than y and also that y be less than $\frac{17}{4}$. Sentence L shows a simpler way to express the compound condition. D8 shows how to read sentence L.

W Is $\{y|\frac{11}{4} < y < \frac{17}{4}\}$ the intersection of the solution set of $\frac{11}{4} < y$ and the solution set of $y < \frac{17}{4}$?

X Look at the graph in D9. Why is the dot for point 0 not contained in the graph of the solution set of $y < \frac{17}{4}$?

Y Would the dots for the points between point $\frac{11}{4}$ and point $\frac{17}{4}$ form the graph of $\{y|\frac{11}{4} < y < \frac{17}{4}\}$? How are these dots indicated?

Z Is $\{y|\frac{11}{4} < y < \frac{17}{4}\}$ an infinite set? Explain your answer.

In this lesson you have reviewed the intersection of sets and have found solution sets of both simple and compound conditions.

On your own

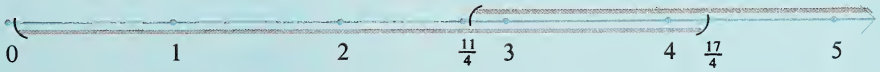
For each of exercises 1 through 4, first tabulate sets A and B. Next tabulate $A \cap B$. Then make a Venn diagram for $A \cap B$. The universe is the set of natural numbers. $U = N$.

1 $A = \{x|x + 3 < 8\}$.
$B = \{x|11 + x > 16\}$.

2 $A = \{m|m - 23 > 15\}$.
$B = \{m|m + 4 > 13\}$.

3 $A = \{z|17 > z + 6\}$.
$B = \{z|15 - z = 7\}$.

$U = Z$.



$\{y|\frac{11}{4} < y < \frac{17}{4}\}$

D9

4 $A = \{x \mid 6 < x < 19\}$.
$B = \{x \mid 10 < x < 25\}$.

For each of exercises 5 through 12, tell whether the sentence expresses a true statement or a false statement. $U = R_a$.

5 $\{d \mid \frac{1}{3} + d = \frac{5}{8}\} = \{\frac{15}{24}\}$.
6 $\{x \mid x - 3.2 > 1.4\} = \{x \mid x > 4.6\}$.
7 $\{w \mid 1\frac{7}{8} - w > \frac{5}{6}\} = \{w \mid 1\frac{1}{24} > w\}$.
8 $\{n \mid n + \frac{5}{12} < \frac{7}{6}\} = \{\frac{9}{12}\}$.
9 $\{y \mid 3\frac{5}{8} < y < 3\frac{3}{4}\} = \{\ \}$.
10 $\{y \mid 5\frac{1}{4} + y > 7\frac{2}{3} \wedge y + 3 < 17\frac{1}{2}\} = \{y \mid 2\frac{5}{12} < y < 14\frac{1}{2}\}$.
11 $\{x \mid 2\frac{3}{7} + x = 14 \wedge x + 19\frac{1}{4} < 40\} = \{x \mid x < 20\frac{3}{4}\}$.
12 $\{g \mid 7\frac{1}{6} - g > 3\frac{1}{4} \wedge 1\frac{3}{4} + g > 2\frac{2}{5}\} = \{g \mid \frac{13}{20} < g < 3\frac{11}{12}\}$.

Make a graph of each solution set described in the following exercises. For exercises 13, 14, and 15, $U = Z$. For exercises 16, 17, and 18, $U = R_a$.

13 $\{x \mid x < 1.5\}$
14 $\{x \mid x + \frac{3}{8} < 2\frac{1}{4}\}$
15 $\{x \mid 2\frac{1}{2} - x > 1\frac{5}{6}\}$
16 $\{x \mid 2\frac{3}{8} < x < 4\frac{1}{2}\}$
17 $\{x \mid x + 3.5 < 5.4 \wedge 7.25 - x > 6.1\}$
18 $\{x \mid x + 1\frac{1}{8} > 2\frac{3}{4}\}$

KEEPING SKILFUL

For each exercise below, find the sum, difference, product, or quotient.

1 $2503 + 4126 + 7899$
2 $25178 - 14367$
3 $3342 + 15374 + 200$
4 $6035 + 7110 + 3425$
5 $4508 \div 49$
6 $897 + 101 + 34 + 25$
7 109×24
8 $22100 + 344 + 6103$
9 $28493 + 85487 + 930$
10 $92843 - 62106$
11 $188922 \div 222$
12 955×312
13 7894×306
14 $27008 - 1245$
15 17256×7
16 $78594 \div 708$
17 504×805
18 $47106 - 3400$
19 $15085 - 13941$
20 $6407 \div 75$
21 $14145 \div 69$
22 $12000 - 185$
23 $195097 \div 780$
24 6248×1179

Compound conditions that include the connective "or"

You know that the solution set of a compound condition that includes the connective "and" is the intersection of the solution sets of the simple conditions. In this lesson you will study a kind of compound condition for which the solution set is the *union* of the solution sets of the simple conditions.

A Sets A, B, and C are tabulated in D1. Describe A. Describe B.

B Is each member of A a member of C? Is each member of B also a member of C? Are there any members of C that are not members of either A or B?

You will remember that the union of two sets is the set that contains all those objects, and only those objects, that belong to either set or to both sets. You can use the symbol $A \cup B$ to express the union of A and B.

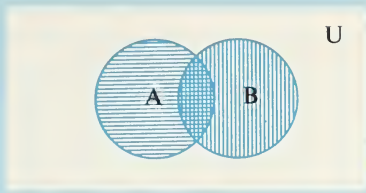
C Is $C = A \cup B$? How do you know?

Now look at the Venn diagram in D2. The diagram represents sets A, B, and C, tabulated in D1.

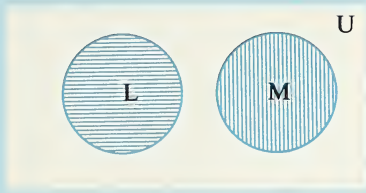
D The universe is the set of natural numbers. Is each of the three sets tabulated in D1 a subset of the universe?

$A = \{0, 2, 4, 6, 8, 10\}$.
$B = \{5, 10, 15\}$.
$C = \{0, 2, 4, 5, 6, 8, 10, 15\}$.

D1



D2



D3

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

A $s + 2 < 7$. B $s + 4 > 6$.

D4

E Sets that have some common members are sets that meet. How does the diagram show that A and B meet?

F Which portion of the Venn diagram represents $A \cup B$?

Look at the Venn diagram in D3. The universe is the set of seventh-, eighth-, and ninth-grade students who attend Willmott Junior High School. L is the set of seventh-grade students at Willmott. M is the set of eighth-grade students at Willmott.

G Is L a subset of the universe? Is M a subset of the universe?

H Do L and M have any common members? How does the diagram in D3 show that L and M are disjoint sets?

I Describe $L \cup M$. Which portion of the diagram represents $L \cup M$?

So far in this lesson, you have studied the union of two sets. In the following exercises, you will use your knowledge of the union of two sets to help you understand a new kind of compound condition.

A What conditions are expressed in D4?

B Describe the universe for s in each condition. Tabulate the solution set of $s + 2 < 7$. Tabulate the solution set of $s + 4 > 6$.

The sentence in D5 expresses a compound condition formed from the two simple conditions expressed in D4 and the connective "or." The connective "or" in the compound condition means that the solution set of the compound condition contains every member of the solution sets of both of the simple conditions.

Mathematicians use the symbol shown below at the right to express the connective "or." D6 shows how to write and read a sentence that includes the symbol for "or."

$\vee$

In a sentence like "Is N a finite set, or is it an infinite set?" the word "or" means "one or the

$s + 2 < 7$ or $s + 4 > 6$.

D5

"s plus two is less than seven

$s + 2 < 7 \vee s + 4 > 6$.

or

s plus four is greater than six."

D6

other, *but not both.*" The word "or" is often used this way in everyday language. However, in a mathematical sentence, the symbol $\vee$ means "one or the other, *or both.*"

Next you will learn how to find the solution set of $s+2<7 \vee s+4>6$. Sentences C, D, and E in D7 express true statements about the solution sets of the two simple conditions and of the compound condition.

C Is each member of $\{s|s+2<7\}$ also a member of $\{s|s+2<7 \vee s+4>6\}$?

D Is each member of $\{s|s+4>6\}$ also a member of $\{s|s+2<7 \vee s+4>6\}$?

E Is each member of the solution set of the compound condition also a member of the solution set of at least one of the simple conditions?

F Tabulate the union of $\{s|s+2<7\}$ and $\{s|s+4>6\}$. Does the union of these two sets contain every member of each set? Does the union of these two sets contain any other members?

G Is the union of the solution set of $s+2<7$ and the solution set of $s+4>6$ the same set as the solution set of $s+2<7 \vee s+4>6$? How do you know?

H Read sentence F in D8. Does the sentence express a true statement?

I What simple conditions are expressed in D9?

J What is the universe for m in each of the simple conditions expressed in D9? Give a standard description of the solution set of each simple condition.

Look again at D9. Condition I is formed from conditions G and H and the connective "or."

K What is the universe for m in condition I? Is the member of $\{m|m=4\frac{1}{2}\}$ also a member of $\{m|m=4\frac{1}{2} \vee m>4\frac{1}{2}\}$? Is each member of $\{m|m>4\frac{1}{2}\}$ also a member of $\{m|m=4\frac{1}{2} \vee m>4\frac{1}{2}\}$?

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

C $\{s|s+2<7\} = \{1, 2, 3, 4\}$.

D $\{s|s+4>6\} = \{3, 4, 5, 6, 7, 8\}$.

E $\{s|s+2<7 \vee s+4>6\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

D7

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

F $\{s|s+2<7\} \cup \{s|s+4>6\} = \{s|s+2<7 \vee s+4>6\}$.

D8

$U = R_a$.

G $m = 4\frac{1}{2}$.

H $m > 4\frac{1}{2}$.

I $m = 4\frac{1}{2} \vee m > 4\frac{1}{2}$.

D9

"m $m \geqq 4\frac{1}{2}$.
is greater than
or equal to
four and one half."

D10

L Does $\{m|m=4\frac{1}{2} \vee m>4\frac{1}{2}\}$ contain any members of R_a that are not members of either $\{m|m=4\frac{1}{2}\}$ or $\{m|m>4\frac{1}{2}\}$?

M Is $\{m|m=4\frac{1}{2}\} \cup \{m|m>4\frac{1}{2}\}$ the same set as $\{m|m=4\frac{1}{2} \vee m>4\frac{1}{2}\}$? Explain your answer.

Look again at sentence I in D9. Notice that one simple condition is that m is equal to $4\frac{1}{2}$, and the other simple condition is that m is

greater than $4\frac{1}{2}$. The compound condition is that m is equal to, or greater than, $4\frac{1}{2}$. You can express this compound condition in a shorter way by using the symbol shown at the right. This symbol expresses the idea of "greater than or equal to." D10 shows how to write and read a sentence that includes this symbol.

N You know that $m \geqq 4\frac{1}{2}$ is the same condition as $m = 4\frac{1}{2} \vee m > 4\frac{1}{2}$. Is $\{m \mid m \geqq 4\frac{1}{2}\}$ the same set as $\{m \mid m = 4\frac{1}{2} \vee m > 4\frac{1}{2}\}$? Is $\{m \mid m \geqq 4\frac{1}{2}\}$ the union of $\{m \mid m = 4\frac{1}{2}\}$ and $\{m \mid m > 4\frac{1}{2}\}$?

O What compound condition is expressed by sentence J in D11? What connective is used in the condition? What is the universe for the variable in the condition?

P How do you know that $\{x \mid x = 3.25\}$ is the solution set of $x + 6 = 9.25$?

Q Give a standard description of the solution set of $x + 2.25 < 5.5$. Remember to give the simpler description.

R Look again at D11. Does condition K have the same solution set as condition J?

Notice that sentence K in D11 expresses the compound condition that x is less than, or equal to, 3.25. You can express this condition in a shorter way. You can use the symbol shown at the right to express the idea of "less than or equal to." D12 shows how to write and read a sentence that includes the symbol. $x \leqq 3.25$ is the same condition as $x < 3.25 \vee x = 3.25$.

S Does $\{x \mid x \leqq 3.25\}$ contain the same members of Z as $\{x \mid x < 3.25 \vee x = 3.25\}$? Is $\{x \mid x \leqq 3.25\}$ the intersection, or is it the union, of $\{x \mid x < 3.25\}$ and $\{x \mid x = 3.25\}$? How do you know?

T Is 0 a member of $\{x \mid x \leqq 3.25\}$? Explain your answer.

U = Z.

J $x + 2.25 < 5.5 \vee x + 6 = 9.25$.

K $x < 3.25 \vee x = 3.25$.

D11

"x

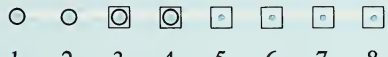
$x \leqq 3.25$.

is less than
or equal to

three and
twenty-five hundredths."

D12

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$.



1 2 3 4 5 6 7 8

$\{s \mid s + 2 < 7 \vee s + 4 > 6\}$

D13

From the preceding exercises, you know that the solution set of a compound condition formed from two simple conditions and the connective "or" is the union of the solution sets of the two simple conditions. Now you will learn how to make graphs of the solution sets of such compound conditions.

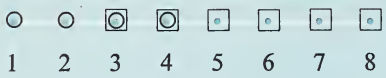
A Look at D13. This display shows a graph of the solution set of $s + 2 < 7 \vee s + 4 > 6$. The encircled dots form the graph of $\{s \mid s + 2 < 7\}$. How are the dots in the graph of $\{s \mid s + 4 > 6\}$ indicated?

The dots that are encircled, the dots that are enclosed by squares, and the dots that are both encircled and enclosed by squares form the graph of $\{s|s+2<7 \vee s+4>6\}$.

B Is the graph of $\{s|s+2<7\}$ included in the graph of $\{s|s+2<7 \vee s+4>6\}$? How do you know?

C Is the graph of $\{s|s+4>6\}$ also included in the graph of $\{s|s+2<7 \vee s+4>6\}$? How do you know?

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$.



$\{s|s+2<7 \vee s+4>6\}$

D13

Next you will study the graph of the solution set of the compound condition $m \geqq 4\frac{1}{2}$. D14 shows this graph.

D Is point $\frac{9}{2}$ the same as point $4\frac{1}{2}$? The vertical stroke through the dot for point $\frac{9}{2}$ indicates the graph of $\{m|m=4\frac{1}{2}\}$. Is the dot for point $4\frac{1}{2}$ also in the graph of $\{m|m \geqq 4\frac{1}{2}\}$?

E How are the dots in the graph of $\{m|m>4\frac{1}{2}\}$ indicated in D14? Are these same dots in the graph of $\{m|m \geqq 4\frac{1}{2}\}$?

F Think about all the points associated with members of R_a that are either greater than or equal to $4\frac{1}{2}$. Does the graph of $\{m|m \geqq 4\frac{1}{2}\}$ contain all the dots for these points?

Next you will study the graph of the solution set of $x \leqq 3.25$. Look at D15.

G Is $\{x|x \leqq 3.25\}$ the solution set of the condition $x<3.25 \vee x=3.25$?

H Why is 0 not a member of $\{x|x \leqq 3.25\}$? How does D15 indicate that 0 is not a member?

$U = R_a$.

$\{m|m \geqq 4\frac{1}{2}\}$

D14

$U = Z$.

$\{x|x \leqq 3.25\}$

D15

I Is 3.25 a member of $\{x|x \leqq 3.25\}$? How is this indicated in D15?

J Think about all the points associated with members of Z that are either less than or equal to 3.25. Does the graph of $\{x|x \leqq 3.25\}$ contain all the dots for these points? How are these dots indicated in D15?

In this lesson you reviewed some ideas about the union of two sets. You also studied compound conditions that included "or," and you learned how to make graphs of the solution sets of compound conditions that include "or."

On your own

For each of exercises 1 through 15, tell whether the sentence expresses a true statement or a false statement. The universe is given for each exercise.

1 $\{n|3+4>n \vee 25-22=n\} = \{0, 1, 2, 3, 4, 5, 6\}$. U = N.

2 $\{p|p-3\frac{3}{4}>10\frac{1}{4} \vee p+16>41\} = \{p|p>6\frac{1}{2}\}$. U = R_a.

3 $\{s|s+6.4>9.9 \vee s+2=5.5\} = \{s|s \geqq 3.5\}$. U = R_a.

4 $\{m|1+2>m \vee 5-m=4\} = \{0, 1, 2\}$. U = C. (The letter C names the set of counting numbers.)

5 $\{a|a+6=10 \vee a+11<15\} = \{0, 1, 2, 3\}$. U = N.

6 $\{x|x+4\frac{2}{3}<5 \vee x+2\frac{1}{2}=2\frac{5}{6}\} = \{x|x \leqq \frac{1}{3}\}$. U = R_a.

7 $\{r|r-6=7\frac{1}{2} \vee 6+r=6\} = \{0, 13\frac{1}{2}\}$. U = R_a.

8 $\{t|8-t>\frac{4}{5} \vee t+3<13\frac{1}{2}\} = \{t|t<7\frac{1}{5}\}$. U = R_a.

9 $\{y|6+3.5=y \vee y-2=7.2\} = \{9.2, 9.5\}$. U = R_a.

10 $\{n|23+n>75 \vee n-253>748\} = \{53, 54, 55, \ldots\}$. U = N.

11 $\{q|9\frac{3}{4}-6\frac{2}{3}<q \vee 7\frac{5}{6}-4\frac{3}{4}=q\} = \{q|q \leqq 3\frac{1}{12}\}$. U = R_a.

12 $\{e|e+5>18 \vee e+7\frac{1}{3}<9\frac{5}{6}\} = \{\ \}$. U = Z.

13 $\{w|3+w<6.2 \vee w-2.3>7.6\} = \{w|w<3.2 \vee w>9.9\}$. U = Z.

14 $\{c|c+2\frac{3}{4}=6 \vee c-5\frac{1}{8}>8\frac{1}{2}\} = \{c|c>13\frac{5}{8}\}$. U = R_a.

15 $\{y|y<12-3 \vee y+\frac{2}{3}>\frac{7}{6}\} = \{y|y<9 \vee y>\frac{1}{2}\}$. U = R_a.

For each of the conditions expressed in exercises 16 through 22, first give a standard description of the solution set of each simple condition and also of the compound condition. Then make a Venn diagram to represent the solution set of the compound condition. Use the letter A to name the solution set of the first simple condition. Use the letter B to name the solution set of the second simple condition. U = R_a.

16 $6-1=x \vee 7\frac{1}{2}+x<12\frac{1}{2}$.

17 $x-3.8>4.3 \vee x+2.9<4.5$.

18 $x+2\frac{1}{3}=6 \vee x+3\frac{1}{5}>5\frac{7}{10}$.

19 $5.75+x<7.75 \vee x-3.5>5$.

20 $9\frac{1}{2}+x>12\frac{1}{2} \vee 32-16<x$.

21 $x-3>4\frac{1}{2} \vee 5-4>x$.

22 $x+6>7\frac{1}{2} \vee x=4-\frac{2}{3}$.

For each of the conditions expressed in exercises 23 through 31, first tabulate the solution set, if possible. If tabulation is not possible, give a standard description of the solution set. Use the symbols $\geqq$ and $\leqq$ wherever you can. Then make a graph of the solution set. U = R_a.

23 $x+3\frac{1}{2}<6 \vee x=2\frac{1}{2}$.

24 $x=1.75 \vee 5.25-3.5<x$.

25 $\frac{4}{3}=x \vee 7-1\frac{1}{3}<x$.

26 $x+5\frac{1}{3}<5\frac{2}{3} \vee x-1\frac{3}{4}>2$.

27 $x<2+3\frac{1}{2} \vee 6-x=\frac{1}{2}$.

28 $5-x>1\frac{1}{2} \vee 2+x=3\frac{1}{2}$.

29 $6+x=7\frac{3}{4} \vee x-2=2\frac{1}{4}$.

30 $2.5+1=x \vee 3+x>3.5$.

31 $12.2=x+4.9 \vee 15.3-x=5.7$.

Compound conditions in two variables

So far in this unit you have studied conditions in one variable that included either the connective "and" or the connective "or." In this lesson you will study compound conditions in two variables that also include these connectives.

A What condition is expressed in D1? Name the variables in the condition. What is the universe for (x, y)?

B Is each member of $N \times N$ an ordered pair? The first component of each member is a member of what set? The second component is a member of what set? $N \times N$ is the Cartesian set of what set of numbers?

REMINDER

An ordered pair is a pair of objects in which the objects occur in a special order. The object that occurs first is the first component of the ordered pair. The object that occurs second is the second component.

See lesson 30, page 123, Book 1.

A Cartesian set is the set of all ordered pairs that can be formed by matching each member of one set in turn with each member of a second set.

See lesson 31, page 127, Book 1.

C How do you know that (2, 3) is not the same as (3, 2)?

Universe for $(x, y) = N \times N$.
$x + y = 7$.

x	0	1	2	3	4	5	6	7
y	7							

D1

Universe for $(x, y) = N \times N$.
$x + y = 7 \wedge y > x + 1$.

D2

In *Book 1*, you used charts to help you find the solutions of conditions like $x + y = 7$. Part of such a chart is shown in D1.

D In the chart, is each number named in the row opposite the letter x the first component of a member of $N \times N$?

E Replace x in $x + y = 7$ by 0. What replacement for y satisfies $0 + y = 7$? Where is the numeral for this number placed in the chart? Is (0, 7) a solution of $x + y = 7$?

F Now replace x in $x + y = 7$ by 8. Is there any member of N that satisfies $8 + y = 7$? Why do you not use a number greater than 7 as a replacement for x in $x + y = 7$?

G Copy the chart in D1 and complete it. Use each number named in the top row as a replacement for x in $x + y = 7$. For each condition that you obtain, find the replacement for y that satisfies the condition.

H Is the solution set of $x + y = 7$ a set of ordered pairs? How many ordered pairs can you obtain from your chart? Is each of these ordered pairs a member of $N \times N$? Do any other members of $N \times N$ satisfy $x + y = 7$? Tabulate $\{(x, y) | x + y = 7\}$.

Now look at D2. Notice that the condition expressed in the display is a compound condition that includes the connective "and." One

of the simple conditions is the same condition as the one expressed in D1.

I What is the universe for (x, y)? How do you know that the solution set of $x + y = 7 \wedge y > x + 1$ is a set of ordered pairs?

J How do you know that each solution of the compound condition must satisfy not only $x + y = 7$, but also $y > x + 1$?

You can use the chart that you made for exercise G to help you find the members of $\{(x, y) | x + y = 7\}$ that satisfy $y > x + 1$.

K Which members of $\{(x, y) | x + y = 7\}$ also satisfy $y > x + 1$?

L Tabulate $\{(x, y) | x + y = 7 \wedge y > x + 1\}$. Is this set a finite set, or is it an infinite set?

M Is $\{(x, y) | x + y = 7 \wedge y > x + 1\}$ the same set as $\{(x, y) | x + y = 7\} \cap \{(x, y) | y > x + 1\}$?

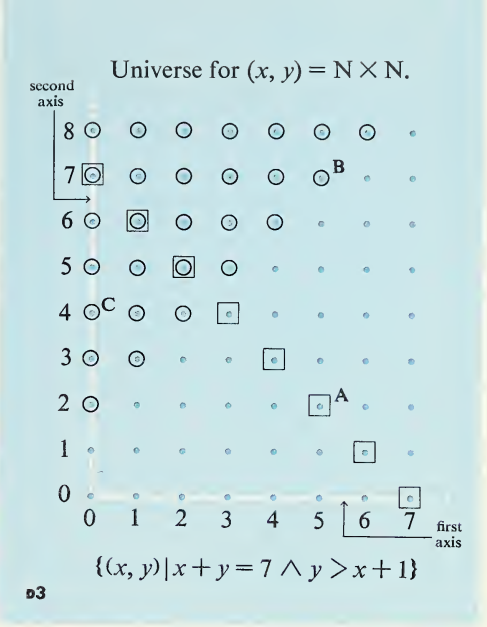
Next you will use a graph of N × N to make graphs of the solution sets of $x + y = 7$ and $y > x + 1$.

N Look at the graph in D3. What is indicated by the three small dots at the end of the picture of each axis?

Remember that each point represented in a graph of N × N is associated with an ordered pair of natural numbers. The first component of each member of N × N is associated with a point in the first axis. The second component is associated with a point in the second axis.

O Look again at the graph in D3. Point A is located directly above point 5 in the first axis and directly to the right of point 2 in the second axis. Is (5, 2) associated with point A?

The number associated with a point in a number line is the *coördinate* of the point. A point represented in a graph of N × N has two coördinates. These coördinates are the components of the ordered pair associated with the point. The first component is the *first coördinate*. The second component is the *second coördinate*.



D3

The components of (5, 2) are the coördinates of point A represented in D3. 5 is the first coördinate of point A, and 2 is the second coördinate.

P Name the first coördinate of point B. Name the second coördinate of point B.

Q Name the coördinates of point C.

R Does each dot that is enclosed by a square represent a point whose coördinates satisfy $x + y = 7$? Do these dots form a complete graph of $\{(x, y) | x + y = 7\}$?

S Does each encircled dot represent a point whose coördinates satisfy $y > x + 1$? How do you know that these dots form an incomplete graph of $\{(x, y) | y > x + 1\}$?

T Find the dots that are in the graph of both $\{(x, y) | x + y = 7\}$ and $\{(x, y) | y > x + 1\}$. How are these dots indicated?

U Do these dots form a complete graph of the solution set of $x + y = 7 \wedge y > x + 1$? How many dots are in the graph of the solution set of the compound condition?

In the following exercises you will use graphs to help you understand compound conditions in two variables that include "or."

A What compound condition is expressed in D4? How many variables are there in this condition? What is the universe for (m, n)?

B How do you know that the solution set of $m + n < 5 \vee m = 3 - n$ is a set of ordered pairs?

C How do you know that each solution of the compound condition must satisfy either or both of the simple conditions?

D You can use a chart like the one begun in D5 to help you find the solutions of $m + n < 5$. Why do you not use any number greater than 4 as a replacement for m in $m + n < 5$?

E Replace m in $m + n < 5$ by 0. What replacements for n satisfy $0 + n < 5$?

F Is $(0, 0)$ a member of the solution set of $m + n < 5$? Is $(0, 1)$ a member? Name three other members of $N \times N$ that have first components of 0 and that satisfy $m + n < 5$.

G Copy and complete the chart in D5.

H How many ordered pairs can you obtain from your chart? Are there any other ordered pairs in $N \times N$ that satisfy $m + n < 5$?

I Is $\{(m, n) | m + n < 5\}$ a finite set, or is it an infinite set? Tabulate $\{(m, n) | m + n < 5\}$.

J Make a chart to help you find the solutions of $m = 3 - n$. Explain why you do not use a number greater than 3 as a replacement for n in $m = 3 - n$.

K How many ordered pairs can you obtain from your completed chart? Is the solution set of $m = 3 - n$ finite, or is it infinite? Tabulate $\{(m, n) | m = 3 - n\}$.

L Is the solution set of $m + n < 5 \vee m = 3 - n$ finite, or is it infinite? Tabulate this set. Use the sets you tabulated for exercises I and K to help you.

Universe for $(m, n) = N \times N$.
$m + n < 5 \vee m = 3 - n$.

D4

Universe for $(m, n) = N \times N$.
$m + n < 5$.

m	0	1	2	3	4
n	0				
	1				
	2				
	3				
	4				

D5

M Is $\{(m, n) | m + n < 5 \vee m = 3 - n\}$ the same set as the union of $\{(m, n) | m + n < 5\}$ and $\{(m, n) | m = 3 - n\}$?

N Now look at D6. How are the dots in the graph of $\{(m, n) | m + n < 5\}$ indicated?

O In what way are the dots in the graph of $\{(m, n) | m = 3 - n\}$ indicated?

P Name the coördinates of the points whose dots are enclosed by circles but not by squares. Whose dots are enclosed by squares and circles. Are there any dots that are enclosed by squares, but not by circles?

Q How do you know that all the dots for the points you named for exercise P are in the graph of $\{(m, n) | m + n < 5 \vee m = 3 - n\}$? Is the graph of this solution set shown in D6 a complete graph, or is it incomplete?

Now you know that the solution set of a compound condition in two variables that includes "or" is the union of the solution sets of the simple conditions. So far, the solution sets of the compound conditions have all been finite, but they may also be infinite.

A What condition is expressed in D7?

B Make a chart to help you find the solutions of $6 - x = y$.

C Is $\{(x, y) | 6 - x = y\}$ an infinite set, or is it a finite set? Tabulate $\{(x, y) | 6 - x = y\}$.

Now you will find the solutions of $x + 3 < 4$.

D How many variables are in $x + 3 < 4$? Is $x + 3 < 4$ satisfied by $(0, 0)$? By $(0, 5)$? By $(0, 600)$? Is $x + 3 < 4$ satisfied by each ordered pair in $N \times N$ whose first component is 0? Explain your answer.

E Is $\{(x, y) | x + 3 < 4\}$ an infinite set?

F Is $\{(x, y) | 6 - x = y \vee x + 3 < 4\}$ an infinite set? How do you know?

G Now look at the graph in D7. How are the dots in the graph of $\{(x, y) | 6 - x = y\}$ indicated? How are the dots in the graph of $\{(x, y) | x + 3 < 4\}$ indicated?

H In what way are the dots in the graph of $\{(x, y) | 6 - x = y \vee x + 3 < 4\}$ indicated? How do you know that the graph of the solution set shown in D7 is an incomplete graph?

Now you know how to find solution sets of compound conditions in two variables.

On your own

For each condition expressed in exercises 1 through 6, tell whether the solution set is finite or infinite. Tabulate each finite solution set. The universe for (x, y) is $N \times N$.

1 $x + y = 6 \vee y = 12 - 2x$.

2 $x + y < 9 \vee y = x$.

3 $x + y > 3 \vee x + y < 4$.

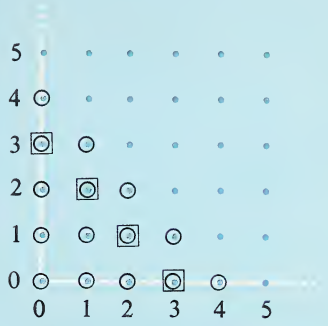
4 $x > y \wedge x + y > 3$.

5 $x + y < 8 \wedge x + y > 6$.

6 $x + y < 8 \vee x + y > 6$.

Make a graph of the solution set of each simple condition expressed in exercises 7 through 12. Tabulate the solution set of the compound condition if it is finite. The universe for (x, y) is $N \times N$.

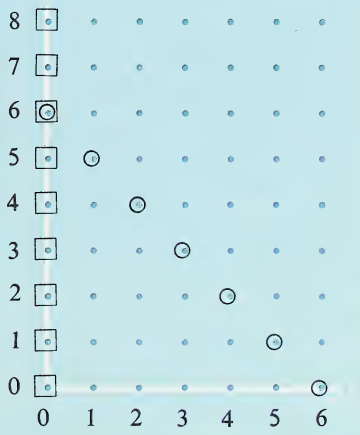
Universe for $(m, n) = N \times N$.



$\{(m, n) | m + n < 5 \vee m = 3 - n\}$

D6

Universe for $(x, y) = N \times N$.
$6 - x = y \vee x + 3 < 4$.



$\{(x, y) | 6 - x = y \vee x + 3 < 4\}$
Incomplete graph

D7

7 $x + y = 7 \vee x + 3y = 9$.

8 $y = 6 - x \wedge y + x < 8$.

9 $2x + y = 8 \vee 7 - y = x$.

10 $2y > x \wedge x + y < 7$.

11 $y > x \vee y - 3 = 2$.

12 $y = 5 - x \vee x = 4 - y$.

The complement of a set

You know how to use two given sets to obtain a new set. For example, you can use two given sets to obtain their intersection or their union. You also know how the solution sets of compound conditions are related to intersection and union. In this lesson you will learn how to use a universe and a subset of that universe to obtain a new set. Then you will learn how the solution set of a new kind of condition is related to the new set.

A Look at the tabulation in D1. Suppose that the universe is the set of all the months in the year. A is the set of all the months that have exactly 30 days. Is A a subset of the universe?

B Read the tabulation in D2. Does B contain all the months in the year that do not have exactly 30 days?

C Is B also a subset of the universe? Is any member of B a member of A?

D Explain how you can use subset A to decide which members of the universe are in subset B.

B is the set that contains all those members of the universe, and only those members, that do not belong to A. Set B is the *complement* of set A.

complement of a set (kom′plə mənt). The complement of a given subset of a universe is the set that contains all the members of the universe, and only those members, that do not belong to the given subset. If U = {1, 2, 3}, then the complement of {2} is {1, 3}.

A = {April, June, September, November}.

D1

B = {January, February, March, May, July, August, October, December}.

D2

U = {1, 2, 3, 4, 5}.
M = {1, 4, 5}.

D3

"The complement of $\overline{\{1, 4, 5\}} = \{2, 3\}$.

the set whose members are one, four, five

is equal to

the set whose members are two, three."

D4

The complement of a given subset of a universe is another subset of the universe that contains all the members of the universe, and only those members, that do not belong to the given subset.

E Describe the universe tabulated in D3. How do you know that M is a subset of the universe?

F Explain why 3 is a member of the complement of M, but 4 is not a member.

G How do you know that {1, 3, 5} is not the complement of M? That {2, 3, 6} is not the complement of M? That {2} is not the complement of M?

H Tabulate the complement of M.

D4 shows how to write and read a sentence about the complement of M. The bar over the symbol {1, 4, 5} means "the complement of."

Now study D5. It shows how to write and read another sentence about the complement of {1, 4, 5}. Notice the bar over the letter M. The symbol $\overline{M}$ is a name for $\overline{\{1, 4, 5\}}$.

I Look at D6. What is the universe for x in $x < 5$? Is D a finite set, or is it an infinite set? Tabulate D.

J Is 4 a member of D? Is 5 a member of D? Is any natural number greater than 4 a member of D?

K Is 5 a member of the complement of D? Is each natural number greater than 4 a member of $\overline{D}$? How do you know that 4 is not a member of $\overline{D}$?

L Is $\overline{D}$ a finite set, or is it an infinite set? Tabulate $\overline{D}$.

M Is $\overline{\{x|x < 5\}} = \{5, 6, 7, \ldots\}$? Remember that U = N.

You have used Venn diagrams to represent the union and the intersection of sets. You can also use Venn diagrams to represent the complement of a set.

N Suppose that U is the set of natural numbers and that A is the set of even numbers. Tabulate A. Tabulate $\overline{A}$.

O Is each set that you tabulated for exercise N a subset of the set of natural numbers?

P Look at the Venn diagram in D7. The universe is represented by the picture of the rectangle and its interior. How is A represented?

Q $\overline{A}$ is represented by the portion of the diagram that has vertical shading. How does the diagram show that A and $\overline{A}$ are disjoint sets?

"The complement of M is equal to the set whose members are two, three."

$$\overline{M} = \{2, 3\}.$$

D5

$$U = N.$$
$$D = \{x|x < 5\}.$$

D6

D7

R What is the union of a given set and its complement? What is the intersection of a given set and its complement?

S If set L is the complement of set K, what is the complement of L?

T If M is a given subset of a universe, what is the complement of $\overline{M}$?

U What is the complement of { }? What is the complement of any given universe?

Next you will learn how to use one condition and a connective to form a new condition. Then you will learn how the solution set of the new condition is related to the idea of complement.

U = N.
$x > 6$.

D8

U = N.
It is *not* the case that $x > 6$.

D9

"It is not the case that $\sim (x > 6)$. x is greater than six."

D10

U = N.
$\{x | x > 6\} = \{7, 8, 9, \ldots\}$.
$\{x | \sim (x > 6)\} = \{0, 1, 2, 3, 4, 5, 6\}$.

D11

A What simple condition is expressed in D8? What is the universe for x? Tabulate $\{x | x > 6\}$.

B Is 3 a member of $\{x | x > 6\}$? Is 0 a member of $\{x | x > 6\}$? There are five other members of the universe that do *not* satisfy $x > 6$. Name these members.

Now look at D9. The sentence in D9 expresses a new condition formed from $x > 6$ and the connective "not." The connective "not" in the sentence means that the solution set of the new condition is the set of all members of the universe that do *not* satisfy the original condition, $x > 6$.

Mathematicians use the symbol shown at the right to express the connective "not." Notice that it is the same symbol that we use for equivalence. However, when you see the symbol used in a sentence, you will be able to tell which idea it expresses, the idea of "not" or the idea of equivalence.

D10 shows how to write and read a mathematical sentence that includes the symbol for "not." Notice that the sentence $x > 6$ is within parentheses.

Now you will learn how to find the solution set of $\sim (x > 6)$. The solution set of $x > 6$ is tabulated in D11. The solution set of the new condition formed from $x > 6$ and the connective "not" is also tabulated.

C Does 6 satisfy $x > 6$? Does any natural number less than 7 satisfy $x > 6$? Does every natural number greater than 6 satisfy $x > 6$?

D Tabulate the set of natural numbers that do not satisfy $x > 6$.

E Is each member of {0, 1, 2, 3, 4, 5, 6} a member of N that does not satisfy $x > 6$?

F Does {0, 1, 2, 3, 4, 5, 6} contain all the members of N that do not satisfy $x > 6$?

G Is $\{0, 1, 2, 3, 4, 5, 6\} = \{x | \sim (x > 6)\}$?

H Tabulate $\overline{\{x | x > 6\}}$. U = N.

I Is the solution set of $\sim (x > 6)$ the same set as the complement of the solution set of $x > 6$? Explain your answer.

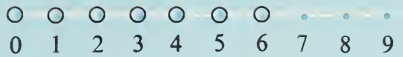
J Now look at the graph in D12. How are the dots in the graph of $\{x | \sim (x > 6)\}$ indicated?

Now you will use the graph in D12 to find a compound condition that includes the connective "or" and whose solution set is the same as $\{x | \sim (x > 6)\}$.

K Which dot forms the graph of $\{x | x = 6\}$? Is $\{x | x = 6\}$ a subset of $\{x | \sim (x > 6)\}$?

L Which dots form the graph of $\{x | x < 6\}$? Is $\{x | x < 6\}$ a subset of $\{x | \sim (x > 6)\}$?

$U = N.$



$\{x | \sim (x > 6)\}.$

D12

$U = R_a.$

$\sim (m = 2).$

D13

M Which dots form the graph of $\{x | x \leqq 6\}$?

N Is the graph of $\{x | x \leqq 6\}$ the same as the graph of $\{x | \sim (x > 6)\}$?

O Is $\{x | x \leqq 6\} = \{x | \sim (x > 6)\}$?

P Is $\overline{\{x | x > 6\}} = \{x | x \leqq 6\}$?

In the following exercises you will learn how to find the solution sets of other conditions that include the connective "not."

A What condition is expressed in D13? What is the universe for the variable?

B Is $\{m | \sim (m = 2)\} = \overline{\{m | m = 2\}}$?

C What member of the universe satisfies $m = 2$? Does $\overline{\{m | m = 2\}}$ contain all the rational numbers of arithmetic except the number 2?

D How is the graph of $\{m | \sim (m = 2)\}$ indicated in D13? Does this graph contain the dot for point 2? How do you know?

E Is the graph of $\{m | \sim (m = 2)\}$ also the graph of $\overline{\{m | m = 2\}}$?

F Is the graph of $\{m | \sim (m = 2)\}$ also the graph of $\{m | m \neq 2\}$? Remember that the symbol $\neq$ expresses the idea of "not equal to."

G Is $\{m | \sim (m = 2)\} = \{m | m \neq 2\}$?

You can develop a compound condition that includes the connective "or" and whose solution set is the same as $\{m | \sim (m = 2)\}$.

H Look again at D13. Which bar indicates the graph of $\{m | m < 2\}$? Which bar indicates the graph of $\{m | m > 2\}$?

I How is the graph of $\{m | m < 2 \vee m > 2\}$ indicated? Is this graph also the graph of $\{m | \sim (m = 2)\}$?

J Is $\{m | m < 2 \vee m > 2\} = \{m | \sim (m = 2)\}$? Is $\overline{\{m | m = 2\}} = \{m | m < 2 \vee m > 2\}$?

K Think about $\sim (b + 1 < 4)$. $U = N$. How do you know that $\{b | \sim (b + 1 < 4)\}$ is the same as $\{b | \sim (b < 3)\}$? The same as $\{b | b \geqq 3\}$? The same as $\overline{\{b | b < 3\}}$?

Now you know what is meant by the complement of a set. You also know how to find the solution set of a condition formed from a simple condition and the connective "not."

On your own

In each of exercises 1 through 8, a universe and a subset A of the universe are given. For each exercise, tabulate $\overline{A}$ if possible. Otherwise, give a standard description of $\overline{A}$.

1 $U = N.$ $A = \{4, 5, 6, \ldots\}.$

2 $U = C.$ $A = \{1, 2, 3\}.$

3 $U = R_a.$ $A = \{x | x \neq \frac{7}{5}\}.$

4 $U = N.$ $A = \{x | x \leqq 1\}.$

5 $U = N.$ $A = \{x | 2 + x > 12\}.$

6 $U = R_a.$ $A = \{x | x + 13 > 21\}.$

7 $U = \{5, 10, 15, \ldots\}.$ $A = \{\ \}.$

8 $U = R_a.$ $A = \{x | 2 + x = 9\}.$

Tell whether each of the sentences in exercises 9 through 13 expresses a true statement or a false statement. $U = N.$

9 $\{x | x \geqq 3\} = \overline{\{x | \sim (x < 3)\}}.$

10 $\overline{\{x | x + 2 = 7\}} = \{x | \sim (x + 2 = 7)\}.$

11 $\{y | y > 4 \wedge y < 4\} = \overline{\{y | y = 4\}}.$

12 $\{m | \sim (m + 1 > 5)\} = \{m | m \leqq 4\}.$

13 $\overline{\{k | k + 3 < 7\}} = \{k | k \geqq 4\}.$

D14

Use the Venn diagram in D14 to help you answer the questions in exercises 14 through 22.

14 Suppose that U is the set of all musical instruments. Suppose also that A is the set of all violins and B is the set of all stringed instruments. Is A a subset of B?

15 How is the universe represented in D14?

16 How is set A represented in the diagram? How is set B represented? How does the diagram show that A is a subset of B?

17 Which set, A or B, is the same as $A \cap B$? Which set, A or B, is the same as $A \cup B$?

18 Which portion of the diagram represents $\overline{B}$? Which portion represents $\overline{A}$?

19 Is $\overline{B}$ a subset of $\overline{A}$? Explain your answer.

20 What set is the same set as $B \cup \overline{B}$? The same set as $A \cup \overline{A}$? The same set as $\overline{A} \cup \overline{B}$?

21 What set is the complement of $\overline{A}$? Of $\overline{B}$?

22 What set is the same set as $\overline{B} \cap B$? The same set as $\overline{A} \cap A$? The same set as $\overline{A} \cap \overline{B}$?

For each condition expressed in exercises 23 through 28, tabulate the solution set, if possible. Otherwise, give a standard description of the solution set. The universe for each variable is given.

23 $\sim(y + 1 > 5)$. $U = R_a$.

24 $\sim(x < 5)$. $U = N$.

25 $\sim(5 + m > 11)$. $U = Z$.

26 $\sim(x - 3 < 2)$. $U = C$.

27 $\sim(16 = 9 + y)$. $U = R_a$.

28 $\sim(y + 1 \neq 7)$. $U = R_a$.

CHECKING UP

If you have trouble with these two tests, you can find help in lessons 109 through 112.

Test 114

A finite universe and two subsets, A and B, are tabulated below. Using these sets, tabulate each of the sets named in exercises 1 through 9.

$U = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$A = \{2, 5, 7, 8\}$. $B = \{1, 8\}$.

1 $\overline{A}$	**4** $A \cup B$	**7** $\overline{B} \cup B$
2 $\overline{B}$	**5** $\overline{A} \cap \overline{B}$	**8** $A \cap \overline{A}$
3 $A \cap B$	**6** $\overline{A} \cup \overline{B}$	**9** $B \cup \overline{A}$

For each of exercises 10 through 15, tell whether the sentence expresses a true statement or a false statement. The universe for the variables is R_a.

10 $\{x|\sim(x < 3)\} = \overline{\{x|x < 3\}}$.

11 $\{x|x \geqq 9\frac{3}{8}\} = \{x|x > 9\frac{3}{8} \vee x = 9\frac{3}{8}\}$.

12 $\{x|x = 6\frac{1}{8}\} = \{x|x > 6\frac{1}{8} \vee x < 6\frac{1}{8}\}$.

13 $\{x|5 < x < 7\} = \{x|5 < x\} \cap \{x|x < 7\}$.

14 $\{x|x \leqq 5\} = \{x|\sim(x > 5)\}$.

15 $\{(x, y)|x + y < 4\} =$
$\{(x, y)|x < 4 \wedge y < 4\}$.

Test 115

For each condition expressed in exercises 16 through 21, give a standard description of the solution set. Then make a graph of the solution set. $U = R_a$.

16 $13\frac{3}{4} < x + 8\frac{5}{8}$.

17 $x + 7\frac{1}{2} > 9 \wedge 16\frac{1}{4} - x > 11$.

18 $11\frac{5}{6} - a > 9\frac{1}{2}$.

19 $7\frac{3}{8} + y = 9\frac{7}{8} \vee y + 2\frac{5}{8} > 5\frac{1}{8}$.

20 $\sim(x + 21\frac{1}{4} > 25\frac{7}{8})$.

21 $17\frac{2}{5} < 14\frac{1}{5} + b \wedge b + 6 < 13\frac{1}{2}$.

For each condition expressed below, make a graph of the solution set. $U = N \times N$.

22 $x + y < 5 \wedge y + 8 < 11$.

23 $x + y = 4 \vee 2x - y = 3$.

24 $x = y \vee x + y < 7$.

25 $y > 3x \wedge x + y > 6$.

End-of-block tests on solution sets of simple and compound conditions and complements of sets

Intervals

In this lesson you will use what you know about compound conditions as you study certain sets of numbers that are called intervals.

Look at D1. Each of the conditions expressed by sentences A, B, and C was obtained from the condition $a < x < b$ by replacing a and b by members of R_a. The universe for each of the variables, a, x, and b, is R_a.

A What replacements were made for a and for b in $a < x < b$ to obtain condition A? Condition B? Condition C? In each condition, is the replacement used for a less than the replacement used for b?

B Look again at D1. The sentence $1\frac{1}{2} < x < 4\frac{2}{3}$ expresses the same compound condition as the sentence $1\frac{1}{2} < x \wedge x < 4\frac{2}{3}$. What sentence that includes the symbol for "and" expresses the same condition as the sentence $3\frac{1}{7} < x < 7\frac{1}{3}$? As the sentence $1.2 < x < 8.6$?

C Does the sentence $a < x < b$ express the same compound condition as the sentence $a < x \wedge x < b$?

D Think about condition A. Is $1\frac{1}{2}$ a solution of $1\frac{1}{2} < x < 4\frac{2}{3}$? Is $4\frac{2}{3}$ a solution? Is each member of R_a that is greater than $1\frac{1}{2}$ and less than $4\frac{2}{3}$ a solution? Explain your answers.

Each solution of $1\frac{1}{2} < x < 4\frac{2}{3}$ is greater than $1\frac{1}{2}$ and less than $4\frac{2}{3}$. Therefore, the solution set of $1\frac{1}{2} < x < 4\frac{2}{3}$ is the set of rational numbers of arithmetic between $1\frac{1}{2}$ and $4\frac{2}{3}$.

E Think about condition B. Is either $3\frac{1}{7}$ or $7\frac{1}{3}$ a solution of $3\frac{1}{7} < x < 7\frac{1}{3}$? Is $\{x \mid 3\frac{1}{7} < x < 7\frac{1}{3}\}$ the set of members of R_a between $3\frac{1}{7}$ and $7\frac{1}{3}$?

$U = R_a$.
$a < x < b$.

A $1\frac{1}{2} < x < 4\frac{2}{3}$.
B $3\frac{1}{7} < x < 7\frac{1}{3}$.
C $1.2 < x < 8.6$.

D1

open interval in R_a (in′tər vəl). The set of all x that satisfy $a < x < b$ when $a < b$ and when the universe for a, x, and b is R_a. a and b are the **limits** of the interval.

F Think about condition C. How do you know that neither 1.2 nor 8.6 is a member of the solution set of $1.2 < x < 8.6$? Describe in words the solution set of condition C. Use the word "between" in your description.

G Now suppose that, in $a < x < b$, you replace a by $6\frac{1}{2}$ and b by 2. What condition do you obtain? Is $6\frac{1}{2}$ a member of the solution set of $6\frac{1}{2} < x < 2$? Is 2 a member? Is there any member of R_a that satisfies $6\frac{1}{2} < x < 2$? Explain your answer.

H Suppose that you replace a and b by any two members of R_a so that a is less than b. Will the set of all x that satisfy $a < x < b$ contain a? Will the set contain b? Is $\{x \mid a < x < b\}$ the set of all members of R_a between a and b when the replacement for a is less than the replacement for b?

The set of all members of R_a between two given rational numbers is an *open interval in R_a*. In this book we will call the two given rational numbers the *limits* of the interval. The limits of an open interval are not members of the interval. $\{x \mid 1\frac{1}{2} < x < 4\frac{2}{3}\}$, $\{x \mid 3\frac{1}{7} < x < 7\frac{1}{3}\}$, and $\{x \mid 1.2 < x < 8.6\}$ are all open intervals in R_a.

I Name the limits of each interval obtained from conditions A, B, and C expressed in D1.

$U = R_a$.

$\{x \mid 1\frac{1}{2} < x < 4\frac{2}{3}\}$

D2

$U = R_a$.

$a \leqq x \leqq b$.

D $4\frac{1}{3} \leqq x \leqq 6\frac{2}{5}$.

E $6.5 \leqq x \leqq 7.2$.

D3

J Look at D2. Do the dots for the points between point $1\frac{1}{2}$ and point $4\frac{2}{3}$ form the graph of the open interval $\{x \mid 1\frac{1}{2} < x < 4\frac{2}{3}\}$? How are these dots indicated in D2?

Notice that, in D2, neither the graph of $\{x \mid 1\frac{1}{2} < x\}$ nor the graph of $\{x \mid x < 4\frac{2}{3}\}$ is indicated. We have indicated the graph of the solution set of the compound condition only. From now on, we will usually use this kind of graph for the graph of the solution set of a compound condition.

K How can you tell from the graph in D2 that the limits of $\{x \mid 1\frac{1}{2} < x < 4\frac{2}{3}\}$ are not members of the interval?

L Make a graph of $\{x \mid 3\frac{1}{7} < x < 7\frac{1}{3}\}$ and a graph of $\{x \mid 1.2 < x < 8.6\}$.

You have learned that an open interval does not contain its limits. Next you will study an interval that does contain its limits.

M Look at D3. Each of the conditions D and E was obtained from the condition $a \leqq x \leqq b$ by replacing a and b by members of the universe. What is the universe for each of the variables a, x, and b?

N What replacements were made for a and for b in $a \leqq x \leqq b$ to obtain condition D? To obtain condition E? In each case, is the replacement for a less than the replacement for b?

Think about condition D in D3. The compound condition $4\frac{1}{3} \leqq x \leqq 6\frac{2}{5}$ is the same as the compound condition $4\frac{1}{3} \leqq x \wedge x \leqq 6\frac{2}{5}$. Therefore, to obtain the solution set of $4\frac{1}{3} \leqq x \leqq 6\frac{2}{5}$, you must decide which members of R_a satisfy both $4\frac{1}{3} \leqq x$ and $x \leqq 6\frac{2}{5}$.

O Is $4\frac{1}{3}$ a solution of both $4\frac{1}{3} \leqq x$ and $x \leqq 6\frac{2}{5}$?
? Is $6\frac{2}{5}$ a solution of both $4\frac{1}{3} \leqq x$ and $x \leqq 6\frac{2}{5}$? Explain your answers.

P Is $4\frac{2}{3}$ a solution of both $4\frac{1}{3} \leqq x$ and $x \leqq 6\frac{2}{5}$? Is 5 a solution? Is $6\frac{1}{5}$ a solution? Is each rational number between $4\frac{1}{3}$ and $6\frac{2}{5}$ a solution?

Q How do you know that no number less than $4\frac{1}{3}$ is a solution of both $4\frac{1}{3} \leqq x$ and $x \leqq 6\frac{2}{5}$? That no number greater than $6\frac{2}{5}$ is a solution of both conditions?

R Is the set of numbers that contains $4\frac{1}{3}$ and $6\frac{2}{5}$ and all the rational numbers between $4\frac{1}{3}$ and $6\frac{2}{5}$ the solution set of $4\frac{1}{3} \leqq x \leqq 6\frac{2}{5}$?

S Now think about condition E. How do you know that $6.5 \leqq x \leqq 7.2$ is the same compound condition as $6.5 \leqq x \wedge x \leqq 7.2$? Does 6.5 satisfy $6.5 \leqq x \leqq 7.2$? Does 7.2 satisfy the condition? Does every member of R_a between 6.5 and 7.2 satisfy the condition?

T Is $\{x \mid 6.5 \leqq x \leqq 7.2\}$ the set of rational numbers that contains 6.5 and 7.2 and all the rational numbers between 6.5 and 7.2?

U Suppose that, in $a \leqq x \leqq b$, you replace a
? and b by any two members of R_a so that a is less than b. Will the set of all x that satisfy

$U = R_a$.

$\{x | 4\frac{1}{3} \leqq x \leqq 6\frac{2}{5}\}$

D4

$a \leqq x \leqq b$ contain both a and b? Describe in words the solution set of $a \leqq x \leqq b$ when the replacement for a is less than the replacement for b.

The set of members of R_a that contains two given rational numbers and all the rational numbers between the two given numbers is a *closed interval in R_a*. We call the two given numbers the *limits* of the interval. The limits of a closed interval are members of the interval. $\{x | 4\frac{1}{3} \leqq x \leqq 6\frac{2}{5}\}$ and $\{x | 6.5 \leqq x \leqq 7.2\}$ are closed intervals in R_a.

V Name the limits of the intervals obtained from conditions D and E expressed in D3.

W Look at D4. Do the dots for point $4\frac{1}{3}$, point $6\frac{2}{5}$, and all the points between these two points form the graph of $\{x | 4\frac{1}{3} \leqq x \leqq 6\frac{2}{5}\}$? How is this graph indicated in D4?

X How does the graph show that the limits of $\{x | 4\frac{1}{3} \leqq x \leqq 6\frac{2}{5}\}$ are members of the interval?

Y Make a graph of $\{x | 6.5 \leqq x \leqq 7.2\}$.

Z Suppose that $\frac{5}{6}$ and $\frac{7}{4}$ are the limits of an interval in R_a. If the interval is an open interval, are $\frac{5}{6}$ and $\frac{7}{4}$ members of the interval? If the interval is a closed interval, what do you know about $\frac{5}{6}$ and $\frac{7}{4}$?

Next you will learn about a third kind of interval in R_a.

A Look at D5. Condition F was obtained from the condition $a < x \leqq b$. Condition G was obtained from $a \leqq x < b$. What is the universe for each of the variables?

$U = R_a$.

$a < x \leqq b$.	$a \leqq x < b$.
F $5\frac{1}{2} < x \leqq 7\frac{1}{4}$.	**G** $2 \leqq x < 8$.

D5

closed interval in R_a. The set of all x that satisfy $a \leqq x \leqq b$ when $a < b$ and when the universe for a, x, and b is R_a. a and b are the **limits** of the interval.

B What replacements were made for a and b in $a < x \leqq b$ to obtain condition F? What replacements were made for a and b in $a \leqq x < b$ to obtain condition G? In each condition, is the replacement for a less than the replacement for b?

C Is condition F the same as the compound condition $5\frac{1}{2} < x \wedge x \leqq 7\frac{1}{4}$? Is $5\frac{1}{2}$ a solution of $5\frac{1}{2} < x \leqq 7\frac{1}{4}$? Is $7\frac{1}{4}$ a solution? Is each number between $5\frac{1}{2}$ and $7\frac{1}{4}$ a solution?

D Is $\{x | 5\frac{1}{2} < x \leqq 7\frac{1}{4}\}$ the set that contains $7\frac{1}{4}$ and all the members of R_a between $5\frac{1}{2}$ and $7\frac{1}{4}$?

E Is condition G the same as the compound condition $2 \leqq x \wedge x < 8$? Is 2 a solution of $2 \leqq x < 8$? Is 8 a solution? Describe in words the solution set of $2 \leqq x < 8$.

The set of members of R_a that contains a given rational number and all the rational numbers between that number and another

half-open interval in R_a. The set of all x that satisfy $a < x \leqq b$, or the set of all x that satisfy $a \leqq x < b$ when $a < b$ and when the universe for a, x, and b is R_a. a and b are the **limits** of the interval.

given rational number is a *half-open interval in R_a*. The two given numbers are the limits of the interval. A half-open interval contains only one of its limits.

F Name the limits of $\{x|5\frac{1}{2} < x \leqq 7\frac{1}{4}\}$. Name the limits of $\{x|2 \leqq x < 8\}$.

G Make a graph of $\{x|5\frac{1}{2} < x \leqq 7\frac{1}{4}\}$ and a graph of $\{x|2 \leqq x < 8\}$.

For each of exercises H through M, first write a standard description of an open interval, then a standard description of a closed interval, and, finally, a standard description of a half-open interval. Use the two numbers named in the exercise as the limits of each interval. $U = R_a$. For exercise H, you could write these descriptions:
$\{x|5 < x < 11\}$, $\{x|5 \leqq x \leqq 11\}$, and $\{x|5 < x \leqq 11\}$.

H	5	11	**J**	$\frac{4}{11}$	$\frac{7}{8}$	**L**	.013 .099
I	7.8	9.5	**K**	$6\frac{1}{2}$	$12\frac{2}{3}$	**M**	162 204

Now you know that there are three kinds of intervals in R_a. Each kind contains all the rational numbers between two given rational numbers that we call the limits of the interval. An open interval in R_a contains neither of its limits. A closed interval contains both of its limits. A half-open interval contains only one of its limits.

On your own

For each of the intervals named in exercises 1 through 10, first name the limits. Then tell whether the interval is open, closed, or half open. Finally, make a graph of the interval. $U = R_a$.

$U = R_a$.

H $\{n|2\frac{1}{2} < n < 4\}$

I $\{n|3 \leqq n < 6\}$

D6

1 $\{m|.6 \leqq m \leqq 4.8\}$

2 $\{x|3.4 < x < 7.9\}$

3 $\{q|2.2 \leqq q < 6.6\}$

4 $\{r|\frac{11}{4} < r \leqq \frac{9}{2}\}$

5 $\{v|\frac{1}{7} < v \leqq 5\frac{1}{3}\}$

6 $\{s|5 \leqq s < 10\}$

7 $\{b|4\frac{1}{8} < b < 9\frac{3}{4}\}$

8 $\{k|5 < k < 7.5\}$

9 $\{c|4.9 \leqq c \leqq 7\}$

10 $\{n|1\frac{5}{6} \leqq n \leqq 3\frac{7}{8}\}$

Use the intervals named in D6 to answer the questions in exercises 11 through 15.

11 What kind of interval is each of the intervals? Name the limits of each interval.

12 How do you know that $\{n|2\frac{1}{2} < n < 4\} \cup \{n|3 \leqq n < 6\} = \{n|2\frac{1}{2} < n < 6\}$?

13 How do you know that 3 is the least number that is a member of both the interval $\{n|2\frac{1}{2} < n < 4\}$ and the interval $\{n|3 \leqq n < 6\}$? Can you name the greatest number that is a member of both intervals?

14 Is each rational number between 3 and 4 a member of both $\{n|2\frac{1}{2} < n < 4\}$ and $\{n|3 \leqq n < 6\}$?

15 How do you know that the intersection of $\{n|2\frac{1}{2} < n < 4\}$ and $\{n|3 \leqq n < 6\}$ is the same set as $\{n|3 \leqq n < 4\}$?

For each of exercises 16 through 21, give a standard description of the interval that is the union of the two intervals named. Then give a standard description of the interval that is the intersection of the two intervals. $U = R_a$.

16 $\{h|5.9 \leqq h \leqq 8.6\}$, $\{h|8 \leqq h \leqq 9.5\}$

17 $\{y|2\frac{1}{4} < y < 5\frac{1}{9}\}$, $\{y|1\frac{3}{4} \leqq y \leqq 3\frac{1}{2}\}$

18 $\{z|\frac{4}{5} < z < \frac{13}{3}\}$, $\{z|\frac{10}{3} < z < \frac{21}{4}\}$

19 $\{m|4 \leqq m < 5\frac{1}{3}\}$, $\{m|4\frac{7}{8} < m < 5\}$

20 $\{x|100 \leqq x \leqq 105\}$, $\{x|98 < x < 101\}$

21 $\{b|\frac{1}{5} < b \leqq \frac{2}{3}\}$, $\{b|\frac{1}{7} \leqq b < \frac{4}{5}\}$

Conditions involving the idea of "approximately equal to"

In this lesson you will use your knowledge of intervals as you study a new kind of condition. First, however, you will review some familiar kinds of conditions.

A What conditions are expressed in D1? What is the universe for the variable in each of the conditions?

B How do you know that condition A is a condition for equality? Tabulate the solution set of condition A.

C Is condition B a condition for equality, or is it a condition for inequality? Which rational number of arithmetic is not a member of $\{y \mid y \neq 20.59\}$?

D What kind of condition is condition C? Is 4 a member of the solution set of $4\frac{1}{3} + s < 12$? Is $7\frac{1}{3}$ a member of the solution set? Is $7\frac{2}{3}$ a member? Is $10\frac{3}{4}$ a member? Each member of the solution set of $4\frac{1}{3} + s < 12$ is less than what number?

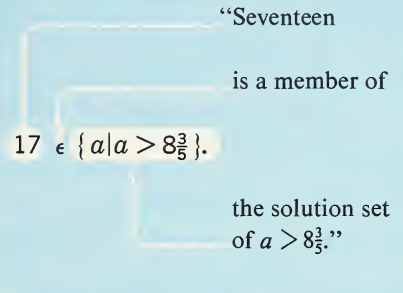
You know that, if an object belongs to a set, it is a member of the set. A member of a set can also be called an *element* (el′ə mənt) of the set. From now on, we will use either "member" or "element" to refer to an object that belongs to a set.

E Look again at D1. What kind of condition is condition D? Is 0 an element of the solution set of $a + 6\frac{2}{5} > 15$? Is 17 an element of the solution set? Is $8\frac{3}{5}$ an element? Every element of the solution set of $a + 6\frac{2}{5} > 15$ is greater than what number?

$U = R_a$.

A $r - 8\frac{1}{2} = 11$.
B $y \neq 20.59$.
C $4\frac{1}{3} + s < 12$.
D $a + 6\frac{2}{5} > 15$.

D1



D2

Mathematicians use the symbol below at the right to express the idea of "member of." This symbol is one form of the Greek letter *epsilon* (ep′sə lon). D2 shows how to write and read a sentence that includes the symbol for "member of." You can say "element of" instead of "member of."

∈

For each of exercises F through I, first read the sentence, using the word "element." Then tell whether the sentence expresses a true statement or a false statement. $U = R_a$.

F $3 \in \{x \mid x > 1\}$.
G $9 \in \{q \mid q \geqq 9\}$.
H $.4 \in \{y \mid .4 < y < .6\}$.
I $\frac{11}{2} \in \{z \mid 2\frac{1}{2} \leqq z \leqq 10\}$.

The exercises you have just completed concern conditions for equality and inequality. The exercises that follow concern a new kind of condition. In everyday speech we often make statements like "I have about 50 records" or "He walked about 3 miles." All such statements involve the idea of "*approximately equal to*." This idea is also used in mathematics, especially in the study of measurement.

The symbol below at the right expresses the idea of "approximately equal to."

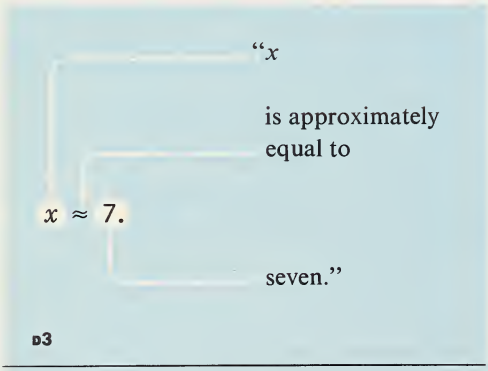
D3 shows how to write and read a sentence that includes this symbol. Study D3.

Read each of the sentences in exercises A through D.

A $z \approx 10$. **B** $d \approx 50.5$ **C** $r \approx 3\frac{1}{2}$. **D** $y \approx 13$.

Suppose that you were to ask several persons if 6 is approximately equal to 7. Each person would have his own idea of what "approximately equal to" means. To decide what numbers satisfy $x \approx 7$ you can use different rules. These rules involve the *absolute difference* (ab′se lüt) of two numbers.

The *absolute difference* of two numbers is the difference obtained when the lesser number is subtracted from the greater number. The absolute difference of 7 and 4 is 3. The absolute difference of 4 and 7 is also 3. The absolute difference of 5 and 5 is 0.



D3

One rule you can use to find the members of $\{x|x \approx 7\}$ is "If the absolute difference of 7 and a given number is equal to 1 or less than 1, then the given number is approximately equal to 7." $U = R_a$.

Use the rule that you have been given to help you answer the questions in exercises E through P.

E The difference of 7 and the replacement for x is to be equal to 1 or less than 1. So subtract 1 from 7. What number do you obtain? What is the absolute difference of 7 and 6? How do you know that 6 is a solution of $x \approx 7$?

F Now add 1 to 7. What number do you obtain? What is the absolute difference of 7 and 8? Is 8 a solution of $x \approx 7$?

G What is the absolute difference of 7 and 6.1? Is .9 less than 1? Is 6.1 an element of $\{x|x \approx 7\}$? What is the absolute difference of 7 and 7.9? Is 7.9 an element of $\{x|x \approx 7\}$?

H What is the absolute difference of 7 and 6.5? What is the absolute difference of 7 and 7.5? Is $6.5 \in \{x|x \approx 7\}$? Is $7.5 \in \{x|x \approx 7\}$?

I Is $6.99 \in \{x|x \approx 7\}$? Is $7.01 \in \{x|x \approx 7\}$? Explain your answers.

J Is 0 less than 1? Is $7 \in \{x|x \approx 7\}$?

K You know, from exercises G through J, that 6.1, 6.5, 6.99, 7, 7.01, 7.5 and 7.9 are solutions of $x \approx 7$. Is each of these numbers between 6 and 8?

L What is the absolute difference of 7 and 5.5? Of 7 and 8.7?

$U = R_a$.

$\{x|6 \leqq x \leqq 8\} = \{x|x \approx 7\}$.

D4

$U = R_a$.

$\{x|6.5 \leqq x \leqq 7.5\} = \{x|x \approx 7\}$.

D5

M Is either 5.5 or 8.7 an element of the solution set of $x \approx 7$? How do you know? Is either 5.5 or 8.7 between 6 and 8?

According to the rule you have been using, both 6 and 8 satisfy $x \approx 7$. Each number between 6 and 8 also satisfies $x \approx 7$. The least number that satisfies $x \approx 7$ is 6; the greatest number is 8. Notice that to obtain the least number, you can subtract 1 from 7; to obtain the greatest number, you can add 1 to 7.

N Is $\{x|x \approx 7\} = \{x|6 \leqq x \leqq 8\}$? How do you know?

O Name the limits of $\{x|6 \leqq x \leqq 8\}$. How do you know that $\{x|6 \leqq x \leqq 8\}$ is a closed interval in R_a?

P Look at D4. Explain how you know that the graph in D4 is a graph of both $\{x|6 \leqq x \leqq 8\}$ and $\{x|x \approx 7\}$.

You have used one rule for finding the solutions of $x \approx 7$. Now you will use a different rule. This second rule is "If the absolute difference of 7 and a given number is equal to .5 or less than .5, then the given number is approximately equal to 7." $U = R_a$.

Use this second rule to help you answer the questions in exercises A through J.

A How can you obtain the least member of the solution set of $x \approx 7$? Name the least member of $\{x|x \approx 7\}$. Is the absolute difference of 7 and 6.5 equal to .5?

B What number do you add to 7 to obtain the greatest member of $\{x|x \approx 7\}$? Name the greatest member. What is the absolute difference of 7 and 7.5?

C How do you know that both 6.5 and 7.5 satisfy $x \approx 7$?

D What is the absolute difference of 7 and 6.55? Of 7 and 7.459? Are both 6.55 and 7.459 elements of $\{x|x \approx 7\}$?

E Is $6.9 \in \{x|x \approx 7\}$? Is $7 \in \{x|x \approx 7\}$? Is $7.35 \in \{x|x \approx 7\}$? Explain your answers.

F You know that each of the numbers 6.55, 6.9, 7, 7.35, and 7.459 satisfies $x \approx 7$. Is each of these numbers between 6.5 and 7.5?

G When you use the second rule, 6 is not a member of $\{x|x \approx 7\}$ because the absolute difference of 7 and 6 is greater than .5. Is 8 a solution of $x \approx 7$? How do you know? Is either 6 or 8 between 6.5 and 7.5?

H When you use the second rule, is $\{x|x \approx 7\}$ the same set as $\{x|6.5 \leqq x \leqq 7.5\}$?

I $\{x|6.5 \leqq x \leqq 7.5\}$ is what kind of interval? Name the limits of the interval.

J Look at the graph in D5. Is it a graph of both $\{x|x \approx 7\}$ and $\{x|6.5 \leqq x \leqq 7.5\}$? Why is the graph of $\{x|x \approx 7\}$ in D5 different from the graph of $\{x|x \approx 7\}$ in D4?

You have used two different rules to obtain two different solution sets of $x \approx 7$. In each case, the solution set was a closed interval in R_a. Now, for exercises K through N, use the rule "If the absolute difference of 7 and a given number is equal to .1 or less than .1, then the given number is approximately equal to 7." $U = R_a$.

K How can you obtain the least number that is a solution of $x \approx 7$? How can you obtain the greatest number? What is the least number that is a solution of $x \approx 7$? What is the greatest number?

L Is $\{x|6.9 \leqq x \leqq 7.1\}$ the solution set of $x \approx 7$? Is $\{x|x \approx 7\} = \{x|6.9 \leqq x \leqq 7.1\}$?

M $\{x|6.9 \leqq x \leqq 7.1\}$ is what kind of interval? Explain your answer.

N Make a graph of $\{x|6.9 \leqq x \leqq 7.1\}$. Is your graph also a graph of $\{x|x \approx 7\}$?

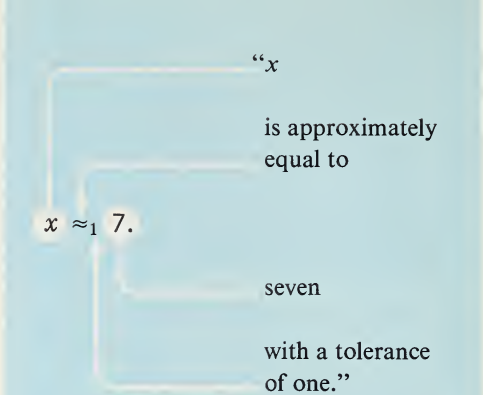
Although each rule you used to obtain the solutions of $x \approx 7$ involved a different given member of R_a, all the rules involved the same procedure. In each case, you subtracted the given rational number from 7 to obtain the least member of $\{x|x \approx 7\}$, and you added the given rational number to 7 to obtain the greatest member of $\{x|x \approx 7\}$. A rational number that is used in this way is a *tolerance* (tol′ər-əns). When you use 1 to determine the solutions of $x \approx 7$, the tolerance is 1. When you use .5, the tolerance is .5. When you use .1, the tolerance is .1.

Look at D6. This display shows how to write and read a sentence that expresses a condition involving the idea of tolerance. Notice that the numeral that expresses the tolerance is placed below and to the right of the symbol for "approximately equal to."

O Study the mathematical sentence in D6. Then write similar sentences to express the conditions "$x \approx 7$ with a tolerance of .5" and "$x \approx 7$ with a tolerance of .1."

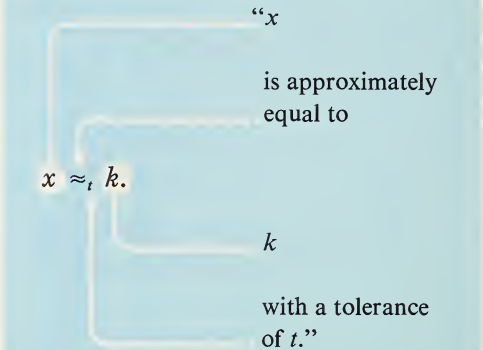
P D7 shows how to write and read a sentence that expresses a condition that has a variable for the tolerance. What is the universe for each of the variables?

The condition $x \approx_t k$ requires the absolute difference of k and x to be either equal to t or less than t.



D6

U = Z.



D7

Q How do you know that, for each replacement of k and t in $x \approx_t k$, $k - t$ is the least number that is a solution of $x \approx_t k$? What is the greatest number?

R When the universe is Z, the solution set of $x \approx_t k$ is $\{x|k - t \leqq x \leqq k + t\}$. Is $\{x|x \approx_t k\}$ the same set as $\{x|k - t \leqq x \leqq k + t\}$?

S Suppose that you replace k in $x \approx_t k$ by 17 and that you replace t by .2. What condition do you obtain?

T What closed interval in R_a is the solution set of the condition you obtained for exercise S?

U What replacements should you make for k and t in $k - t \leqq x \leqq k + t$ to obtain a condition that has the same solution set as $x \approx_4 11$? Write a standard description of the interval that is the solution set of $x \approx_4 11$.

V Does each pair of replacements for k and t in $k - t \leqq x \leqq k + t$ determine an interval in R_a if the replacement for k is greater than the replacement for t?

In this lesson you learned to find solution sets of conditions that involve the idea of "approximately equal to."

On your own

For each of exercises 1 through 10, tell whether the sentence expresses a true statement or a false statement. $U = R_a$.

1 $5 \in \{x | x + 2 < 6\}$.
2 $10 \in \{x | x - 1 > 8\}$.
3 $.7 \in \{x | x + .3 < .11\}$.
4 $9 \in \{x | 3 + x > 14\}$.
5 $\frac{2}{3} \in \{x | \frac{2}{3} + x < \frac{4}{3}\}$.
6 $.3 \in \{x | x \leqq .3\}$.
7 $4 \in \{x | 4 > x\}$.
8 $8 \in \{x | x + 1 = 9\}$.
9 $5 \in \{x | 1 + x < 6\}$.
10 $0 \in \{x | .1 < x\}$.

For each condition expressed in exercises 11 through 22, tell what the tolerance is. Name the least number and the greatest number in the interval that is the solution set. Then write a standard description of this interval. $U = R_a$. For exercise 11, you could write: tolerance, .3; 3.7, 4.3; $\{x | 3.7 \leqq x \leqq 4.3\}$.

11 $x \approx_{.3} 4$.
12 $z \approx_7 50$.
13 $x \approx_{\frac{1}{4}} 17$.
14 $a \approx_4 25$.
15 $n \approx_{.5} 3$.
16 $m \approx_1 21$.
17 $x \approx_{.6} 9$.
18 $a \approx_{.09} 1.5$.
19 $b \approx_{\frac{1}{8}} 71$.
20 $q \approx_{10} 100$.
21 $x \approx_{.25} 4$.
22 $m \approx_{.05} .6$.

For each exercise below, make a graph of the solution set of the condition expressed. $U = R_a$.

23 $x \approx_2 4.1$.
24 $y \approx_3 5.5$.
25 $r \approx_{.5} 9$.
26 $w \approx_{.1} 2.7$.
27 $m \approx_{\frac{3}{16}} 1$.
28 $s \approx_{\frac{1}{4}} 2\frac{3}{4}$.

KEEPING SKILFUL

For each exercise below, name the basic fraction that indicates the rational number of arithmetic named in the exercise.

1 $\frac{10}{12}$
2 $\frac{25}{5}$
3 $\frac{148}{12}$
4 $\frac{84}{90}$
5 $\frac{132}{144}$
6 $\frac{95}{105}$
7 $\frac{14}{22}$
8 $\frac{80}{80}$
9 $\frac{100}{8}$
10 $\frac{176}{16}$
11 $\frac{45}{30}$
12 $\frac{10}{160}$

For each exercise below, write a mixed numeral that expresses the rational number of arithmetic named in the exercise.

13 $\frac{49}{12}$
14 $\frac{53}{5}$
15 $\frac{60}{40}$
16 $\frac{203}{2}$
17 $\frac{109}{9}$
18 $\frac{55}{3}$
19 $\frac{341}{20}$
20 $\frac{20}{6}$
21 $\frac{31}{19}$
22 $\frac{30}{25}$
23 $\frac{2000}{12}$
24 $\frac{99}{8}$

For each of exercises 25 through 44, find the sum, difference, product, or quotient.

25 $17\frac{1}{2} + 8\frac{2}{3}$
26 $10\frac{5}{6} - 1\frac{2}{3}$
27 $\frac{7}{12} \times 5$
28 $25\frac{3}{20} + 3\frac{9}{10}$
29 $\frac{5}{6} \times \frac{3}{100}$
30 $1\frac{14}{15} - \frac{1}{5}$
31 $\frac{14}{15} \div \frac{7}{20}$
32 $3\frac{1}{2} + 16\frac{4}{7} + 12\frac{9}{10}$
33 $\frac{24}{25} \div \frac{1}{3}$
34 $1\frac{5}{6} \times 3\frac{1}{5}$
35 $10\frac{16}{17} \div 6$
36 $31\frac{3}{4} + 8\frac{9}{10}$
37 $25\frac{1}{2} \div \frac{3}{4}$
38 $1\frac{21}{25} - \frac{3}{4}$
39 $8\frac{1}{3} \times 1\frac{1}{5}$
40 $18\frac{2}{15} + 60$
41 $10\frac{5}{6} \div \frac{1}{12}$
42 $4 \times 3\frac{12}{19}$
43 $35\frac{15}{16} - 30$
44 $22\frac{1}{6} + 12\frac{1}{10}$

For each exercise below, find the sum, difference, product, or quotient named in the exercise.

45 $463 + 287 + 1899$
46 $346 \div 29$
47 $8291 - 2445$
48 676×82
49 $46588 + 8319$
50 $1082 \div 321$
51 6617×462
52 $8003 - 997$
53 816×1043
54 $23 + 1837 + 159$
55 6553×12
56 $2153 + 608 + 4839$
57 $6715 \div 56$
58 $195247 - 3684$
59 $23 + 16 + 89 + 77$
60 114×627

Conditions in three variables

In *Book 1* you used ordered pairs to make replacements in conditions in two variables. In this lesson you will learn to make replacements in conditions in three variables.

A Look at D1. Is it possible to decide whether the idea expressed by sentence A is true or false? Explain your answer.

B How many variables are in condition A?

C When you replace x by 2, y by 4, and z by 6 in $x + 2y + 3z < 20$, do you obtain the statement expressed by sentence B? Is it possible to decide whether this statement is a true statement or a false statement?

D When a condition contains three variables, how many of the variables must you replace before you obtain a statement?

E Look at D2. What are the variables in the condition?

F Read the tabulations of sets Q, V, and W in D2. The members of Q will be used to replace x in $x + y + z < 12$; the members of V, to replace y; and the members of W, to replace z. How many possible replacements are there for x? For y? For z?

G The sentences in D3 express all the statements obtained from $x + y + z < 12$ when you use the elements of Q, V, and W as replacements for x, y, and z. To obtain statement C, you replace x by 4, y by 2, and z by 3. What replacements do you make for x, y, and z to obtain statement D?

Notice that, to obtain statement D, you used a *triple* of numbers, 4, 2, and 6, as replacements for the three variables, x, y, and z in $x + y + z < 12$. A triple of objects like this is an *ordered triple*. 4 is the first component of the ordered triple; 2 is the second component; and 6 is the third component.

A $x + 2y + 3z < 20$.
B $2 + 2(4) + 3(6) < 20$.

D1

$$x + y + z < 12.$$

$Q = \{4, 7\}$.
$V = \{2, 3, 5\}$.
$W = \{3, 6\}$.

D2

C $4 + 2 + 3 < 12$.	**I** $7 + 2 + 3 < 12$.
D $4 + 2 + 6 < 12$.	**J** $7 + 2 + 6 < 12$.
E $4 + 3 + 3 < 12$.	**K** $7 + 3 + 3 < 12$.
F $4 + 3 + 6 < 12$.	**L** $7 + 3 + 6 < 12$.
G $4 + 5 + 3 < 12$.	**M** $7 + 5 + 3 < 12$.
H $4 + 5 + 6 < 12$.	**N** $7 + 5 + 6 < 12$.

D3

"The ordered triple

(4, 2, 6)

four, two, six"

D4

ordered triple (trip′əl). A triple of objects in which the objects occur in a special order. The object that occurs first is the **first component** of the ordered triple. The object that occurs second is the **second component.** The object that occurs third is the **third component.**

D4 shows how to write and read the name of the ordered triple whose components are 4, 2, and 6.

Now look at D5. The diagram shows one way to determine all the ordered triples that can be formed from the members of sets Q, V, and W. Notice that each member of Q is matched with each member of V. Then each *pair* of members of Q and V is matched with each member of W in turn.

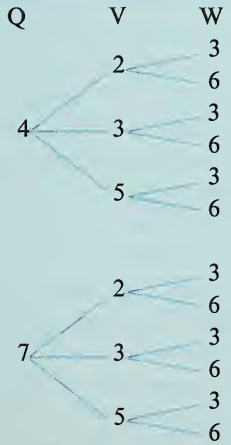
H Tabulate the set of ordered triples that you can obtain from the diagram in D5.

I Look again at D2. How many elements are there in Q? In V? In W? How many ordered triples do you obtain from the diagram in D5?

J How can you use the number of elements in each of sets Q, V, and W to determine the number of ordered triples that you can form from the elements of Q, V, and W?

The members of the set that you tabulated for exercise H were used as replacements for x, y, and z in $x + y + z < 12$. This set is the universe for the ordered triple of variables x, y, and z. You may use the symbol at the right to show that you are working with an ordered triple of variables.

(x, y, z)



D5

"The set whose members are

$$\{(x, y, z) \mid x \in Q \wedge y \in V \wedge z \in W\}$$

all ordered triples x, y, z

that satisfy the condition that

x is a member of Q

and y is a member of V

and z is a member of W"

D6

The universe we have been discussing is the set of all (x, y, z) that satisfy the condition that $x \in Q \wedge y \in V \wedge z \in W$. D6 shows how to write and read a symbol for a standard description of this universe. Study D6.

K Which statements expressed in D3 are true? Tabulate $\{(x, y, z) \mid x + y + z < 12\}$.

So far you have used a finite set as a universe for an ordered triple of variables. In the following exercises you will use as a universe the infinite set of ordered triples formed from the set of natural numbers.

A Suppose that the universe for (x, y, z) in a condition is the set of all ordered triples whose three components are natural numbers. Is $(2, 1, 7)$ a member of this set? Is $(0, 0, 0)$ a member? Is $(4, 3, \frac{1}{2})$ a member?

B Look at D7. How many variables are in condition A? What is the universe for (x, y, z)?

C Is $\{(x, y, z) | x \in \mathrm{N} \wedge y \in \mathrm{N} \wedge z \in \mathrm{N}\}$ a finite set, or is it an infinite set? Explain your answer.

D What three simple conditions make up condition A? What connective is used in condition A? Notice that the sentence for each simple condition is within parentheses.

E How do you know that the solution set of condition A is a set of ordered triples?

F How do you know that each solution of condition A must satisfy all three of the simple conditions that make up condition A?

G Which simple condition in condition A contains only one variable?

H Name the second component of each ordered triple that satisfies $y = 3$. Is $\{(x, y, z) | y = 3\}$ a finite set, or is it an infinite set?

I Which simple condition in condition A contains exactly two variables?

Next you will decide which ordered triples that satisfy $y = 3$ also satisfy $x + y < 7$.

J Replace y in $x + y < 7$ by 3. Does the condition $x + 3 < 7$ have the same solution set as $x < 4$? What elements of N satisfy $x < 4$?

K What natural numbers are first components of the ordered triples that satisfy both $x + y < 7$ and $y = 3$? How do you know that 3 is the second component of each of these ordered triples?

Now you will decide which ordered triples satisfy $x + y + z = 12$ as well as $x + y < 7$ and $y = 3$. You know that each ordered triple that satisfies both $x + y < 7$ and $y = 3$ has a first component that is less than 4 and a second component that is equal to 3. The sentences in D8 express all the conditions that you can obtain from $x + y + z = 12$ by replacing x by a natural number less than 4, and y by 3.

L Look at D8. What member of N satisfies condition B? Condition C? Condition D? Condition E? Is each of the numbers that you just named the third component of an ordered triple that satisfies the compound condition?

M Name the third component of the solution of condition A whose first component is 0 and whose second component is 3.

N Name the third component of the solution of condition A whose first component is 1 and whose second component is 3.

O You have found that both $(0, 3, 9)$ and $(1, 3, 8)$ satisfy condition A. Name the two other ordered triples that satisfy condition A.

P Tabulate the solution set of condition A. Is the set you tabulated a subset of the universe?

Universe for $(x, y, z) = \{(x, y, z) | x \in \mathrm{N} \wedge y \in \mathrm{N} \wedge z \in \mathrm{N}\}$.

A $(x + y + z = 12) \wedge (x + y < 7) \wedge (y = 3)$.

D7

B $0 + 3 + z = 12$.
C $1 + 3 + z = 12$.
D $2 + 3 + z = 12$.
E $3 + 3 + z = 12$.

D8

Universe for the variables in $(x, y, z) = \mathrm{N}$.

F $(x + y = 5) \wedge (y < 4) \wedge (x + z < 3)$.

D9

In the following exercises you will find the solution set of another compound condition in three variables. This time each of the simple conditions will contain only one or two variables.

A Look at D9. How do you know that condition F is a compound condition? How many different variables are in condition F?

B In D9 the sentence "Universe for the variables in $(x, y, z) = \mathrm{N}$" means that the universe for the ordered triple of variables is the set of all ordered triples whose first, second, and third components are natural numbers. Write a standard description of this universe.

C Which simple condition in condition F contains only one variable?

D What do you know about the second component of each member of the universe that satisfies $y < 4$? Is $\{(x, y, z) | y < 4\}$ a finite set, or is it an infinite set?

Now you can decide which solutions of $y < 4$ also satisfy $x + y = 5$. The sentences in D10 express all the conditions that you can obtain from $x + y = 5$ by replacing y by a natural number less than 4.

E What member of N satisfies condition G? Are both $y < 4$ and $x + y = 5$ satisfied by each ordered triple whose first component is 5 and whose second component is 0?

F What member of N satisfies condition H? Are both $y < 4$ and $x + y = 5$ satisfied by each ordered triple whose first component is 4 and whose second component is 1?

G Read sentences I and J. How do you know that each ordered triple with a first component of 3 and a second component of 2 satisfies both $y < 4$ and $x + y = 5$? How do you know that each ordered triple with a first component of 2 and a second component of 3 satisfies both simple conditions?

G $x + 0 = 5$.
H $x + 1 = 5$.
I $x + 2 = 5$.
J $x + 3 = 5$.

D10

K $5 + z < 3$.
L $4 + z < 3$.
M $3 + z < 3$.
N $2 + z < 3$.

D11

Now that you have a set of ordered triples that satisfy both $y < 4$ and $x + y = 5$, you can find the ordered triples that satisfy $x + z < 3$. The sentences in D11 express all the statements you can obtain from $x + z < 3$ by replacing x by each of the first components you obtained for exercises E, F, and G.

H Is there a natural number that satisfies condition K? Condition L? Condition M?

I What natural number satisfies condition N? What is the third component of the solution of $(x + y = 5) \wedge (y < 4) \wedge (x + z < 3)$ that has a first component of 2 and a second component of 3? Is (2, 3, 0) the only ordered triple in the universe that satisfies all three of the simple conditions in condition F expressed in D9.

J Tabulate the solution set of condition F.

In this lesson you learned what ordered triples are. You also learned how to use a set of ordered triples as the universe for three variables in simple and compound conditions.

On your own

Write a name for each of the ordered triples described in exercises 1, 2, and 3.

1 The ordered triple whose first component is 10:15 P.M.; whose second component is 9:15 P.M.; and whose third component is 8:15 P.M.

2 The ordered triple whose first component is $4\frac{2}{3}$ ft.; whose second component is 7 ft.; and whose third component is $5\frac{1}{2}$ ft.

3 The ordered triple whose first component is the number of prime numbers less than 10; whose second component is the number of even numbers less than 10; and whose third component is the number of members in the intersection of the set of prime numbers and the set of even numbers

4 Is (1, 3, 2) the same ordered triple as (3, 2, 1)? Explain your answer.

For each of exercises 5 through 10, supply the missing component or components of an ordered triple and write the name of the completed ordered triple. The second component of the ordered triple should be 2 times the first component, and the third component should be the sum of the first and second components.

5 (, 4, 6)
6 (5, 10,)
7 $(1\frac{1}{2}, \ , 4\frac{1}{2})$
8 $(\frac{2}{3}, \ , \ )$
9 (, .50,)
10 (, , 45)

11 Sets A, B, and C are tabulated below. Tabulate $\{(x, y, z) | x \in A \wedge y \in B \wedge z \in C\}$.

$A = \{2.5, 3.4\}$. $B = \{.3, .1\}$. $C = \{4, .25\}$.

Write sentences for all the statements that you can obtain from each condition expressed in exercises 12 and 13. Then write "T" after each sentence that expresses a true statement. The universe for (x, y, z) is the set that you tabulated for exercise 11.

12 $2x + 3y - z < 5$.
13 $x + y + z = 6.8$.

14 Set $A = \{0, 1, 2, 3, 4\}$. How many ordered triples are in $\{(x, y, z) | x \in A \wedge y \in A \wedge z \in A\}$?

Tabulate the solution set of each condition expressed in exercises 15 through 20. The universe for the variables in $(x, y, z) = N$.

15 $(2x = 6) \wedge (x + y = 12) \wedge (2x + 2y + z = 25)$.
16 $(z = 2) \wedge (x + z = 7) \wedge (x + y + z < 10)$.
17 $x + y + z = 4$.
18 $(x + y + z < 3) \wedge (y < 2) \wedge (z < 2)$.
19 $(x > 3) \wedge (x + y = 5) \wedge (5 - z > 3)$.
20 $(x < 3) \wedge (x + y < 4) \wedge (y + z = 5)$.

CHECKING UP

If you have trouble with these two tests, you can find help in lessons 113, 114, and 115.

Test 116

1 What mathematical sentence that includes the symbol for "and" expresses the same condition as the sentence $r < s < t$?

2 What idea is expressed by the symbol $\approx$?

For each of exercises 3 through 10, first tell what kind of interval in R_a is described. Then give the limits of the interval. $U = R_a$.

3 $\{x | \frac{1}{2} \leqq x \leqq \frac{3}{4}\}$
4 $\{y | 2.5 < y \leqq 7.6\}$
5 $\{z | 50 \leqq z < 51\}$
6 $\{p | 2\frac{1}{6} \leqq p \leqq 8\frac{1}{3}\}$
7 $\{s | 4.9 < s < 5.7\}$
8 $\{w | w \approx_{\frac{1}{8}} 3\frac{5}{8}\}$
9 $\{r | r \approx_{.5} 9\}$
10 $\{q | \frac{1}{980} < q < \frac{1}{713}\}$

For each of the intervals described in exercises 11 through 16, tell which of the limits, if any, are contained in the interval. $U = R_a$.

11 $\{x | 5 < x \leqq 8\}$
12 $\{y | \frac{2}{3} \leqq y \leqq 2\frac{1}{3}\}$
13 $\{m | \frac{1}{25} < m < \frac{1}{20}\}$
14 $\{z | z \approx_{.1} 4.6\}$
15 $\{n | n \approx_{\frac{1}{2}} 3\frac{1}{6}\}$
16 $\{b | \frac{1}{8} \leqq b < 5\frac{6}{7}\}$

17 Using the symbol $\leqq$, write another standard description of $\{x | x \approx_{.4} 3\}$. $U = R_a$.

18 Make a graph of $\{x | x \approx_{.4} 3\}$. $U = R_a$.

19 Give a standard description of the intersection of $\{y | 1\frac{1}{2} \leqq y \leqq 2\}$ and $\{y | 1\frac{7}{8} < y < 2\frac{1}{3}\}$. $U = R_a$.

20 What kind of interval is the set that you described for exercise 19? What are its limits?

Test 117

21 Set A has 3 elements; B has 5 elements; and C has 8 elements. How many elements are in $\{(x, y, z) | x \in A \wedge y \in B \wedge z \in C\}$?

Tabulate the solution set of the condition expressed in each of exercises 22 through 25. The universe for the variables in $(x, y, z) = N$.

22 $x + y + z = 3$.
23 $x < 2 \wedge y - x = 3 \wedge x + y + z < 5$.
24 $y - 2 = 4 \wedge y - x > 2 \wedge z = xy$.
25 $x + y - z = 8 \wedge z < 1 \wedge \sim(x < y)$.

Problems involving conditions in one and two variables

In *Book 1* you studied many new mathematical ideas and used these ideas when you solved problems. In the next few lessons, you will first review problems like those you solved in *Book 1*. Then you will learn to solve problems by using some of the new ideas that have been presented in this unit.

Read the problem in D1. U = Z. Remember that Z is the set of non-zero rational numbers of arithmetic.

A Do you know how many feet of copper tubing Mr. Ames had before he bought more? Use x as a variable for the number of feet of copper tubing he had to begin with. x

B How many feet of copper tubing did he buy? To what does $x + 27$ refer? $x + 27$

C You know that he had less than $40\frac{3}{8}$ ft. of tubing after he bought more tubing. How do you know that the sentence at the right expresses a condition for the problem? $x + 27 < 40\frac{3}{8}$.

D Why is the symbol $\{x|x < 13\frac{3}{8}\}$ a standard description of the solution set of the condition $x + 27 < 40\frac{3}{8}$?

E Is 0 a member of $\{x|x < 13\frac{3}{8}\}$? Why can you not tabulate $\{x|x < 13\frac{3}{8}\}$?

You can use $\{x|x < 13\frac{3}{8}\}$ to get the answer to the problem.

Mr. Ames had less than $13\frac{3}{8}$ ft. of copper tubing before he bought the 27 ft. of tubing.

Mr. Ames had some copper tubing. After he had bought 27 ft. more, he still had less than $40\frac{3}{8}$ ft. of tubing. How much copper tubing did he have before he bought the 27 ft. of tubing?

D1

Sam had less than \$30.00. After he had spent \$5.25, he had more than \$20.00 left. How much money did he have before he spent the \$5.25?

D2

F Now you can verify the answer to the problem. If Mr. Ames had less than $13\frac{3}{8}$ ft. of copper tubing and bought 27 feet more, did he then have less than $40\frac{3}{8}$ ft. of tubing?

You solved the problem in D1 by using one simple condition in one variable. In the following exercises, you will solve a problem by using a compound condition in one variable.

Read the problem in D2. U = R_a.

A Explain why $n - 5.25 > 20.00$ is one of the simple conditions for the problem. For what is n a variable?

B How do you know that $n < 30.00$ is another simple condition for the problem?

C Does the sentence below in blue express the compound condition for the problem in D2?

$$n - 5.25 > 20.00 \wedge n < 30.00.$$

D How do you know that $\{n|n > 25.25\}$ is the solution set of $n - 5.25 > 20.00$?

E Does each member of $\{n|n > 25.25\}$ satisfy $n < 30.00$? Explain your answer.

F Is $\{n|25.25 < n < 30.00\}$ the solution set of $n - 5.25 > 20.00 \wedge n < 30.00$? Explain your answer.

G Is the solution set of the compound condition an open interval in R_a, or is the solution set a closed interval in R_a? What are the limits of this interval? Explain your answers.

H You know that any member of R_a between 25.25 and 30.00 is a solution of the compound condition for the problem in D2. Is 25.6535 between 25.25 and 30.00? Can you use a solution like 25.6535 to get a sensible answer to the problem?

I Is 25.26 the least number that you can use to get a sensible answer to the problem? What is the greatest number?

Now you can give the answer to the problem in D2.

Sam had between \$25.25 and \$30.00 before he spent the \$5.25.

So far you have used simple and compound conditions in one variable to solve problems. In the following exercises you will solve problems by using simple and compound conditions in two variables.

Read the problem in D3. The universe for each variable in (x, y) is N.

A Do you know how many hits Nick made? Use x as a variable for the number of hits Nick made. x

B Do you know how many hits Scott made? Use y as a variable for the number of hits Scott made. y

C What does $x + y$ refer to in the problem? $x + y$

D How do you know that the sentence at the right expresses the condition for the problem? How many variables are there in the condition? $x + y < 4.$

E Now you can find the solution set of the condition. How do you know that the solution set of the condition is a set of ordered pairs?

Sam had less than \$30.00. After he had spent \$5.25, he had more than \$20.00 left. How much money did he have before he spent the \$5.25?

D2

Nick and Scott are members of the baseball team at Central School. Together, they made fewer than 4 hits during the first game of the season. How many hits did each boy make?

D3

There are fewer than 9 boys and girls on the staff of the school paper. There are fewer girls than boys on the staff. How many girls and how many boys are on the staff?

D4

If the solution set is a set of ordered pairs, why must the universe be a set of ordered pairs?

F What is the greatest number of hits that either boy could have made? What numbers do you use as replacements for x in $x + y < 4$?

G Make a chart to help you determine the solutions of $x + y < 4$. Tabulate the solution set of $x + y < 4$.

H To what does the first component of each member of the solution set refer? To what does the second component of each member refer?

Now you can use the ordered pairs that satisfy the condition for the problem in D3 to get the answers to the problem.

I If Nick made 0 hits, then Scott made 0 hits, 1 hit, 2 hits, or 3 hits. What are the other answers to the problem in D3?

Now you will solve a problem by using a compound condition in two variables.

Read the problem in D4. The universe for each variable is C.

J Do you know how many girls are on the staff? Use a as a variable for the number of girls who are on the staff. a

K Do you know how many boys are on the staff? Use b as a variable for the number of boys who are on the staff. b

L What does $a + b$ refer to in the problem? $a + b$

M How do you know that the sentence below at the right expresses one of the conditions for the problem? $a + b < 9.$

N How do you know that $a < b$, and not $b < a$, is another simple condition for the problem? $a < b.$

O Does the sentence below in blue express a compound condition for the problem?

$$a + b < 9 \wedge a < b.$$

P Now you can find the solution set of the compound condition. Is the universe for (a, b) the same as $\{(a, b) | a \in C \wedge b \in C\}$? Describe the universe in words.

Q Why are the numbers from 1 through 7 the only numbers that you can use as replacements for a in $a + b < 9$? Make a chart to help you determine the solutions of the simple condition $a + b < 9$.

With the help of your chart, you can find the solutions of $a + b < 9$ that are also solutions of $a < b$.

R How many of the ordered pairs expressed in your chart have first components that are less than their second components?

S Tabulate $\{(a, b) | a + b < 9 \wedge a < b\}$.

T Give the answers to the problem in D4.

Beverly and Jane collect stamps. Jane has 35 more stamps than Beverly has. Beverly has more than 145 stamps, but she also has fewer than 150 stamps. How many stamps does each girl have?

D5

The problems you have studied so far are like problems that you solved in *Book 1*. The next problem, however, involves a new idea. To find the answer to the problem, you will use a new kind of condition.

Read the problem in D5. The universe for each variable is N.

A Do you know how many stamps Beverly has? Use m as a variable for the number of stamps she has. m

B Do you know how many stamps Jane has? Use n as a variable for the number of stamps Jane has. n

C How do you know that the sentence below at the right expresses one of the conditions for the problem? $n - m = 35.$

D Is 145 less than the number of stamps that Beverly has? Is the number of stamps that Beverly has less than 150? Explain why the sentence at the right expresses another condition for the problem. $145 < m < 150.$

E Does the sentence below in blue express a compound condition for the problem?

$$n - m = 35 \wedge 145 < m < 150.$$

F Now you can find the solution set of the compound condition. Write a standard description of the universe for (m, n).

G What natural numbers satisfy $145 < m < 150$? What numbers do you use as replacements for m in $n - m = 35$? Make a chart to help you determine the solutions of the compound condition.

H How do you know that each ordered pair named in your chart will satisfy the compound condition $n - m = 35 \wedge 145 < m < 150$?

I Tabulate the solution set of $n - m = 35 \wedge 145 < m < 150$.

J What are the answers to the problem in D5?

Read the problem in D6. The universe for each variable is C.

K Do you know how many stoves were installed? Use x as a variable for the number of stoves installed.

L Do you know how many refrigerators were installed? Use y as a variable for the number of refrigerators installed.

M What does $x + y$ refer to in the problem?

N Is $x + y = 7$ one of the conditions for the problem? Why is the idea of equality used?

O You know that at least 3 stoves were installed. This means that the number of stoves installed was either equal to 3 or greater than 3. Which sentence, $x \geqq 3$ or $y \geqq 3$, expresses another condition for the problem? How do you know?

P Is $x + y = 7 \wedge x \geqq 3$ the compound condition for the problem?

Q Now you can find the solution set of the compound condition. How do you know that 5 is not a member of the universe for (x, y)? Is $(5, 5)$ a member of the universe? Write a standard description of the universe for (x, y).

R What does the condition $x \geqq 3$ tell you about the first component of each solution of $x + y = 7 \wedge x \geqq 3$?

S How do you know that you do not need to use 1 and 2 as replacements for x in $x + y = 7$? Why do you not use any numbers greater than 6 as replacements for x? Make a chart to help you determine the solutions of $x + y = 7$ that also satisfy $x \geqq 3$.

T Tabulate $\{(x, y) | x + y = 7 \wedge x \geqq 3\}$.

U Give the answers to the problem in D6.

D5

Beverly and Jane collect stamps. Jane has 35 more stamps than Beverly has. Beverly has more than 145 stamps, but fewer than 150 stamps. How many stamps does each girl have?

D6

Last year 7 new stoves and refrigerators were installed in the home economics department at Union High School. At least 3 of these appliances were stoves. How many stoves and how many refrigerators were installed?

D7

The sum of two numbers is 5. The first number is not 3. What are the numbers?

So far in this lesson you have solved problems that concern persons and things. Now you will solve a problem that concerns numbers only, and you will use ordered pairs as answers.

A Read the problem in D7. The universe for each variable in (x, y) is N. Use x as a variable for the first number. Use y as a variable for the second number. Is $x + y = 5$ one of the conditions for the problem?

B From the problem you know that it is not the case that the first number equals 3. Which

sentence, $\sim (x = 3)$ or $\sim (y = 3)$, expresses another condition for the problem?

C Does the sentence below in blue express the compound condition for the problem in D7?

$$x + y = 5 \wedge \sim (x = 3).$$

D Tabulate the set of ordered pairs that satisfy $x + y = 5$.

E What does the condition $\sim (x = 3)$ tell you about the first component of each ordered pair that satisfies $x + y = 5 \wedge \sim (x = 3)$? How many members of the set that you tabulated for exercise D satisfy $\sim (x = 3)$?

F Tabulate $\{(x, y) | x + y = 5 \wedge \sim (x = 3)\}$. Are the members of the set you just tabulated the answers to the problem in D7?

In this lesson you used both simple and compound conditions in one and two variables to solve problems.

On your own

For each of problems 1 through 4, first write a sentence that expresses the condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The universe is given for each problem.

1 After Mr. Butler had planted 48 tulip bulbs in his garden, he had fewer than 200 tulip bulbs left. How many tulip bulbs did he have before he planted the 48? U = C.

2 The area of Jamaica is 4411 sq. mi. The area of Puerto Rico is 976 sq. mi. less than the area of Jamaica. What is the area of Puerto Rico? U = Z.

3 Sally has less than $25.00. If she had $10.00 more, she would have more than $32.50. How much money does Sally have now? U = R_a.

4 Paul had $6\frac{3}{4}$ ft. of copper wire. He used some, but not all, of the wire and then had less than $1\frac{1}{2}$ ft. of wire left. How much wire had he used? U = Z.

THINK Why is $x < 6\frac{3}{4}$ one simple condition for the problem?

For each of problems 5 through 10, first write a sentence that expresses the condition. Then write a standard description of the universe for the variable or variables. Finally, tabulate the solution set of the condition and give the answer to the problem.

For problems 5, 6, and 7, the universe for each variable is C. For problems 8, 9, and 10, the universe for each variable is N.

5 Together, Sandy and Marcia made fewer than 6 place mats. How many place mats did each of the girls make?

6 Kay sold 15 tickets for the two performances of a school play. She sold at least 7 tickets for the first performance. How many tickets did she sell for the second performance?

7 The Peerless Letter Service had 35 typewriters. Then the company bought at most 5 typewriters. How many typewriters did they buy and how many did they then have in all?

THINK "At most 5 typewriters" means "5 or fewer than 5 typewriters."

8 Altogether, Hill High School ordered 427 graduation gowns. The school ordered more than 200, but fewer than 215, girls' gowns. How many boys' gowns and how many girls' gowns did the school order?

9 The sum of two numbers is 6886. The first number is 1238 more than 2829. What are the numbers?

10 When the first of two numbers is subtracted from the second, the difference is 8. The sum of the first number and 2 is not greater than 6. What are the numbers?

THINK How do you know that it is not the case that the sum of the first number and 2 is greater than 6?

Problems involving conditions for equivalence

In this lesson you will solve problems about rates and comparisons. To solve such problems, you will find solution sets of conditions for equivalence.

Read the problem in D1. The exercises that follow will help you develop a condition for equivalence for the problem. The universe for each variable is Z.

A You know that the helicopter can fly at a rate of 16 miles in 9 minutes. You can represent this rate by using an ordered pair of numbers called a rate pair. You can express the rate pair by the symbol at the right above. To what does the first component of 16/9 refer? To what does the second component refer?

16/9

B You do not know how many miles the helicopter can fly in 60 min. You must find another rate pair that represents the rate at which the helicopter flies and that is equivalent to 16/9. Use x as a variable for the number of miles the helicopter can fly in 60 min. Why is x a variable for the first component of the rate pair you are to find? Why is 60 the second component of the rate pair that you are to find?

$x/60$

The distance between two airports is 16 mi. A helicopter regularly flies the distance in 9 min. At this rate, how many miles can the helicopter fly in 60 min.?

D1

A $9x = 960$.
B $\frac{1}{9}(9x) = \frac{1}{9}(960)$.
C $(\frac{1}{9} \cdot 9)x = \frac{1}{9}(960)$.
D $1x = \frac{1}{9}(960)$.
E $x = 106\frac{2}{3}$.

D2

REMINDER

Equivalent ordered pairs are ordered pairs that are related as follows:
In a/b and c/d, the universe for each variable is Z. If $ad = cb$, then $a/b \sim c/d$.
Also, if $a/b \sim c/d$, then $ad = cb$.

See lesson 41, page 168, and lesson 92, page 402, Book 1.

C The sentence at the right expresses a condition for the problem. What does the condition tell you about $x/60$?

$16/9 \sim x/60.$

To find the number that satisfies the condition $16/9 \sim x/60$, you can use the definition of equivalent ordered pairs, which is given in the reminder above.

Now look at D2. Condition A was obtained from $16/9 \sim x/60$ by using the definition of equivalent ordered pairs.

D Does the replacement for x that satisfies $9x = 960$ also satisfy $16/9 \sim x/60$?

Now you will find the number that satisfies $9x = 960$. A definition and some properties that you will need for this work are given in the reminder on the next page.

REMINDER

Definition of reciprocal:
If the product of two rational numbers is 1, each number is the reciprocal of the other.
See lesson 84, page 363, Book 1.

Well-defined property of multiplication:
If one or both of the numbers used in forming the product of two rational numbers are replaced by equal rational numbers, the product is still the same rational number.
See lesson 74, page 326, Book 1.

Associative property of multiplication:
For each replacement of the variables, $(xy)z = x(yz)$. $U = R_a$.
See lesson 81, page 352, Book 1.

Identity-element property of multiplication:
For each replacement of the variable, $1 \cdot x = x$. $U = R_a$.
See lesson 83, page 362, Book 1.

E Look again at D2. First you will obtain a condition that involves $1x$ instead of $9x$. In condition B, how do you know that $9x$ has been multiplied by the reciprocal of 9?

You can obtain condition B by using the well-defined property of multiplication of rational numbers. You know, from condition A, that 960 is equal to $9x$. Therefore, in the product $\frac{1}{9}(9x)$, you can replace the $9x$ by 960 and obtain the product $\frac{1}{9}(960)$. The product $\frac{1}{9}(960)$ is the same as the product $\frac{1}{9}(9x)$.

F Compare sentences B and C. What property of multiplication tells you that, for each replacement of x, $(\frac{1}{9} \cdot 9)x = \frac{1}{9}(9x)$?

G Compare sentences C and D. For each replacement of x, is $1x = (\frac{1}{9} \cdot 9)x$?

H Compare sentences D and E. From the identity-element property, you know that, for each x, $1x = x$. What is the product of $\frac{1}{9}$ and 960? Do all the conditions expressed in D2 have the same solution?

I What number satisfies $x = 106\frac{2}{3}$? What number satisfies $9x = 960$? Does this number also satisfy $16/9 \sim x/60$?

J Tabulate $\{x \mid 16/9 \sim x/60\}$.

Now you can give the answer to the problem in D1.

The helicopter can fly $106\frac{2}{3}$ mi. in 60 min.

Read the problem in D3. The universe for each variable is Z.

K What are you asked to find in the problem?

L The enrollment has increased by how many students for each 100 students that were enrolled last year?

M Is the rate pair named at the right below the same rate pair as 31%?

$\frac{?}{31 / 100}$

What comparison is represented by 31/100?

To answer the question in the problem, you should first find a rate pair that is equivalent to 31/100.

N You know that the enrollment increased by 155 students. Is 155 the first component, or is it the second component of the rate pair you are to find? How do you know? For what is x a variable?

$\frac{155}{x}$

At Wrigley School, 155 more students are enrolled in summer classes this summer than were enrolled last summer. The enrollment has increased by 31%. What was the enrollment last summer?

D3

O What does the condition expressed at the right tell you about the rate pair whose second component you are to find?

$31/100 \sim 155/x.$

P Does $U = \{x | x \in Z\}$? Tabulate the solution set of $31/100 \sim 155/x$.

Q You can use the solution of $31/100 \sim 155/x$ to get the answer to the problem. What was last summer's enrollment?

The conditions for the first two problems in this lesson are simple conditions for equivalence. Next you will work with compound conditions that include conditions for equivalence.

Read the problem in D4. The universe for each variable in (x, y) is Z.

A You are asked to find the average length of the portages, that is, the length of one portage if you assume that all the portages were the same length. You need to find the first component of a rate pair that has a second component of 1. Use x as a variable for the average length. Does the rate pair expressed at the right represent the average length per portage?

$x/1$

B Do you know the total length of the three portages? Use y as a variable for the total length. To what does $y / 3$ refer?

$y/3$

C How do you know that the sentence at the right expresses one of the simple conditions for the problem?

$x/1 \sim y/3.$

D How do you know that the sentence at the right expresses another simple condition for the problem? Why is y, and not x, used in the condition?

$y = 120 + 174 + 212.$

At Wrigley School, 155 more students are enrolled in summer classes this summer than were enrolled last summer. The enrollment has increased by 31%. What was the enrollment last summer?

D3

On a canoe trip, Howard and his father made three portages. The lengths of the portages were 120 yd., 174 yd., and 212 yd. What was the average length of the portages?

D4

Linda took a history test. She had 5 times as many correct as incorrect answers. She had at most 20 correct answers. How many incorrect answers did she have?

D5

The sentence below in blue expresses a compound condition for the problem.

$$x/1 \sim y/3 \wedge y = 120 + 174 + 212.$$

E Why is the connective "and" used in the compound condition for the problem?

F Now you will find the solution set of the compound condition. What number satisfies $y = 120 + 174 + 212$?

G Replace y by 506 in $x/1 \sim y/3$. What number satisfies $x/1 \sim 506/3$?

H Tabulate the solution set of $x/1 \sim y/3 \wedge y = 120 + 174 + 212$.

I ? Which number, $168\frac{2}{3}$ or 506, do you use to get the answer to the problem? How do you know?

J Give the answer to the problem in D4.

Now read the problem in D5. The universe for each variable in (x, y) is C.

K You know that Linda had 5 times as many correct as incorrect answers. For each 5 correct answers, how many incorrect answers did she have? Does the rate pair 5/1 represent this comparison? 5/1

L You know one rate pair that represents the comparison. You want to find other rate pairs equivalent to 5/1. Use x as a variable for the number of correct answers Linda had. Why is x a variable for the first component of each rate pair that you are to find? x/y

M Notice that y is used as a variable for the second component. To what does y refer?

N How do you know that the sentence at the right expresses a simple condition for the problem? $5/1 \sim x/y.$

O How do you know that Linda had 20 or fewer correct answers? How do you know that the sentence at the right above expresses another simple condition for the problem? Why is x, and not y, used in this condition? $x \leqq 20.$

The sentence below in blue expresses a compound condition for the problem.

$$5/1 \sim x/y \wedge x \leqq 20.$$

Now you can find the solution set of the compound condition.

P Is the proportional relation tabulated in D6 the solution set of $5/1 \sim x/y$? Is the solution set of $5/1 \sim x/y$ a finite set, or is it an infinite set?

Q Name five rate pairs that are members of $\{5/1, \ldots\}$. Is each rate pair that you named a member of $\{(x, y) | 5/1 \sim x/y\}$?

$\{5/1, \ldots\}$

D6

Mr. Owen needs some new camera equipment. He can buy this equipment at either of two stores for \$126. In one store, he can pay $\frac{1}{12}$ of the amount of the bill each month. In the other store, he can pay $\frac{1}{3}$ of the amount each month. What can his monthly payment be?

D7

R Each solution of the compound condition must satisfy both $5/1 \sim x/y$ and $x \leqq 20$. Which members of $\{5/1, \ldots\}$ have first components that are less than or equal to 20?

S Tabulate $\{(x, y) | 5/1 \sim x/y \wedge x \leqq 20\}$.

T How do you know that you can use the second component of each solution of $5/1 \sim x/y \wedge x \leqq 20$ to obtain an answer to the problem?

U How many incorrect answers did Linda have?

In the following exercises, you will solve a problem by finding the solutions of a compound condition that includes the connective "or."

Read the problem in D7. The universe for each variable is Z.

You know that Mr. Owen's monthly payment can be either $\frac{1}{12}$ of the total amount of the bill or $\frac{1}{3}$ of the amount. You can use the solutions of a compound condition that includes "or" to find what his payment can be. Remember that the solutions of such a condition are solutions of one or both of the simple conditions.

Mr. Owen needs some new camera equipment. He can buy this equipment at either of two stores for \$126. In one store, he can pay $\frac{1}{12}$ of the amount of the bill each month. In the other store, he can pay $\frac{1}{3}$ of the amount each month. What can his monthly payment be?

D7

A In one case, Mr. Owen can pay $\frac{1}{12}$ of the amount of the bill each month. This means that he can pay $\frac{1}{12}$ of a dollar for each one dollar of the amount of the bill.
Does the rate pair $\frac{1}{12}/1$ represent this comparison? $\frac{1}{12}/1$

B You want to find a rate pair that is equivalent to $\frac{1}{12}/1$. Use x as a variable for the amount that Mr. Owen's monthly payment can be. You know that the amount of the bill is \$126.
Why is x the first component of the rate pair you are to find? $x/126$
What is the second component?

C How do you know that the sentence at the right above expresses one of the simple conditions for the problem? $\frac{1}{12}/1 \sim x/126.$

D In the other case, he can pay $\frac{1}{3}$ of the amount of the bill each month. What comparison is represented by the rate pair expressed at the right? $\frac{1}{3}/1$

E How do you know that the sentence at the right expresses another simple condition for the problem? $\frac{1}{3}/1 \sim x/126.$

F Does the sentence below in blue express a compound condition for the problem? Explain why "or" is used in the condition.

$$\frac{1}{12}/1 \sim x/126 \vee \frac{1}{3}/1 \sim x/126.$$

Now you can find the solution set of the compound condition.

G Is the solution set of the compound condition the intersection, or is it the union, of the solution sets of the simple conditions?

H What member of Z satisfies $\frac{1}{12}/1 \sim x/126$? What member of Z satisfies $\frac{1}{3}/1 \sim x/126$?

I Tabulate $\{x|\frac{1}{12}/1 \sim x/126 \vee \frac{1}{3}/1 \sim x/126\}$.

J How much can Mr. Owen's monthly payment be?

In this lesson you solved problems about rates and comparisons, and you used compound conditions that involve simple conditions for equivalence and the connective "or."

On your own

For each problem, first write a sentence that expresses a condition. Then write a standard description of the universe for the variable or variables. Finally, tabulate the solution set of the condition and give the answer to the problem. The universe for each variable is Z.

1 Mr. and Mrs. Frank received a discount of \$94.75 on an electric stove. The discount was 25% of the regular price. What was the regular price of the stove?

2 Some authorities on nutrition say that the minimum bread requirement for an average adult is 3 slices per day. A loaf of a certain kind of bread contains 26 slices. The number of slices in such a loaf is how many times the minimum daily requirement for an adult?

3 Mr. Winthrop receives a 4% commission on each piano that he sells. On Friday he sold a piano priced at \$750. How much commission did he receive?

4 Last year, John read 18 books of science fiction, and Allen read 24. The number of science fiction books that John read was what fraction of the number of these books that Allen read?

5 The golf balls that Dave wants to buy cost \$.75 each. He has \$5.00 to spend for golf balls. How many golf balls can he buy?

THINK Can Dave buy $6\frac{2}{3}$ golf balls?

6 Two trains travel the 175 miles between Elmwood and Greenvale daily. The night train travels at an average rate of 60 miles per hour. The day train travels at an average rate of 75 miles per hour. How long can it take to go from Elmwood to Greenvale by train?

7 Last year, Doug's automobile insurance cost \$150. This year, because he has reached the age of twenty-five, his insurance costs only \$120. What was the per cent of decrease in the cost of his insurance?

8 The tennis racket that Karen wants to buy is on sale at two different stores. The regular price of the racket is \$12. One store offers a 25% discount on the racket, and the other store offers a 15% discount. How many dollars' discount is it possible for Karen to receive if she buys the tennis racket on sale?

9 After deductions for taxes and insurance are made from Mr. Rush's monthly salary, his monthly take-home pay is \$523.45. What is Mr. Rush's yearly take-home pay?

10 Tom gave some of his stamps away. He gave Jim $\frac{1}{3}$ of the stamps that he gave away. He gave Jim at most 10 stamps. How many stamps did Tom give away, and how many stamps did he give to Jim?

11 Mrs. Campbell's kitchen curtains shrank 3% in length when she washed them. The curtains shrank $3\frac{1}{4}$ in. How long were the curtains after she washed them?

12 A certain rate pair is equivalent to 3/5. The difference of the second component and the first component is 12. What is the rate pair?

13 One number is 20% of a second number. The first number is less than or equal to 12. What are the two numbers?

KEEPING SKILFUL

For each exercise below, find the basic fraction that indicates the member of R_a named.

1 .25	**5** .0015	**9** 60.4	**13** .830
2 1.68	**6** .001	**10** .0017	**14** .075
3 10.3	**7** 25.5	**11** .055	**15** .0125
4 .0096	**8** .875	**12** .0625	**16** 100.2

For each exercise below, use a decimal to express the member of R_a named in the exercise.

17 $\frac{13}{20}$	**21** $\frac{300}{400}$	**25** $\frac{9}{25}$	**29** $\frac{33}{50}$
18 $\frac{18}{45}$	**22** $\frac{3}{8}$	**26** $\frac{105}{125}$	**30** $\frac{9}{60}$
19 $\frac{26}{5}$	**23** $\frac{29}{4}$	**27** $\frac{13}{1000}$	**31** $\frac{39}{1000}$
20 $\frac{54}{4}$	**24** $\frac{9}{10}$	**28** $\frac{21}{2}$	**32** $\frac{1}{500}$

For each of exercises 33 through 54, use a decimal to express the sum, difference, product, or quotient.

33 $10.2 + 83.4$	**41** $16.305 + 3.2 + .89$
34 $8.684 - 3.217$	**42** $.1889 - .16$
35 $.25 \times .13$	**43** $10 + .75 + 8.2$
36 $.8 + .9 + .1$	**44** $.8 \times .9 \times 10$
37 $.62 + .03 + .4$	**45** $21 - 18.05$
38 $.742 - .105$	**46** $.1 + .01 + .001$
39 $.185 \times 9.4$	**47** $32 - 13.4289$
40 $1.675 \times .32$	**48** $.69 \times .132 \times 1.5$

49 $16.67 \div 6$ (Thousandths)

50 $46 \div .3$ (Tenths)

51 $.0843 \div .9$ (Ten-thousandths)

52 $100.457 \div 24$ (Thousandths)

53 $45 \div .62$ (Hundredths)

54 $600.4 \div .9$ (Tenths)

For each exercise below, find the sum, difference, product, or quotient.

55 $25 - 13\frac{1}{4}$	**62** $32\frac{1}{2} \div 6\frac{1}{2}$
56 $38\frac{1}{4} - 27\frac{1}{8}$	**63** $4\frac{2}{15} - 1\frac{2}{3}$
57 $21\frac{1}{4} \times 2\frac{2}{5}$	**64** $12\frac{3}{8} + 1\frac{1}{8} + 3\frac{1}{2}$
58 $6\frac{13}{30} \div 3$	**65** $6\frac{1}{12} \times \frac{3}{4}$
59 $73\frac{3}{13} - 55\frac{1}{2}$	**66** $25\frac{1}{2} \div 1\frac{3}{8}$
60 $\frac{16}{17} \times \frac{1}{32}$	**67** $40 + 16\frac{14}{25}$
61 $12\frac{2}{3} + 4\frac{1}{5} + 6\frac{5}{6}$	**68** $16\frac{1}{3} \div 3\frac{1}{4}$

Problems involving conditions in three variables

You have learned how to find solution sets of compound conditions in three variables. In this lesson you will learn how to solve problems by using compound conditions in three variables.

Read the problem in D1. The exercises that follow will help you develop a compound condition for the problem. The universe for each variable in (x, y, z) is Z.

A Do you know the weight of any of the three fish? Use x as a variable for the number of pounds the pike weighed; use y as a variable for the number of pounds the bass weighed; and use z as a variable for the number of pounds the catfish weighed.

$x \quad y \quad z$

B What was the total weight of the three fish? How do you know that the sentence at the right above expresses one of the simple conditions for the problem?

$x + y + z = 22\frac{1}{2}.$

C How do you know that the pike weighed $1\frac{1}{2}$ lb. for each pound that the bass weighed? Is this comparison represented by the rate pair named at the right above?

$1\frac{1}{2}/1$

D To what does each component of x/y refer?

x/y

E How do you know that the sentence at the right expresses another simple condition for the problem?

$1\frac{1}{2}/1 \sim x/y.$

Mr. James caught a pike, a bass, and a catfish. The total weight of these three fish was $22\frac{1}{2}$ lb. The pike weighed $1\frac{1}{2}$ times as much as the bass. The pike also weighed 15 lb. less than the total weight of the three fish. What was the weight of each of the fish?

D1

F The pike weighed how much less than the total weight of the three fish? How do you know that the sentence at the right expresses another simple condition for the problem?

$22\frac{1}{2} - x = 15.$

G Does the sentence below in blue express a compound condition for the problem?

$(x + y + z = 22\frac{1}{2}) \wedge (1\frac{1}{2}/1 \sim x/y) \wedge (22\frac{1}{2} - x = 15).$

Now you can find the solution set of the compound condition. Remember that each member of the solution set must satisfy all three of the simple conditions.

H Why is it a good idea to find the solution set of $22\frac{1}{2} - x = 15$ first?

I What member of Z satisfies $22\frac{1}{2} - x = 15$?

J What do you now know about the first component of each solution of the compound condition? Explain your answer.

K What condition do you obtain when you replace x in $1\frac{1}{2}/1 \sim x/y$ by $7\frac{1}{2}$? What replacement for y satisfies the condition that you obtain when you replace x?

L What is the second component of each member of the solution set of the compound condition?

M When you replace x by $7\frac{1}{2}$ and y by 5 in $x + y + z = 22\frac{1}{2}$, what replacement for z satisfies the condition you obtain?

N What is the third component of the member of the solution set of the compound condition?

O Tabulate the solution set of the compound condition for the problem in D1.

You can use the solution of the compound condition to get the answer to the problem.

The pike weighed $7\frac{1}{2}$ lb.; the bass weighed 5 lb.; and the catfish weighed 10 lb.

P You can verify the answer to the problem. If the pike weighed $7\frac{1}{2}$ lb., the bass weighed 5 lb., and the catfish weighed 10 lb., was the total weight of the three fish $22\frac{1}{2}$ lb.? Did the pike weigh 15 lb. less than all three fish weighed? Did the pike weigh $1\frac{1}{2}$ times as much as the bass?

Read the problem in D2. First you will develop a compound condition for the problem. The universe for each variable in (x, y, z) is Z.

A Do you know how much the rent was for the regular bicycle for 3 days? Use x as a variable for the number of dollars in this amount.

B Do you know how much the rent was for the tandem bicycle for 3 days? Use y as a variable for the number of dollars in this amount.

C To what does $x + y$ refer?

D Do you know the number of dollars it cost to rent both bicycles for 3 days? Use z as a variable for this number.

E Why does the sentence below at the right express one of the simple conditions for the problem? $x + y = z.$

F The cost of renting the tandem bicycle was how many times the cost of renting the regular bicycle? How does the rate pair 2/1 represent this comparison?

G Why do you use y/x, and not x/y, to represent rate pairs equivalent to 2/1?

H Why does the sentence below at the right express another simple condition for the problem? $2/1 \sim y/x.$

A group of girls rented a tandem bicycle and a regular bicycle for 3 days. The rent for the tandem bicycle was 2 times the rent for the regular bicycle. The rent for the regular bicycle was \$1.50 per day. How much rent did the girls pay for both bicycles for 3 days?

D2

I What was the rent for the regular bicycle per day? How does the rate pair 1.50/1 represent this rate?

J How do you know that the sentence at the right expresses another simple condition for the problem? What does the first component of $x/3$ refer to? $1.50/1 \sim x/3.$

K Does the sentence below in blue express a compound condition for the problem?

$$(x + y = z) \wedge (2/1 \sim y/x) \wedge (1.50/1 \sim x/3).$$

Now you can find the solution set of the compound condition.

L Which of the simple conditions will you use first to help you find the solution set of the compound condition? Explain why.

M What member of Z satisfies $1.50/1 \sim x/3$? What is the first component of each solution of the compound condition?

N When you replace x in $2/1 \sim y / x$ by 4.50, what replacement for y satisfies the condition you obtain?

O What are the first and second components of each solution of the compound condition?

P When you replace x by 4.50 and y by 9 in $x + y = z$, what replacement for z satisfies the condition you obtain?

Q Tabulate the solution set of the compound condition.

The sum of three numbers is less than 5. The first number is less than 3. The difference of the first number and the second number is 2. What are the three numbers?

D3

A $0 - s = 2$.
B $1 - s = 2$.
C $2 - s = 2$.

D4

R How much was the rent for both bicycles for 3 days? Which component of the solution did you use to get the answer to the problem?

Read the problem in D3. The universe for each variable in (r, s, t) is N.

A Use r as a variable for the first number. Use s as a variable for the second number. Use t as a variable for the third number. How do you know that $r + s + t < 5$ is one of the simple conditions for the problem?

B How do you know that $r < 3$ is another simple condition for the problem?

C What is a third simple condition for the problem? How do you know?

D Is $(r + s + t < 5) \wedge (r < 3) \wedge (r - s = 2)$ a compound condition for the problem in D3?

E What members of N satisfy $r < 3$?

F If you use the solutions of $r < 3$ as replacements for r in $r - s = 2$, do you obtain the conditions expressed in D4?

G Does any member of N satisfy $0 - s = 2$? Does any member of N satisfy $1 - s = 2$? What member of N satisfies $2 - s = 2$?

H What is the first component of each member of the solution set of the compound condition? What is the second component?

I What condition do you obtain when you replace r by 2 and s by 0 in $r + s + t < 5$? What replacements for t satisfy the condition that you obtained?

J How many ordered triples satisfy all three simple conditions? Tabulate the solution set of the compound condition.

K Is each member of the solution set that you tabulated for exercise J an answer to the problem in D3?

Now you know how to solve problems that involve compound conditions in three variables.

On your own

For each problem, first write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. The universe is given for each problem.

1 Rita bought 5 packages of bias tape priced at \$.12 per package, and 6 spools of thread priced at 2 spools for \$.15. How much did she spend in all for the tape and thread? The universe for each variable is Z.

2 Bud has three kinds of coins. He has fewer nickels than dimes. If he had 3 more pennies, he would have 15 pennies. He has 5 fewer dimes than pennies. How many coins of each kind does he have? The universe for each variable is C.

3 The sum of Marjorie's age and Sue's age is the same as Donna's age. Marjorie is 2 years older than Sue. In 3 years, Marjorie will be 18 years old. How old is each of the girls now? The universe for each variable is C.

4 Altogether, Mrs. Olsen bought fewer than 10 cans of beets, peas, and carrots. She bought fewer than 3 cans of beets. She bought 7 cans in all of beets and peas. How many cans of each kind of vegetable did she buy? The universe for each variable is C.

5 Mr. Angeletti borrowed \$600 from a credit union and agreed to pay 1% interest per month on the unpaid balance. At the end of the first month, he paid the credit union $\frac{1}{12}$ of the amount he had borrowed, with interest. How much in all did Mr. Angeletti pay the credit union at the end of the first month? The universe for each variable is Z.

6 During summer vacation, Judy lost twice as much weight as Harriet. Carol lost twice as much weight as Judy. Together, Harriet and Judy lost 12 lb. How many pounds did each of the girls lose? The universe for each variable is C.

7 The first year that Mr. Harper worked as a salesman, he received a commission of \$700 on his total sales. The second year his commission increased 17%. What was the total amount of commission that Mr. Harper received for the two years? The universe for each variable is Z.

8 A first number is greater than 8. When this first number and a second number are added, the sum is equal to 10. When a third number is subtracted from 7, the difference you obtain is equal to the second number. What are the three numbers? The universe for each variable is N.

9 A first number is less than 4. A second number is the difference of 4 and the first number. When the second number and the third number are added, the sum is less than 3. What are the three numbers? The universe for each variable is N.

10 The sum of two numbers is 6. The first number is equal to the product of 2 and the second number. When the second number is subtracted from the first number, the difference you obtain is equal to a third number. What are the three numbers? The universe for each variable is N.

APPLYING MATHEMATICS

For each problem, write a sentence that expresses a condition. If possible, tabulate the solution set of the condition; otherwise, give a standard description of the solution set. Then give the answer to the problem.

For problems 1 through 4, the universe for each variable is N.

1 Marge has 12 albums of records. She loaned some, but not all, of these albums to Jane. How many albums did Marge loan to Jane and how many did Marge have left?

2 Tom had more than 300 stamps. After he had sold 245 of them, he had fewer than 75 stamps left. How many stamps did Tom have before he sold the 245 stamps?

3 Mr. Miller sells new and used cars. Last month he sold 14 cars. He sold at most 5 new cars. How many new cars and how many used cars did he sell?

4 The sum of three numbers is equal to 15. The first number is less than 5. The second number is the successor of the first number. What are the three numbers?

THINK Why is $y = x + 1$ one of the simple conditions for the problem?

For problems 5 through 8, the universe for each variable is C.

5 Twenty-two seventh- and eighth-grade students belong to the Ames School Camera Club. At least 15 of the 22 members are seventh-grade students. How many seventh-grade students and how many eighth-grade students are members of the Camera Club?

6 Mr. Hanley operates a sporting goods store. With every tennis racket that he sells, he gives the customer 4 tennis balls. On Saturday he sold at most 8 tennis rackets. How many tennis rackets did he sell, and how many tennis balls did he give away?

7 The attendance at a play on 3 consecutive nights was 200, 264, and 286. The attendance on the third night was how much greater than the average attendance per night?

8 A certain rate pair is equivalent to 14/15. The second component of the rate pair is 5 more than the first component. What is the rate pair?

For problems 9 through 18, the universe for each variable is Z.

9 Jim can do $\frac{2}{5}$ as many consecutive push-ups as his older brother can. His brother can do 20. How many consecutive push-ups can Jim do?

10 Mr. Bond's monthly salary is $420. His take-home pay amounts to $82.80 out of each $100 of his salary. What is his monthly take-home pay?

11 The distance from Toronto to Calgary is about 1680 mi. This is about 2.5 times the distance from Toronto to Saint John, N.B. About what is the distance from Toronto to Saint John?

12 During a sale, a photographer sold wallet-size photographs for $2.70 per dozen. At this rate, what was the price of 18 photographs?

13 Last year the Sanford family spent 25.5% of their income for food. They spent $1530 for food. What was their income last year?

14 The radio that Ed wants to buy costs $36. He plans to save either $1.50 every week or $1.80 every week until he has saved the $36. In how many weeks will Ed save enough money to buy the radio?

15 A department store had 1500 hats to sell. During July, $66\frac{2}{3}\%$ of these hats were sold. During August, 80% of the remaining hats were sold. How many hats were sold during each of these months?

16 90.9 is what per cent of 121.2?

17 What number is $12\frac{1}{2}\%$ of 84?

18 $318\frac{2}{5}$ is $66\frac{2}{3}\%$ of what number?

CHECKING UP

If you have trouble with this test, you can find help in lessons 116 through 118.

Test 118

For each problem, write a sentence that expresses a condition. Tabulate the solution set of the condition, if possible; otherwise, give a standard description of the solution set. Then give the answer to the problem.

For problems 1 and 2, the universe for each variable is C. For the other problems, the universe for each variable is Z.

1 There were spaces for 50 autographs in Sam's yearbook. After Autograph Day, Sam had at most 12 unused spaces in his book. How many autographs did he collect in his yearbook?

2 There are 28 fire extinguishers on the first and second floors of the Adams High School. The first floor has at most 12 fire extinguishers. How many fire extinguishers are on each of the two floors?

3 Jean timed how long it took her to wash dishes each day for one week. Her times in minutes were: 17, 9, 12, 14, 10, 18, and 26. What was the average time she spent washing dishes per day that week?

4 Bill had $64.00 in his savings account. After interest had been added to the account on July 1, he had more than $64.50 but less than $65.50 in his account. How much interest was added to Bill's account on July 1?

5 In three games, a baseball pitcher allowed 2 runs in 9 innings; 5 runs in 12 innings; and 6 runs in 4 innings. At this rate, he allowed how many runs for each 9 innings he pitched? (Hundredths)

6 What number is 49% of 23? (Hundredths)

7 35% of what number is $53\frac{1}{2}$?

8 24 is what fraction of 38?

Intersections of lines and circles

In *Book 1* you worked not only with sets of numbers, but also with sets of points. You learned that geometry is a branch of mathematics that deals with points, lines, planes, and space. In this lesson you will review some of the geometric ideas that were presented in *Book 1*. You will also study the intersections of lines and circles.

A Look at D1. Is point E represented by a dot? Is F represented by a dot? Tabulate the set of points that are represented by dots in D1.

B Point G is in the plane represented in D1. Which of the other points represented by dots are in this plane?

C Is a plane a finite set of points? Is it an infinite set of points?

D Look again at D1. Line ℓ_1 contains points E, F, and G. How do you know that ℓ_1 is included in the plane represented in D1?

REMINDER

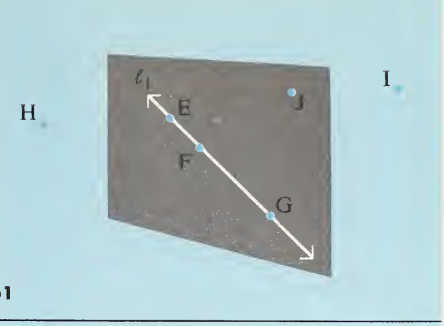
If a line contains two points of a plane, the line is included in the plane.

See lesson 9, page 34, Book 1.

E How do you know that E, F, and G are collinear points?

F Is a line a finite set of points? Is it an infinite set of points?

G Look at D1. Point I is not in the plane represented in D1. What other point named in D1 is not in this plane? How do you know that these two points are members of space?



D1

Universe for X is plane STN.

A $\overline{SX} \cong \overline{ST}$. **B** $\overline{SX} < \overline{ST}$. **C** $\overline{SX} > \overline{ST}$.

D2

You have learned that the universe for the variable in a condition may be a set of points. Remember that the solution set of a condition that involves such a variable is also a set of points.

H Look at D2. All of the points represented in the display are in the same plane. Explain why you can use "plane STN" to name this plane. Explain why you can also use "plane VFR" to name this plane. Explain why you cannot use "plane VST" to name this plane.

Condition A expressed in D2 requires that each point that you are to find, together with point S, must determine a segment congruent to segment ST. The variable in the condition is X.

REMINDER

A segment is a set of points whose members are two given points and all points between these two. Segments that are the same size are congruent segments.

See lesson 10, pages 38 and 39, Book 1.

I What is the universe for X in condition A?

J $\overline{SF} \cong \overline{ST}$. How do you know that point F is a solution of $\overline{SX} \cong \overline{ST}$? How do you know that T is a solution of $\overline{SX} \cong \overline{ST}$?

K You know that T and F are solutions of $\overline{SX} \cong \overline{ST}$. Which of the other points named in D2 appear to be elements of $\{X | \overline{SX} \cong \overline{ST}\}$?

L What geometric figure is $\{X | \overline{SX} \cong \overline{ST}\}$? Is this set finite, or is it infinite?

REMINDER

The set of all points in a plane that satisfy $\overline{RX} \cong \overline{RT}$ is a circle. The centre of this circle is point R.

See lesson 12, page 46, Book 1.

M Name the centre of the circle represented in D2.

You may use a special symbol to name a circle. D3 shows how to write and read a symbol that names circle S, represented in D2.

N Look again at D2. Is S $\in$ $\odot$S? Is V $\in$ $\odot$S? Explain your answers.

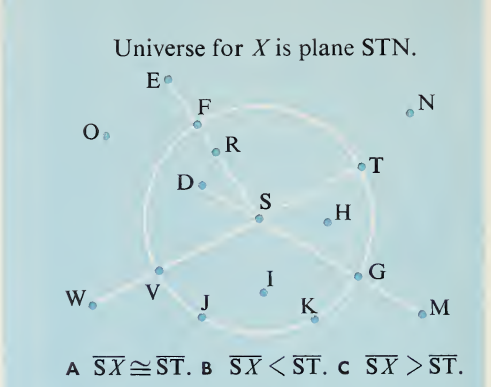
O How do you know that $\overline{ST}$ is a radius of $\odot$S? Name the other radii of $\odot$S that are represented in D2.

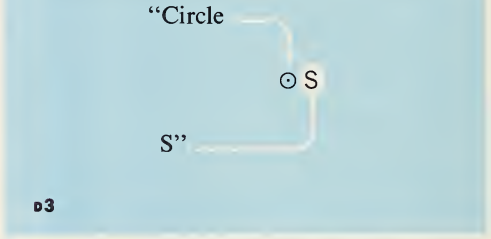
P Condition B requires that each X, together with the centre of the circle, must determine a segment that is less than a radius. What segment is determined by R and the centre of the circle? How do you know that R is a solution of $\overline{SX} < \overline{ST}$?

Q The set that consists of the centre of $\odot$S and all of the points in the plane that satisfy $\overline{SX} < \overline{ST}$ is the interior of $\odot$S. Which points named in D2 are members of $\{X | \overline{SX} < \overline{ST}\}$? Is the interior of a circle a finite set of points? An infinite set of points?

R What does condition C require? How do you know that M is a solution of $\overline{SX} > \overline{ST}$?

S The solution set of $\overline{SX} > \overline{ST}$ is the exterior of $\odot$S. Which points named in D2 are members of $\{X | \overline{SX} > \overline{ST}\}$? Is the exterior of a circle a finite set of points? An infinite set of points?





In the following exercises, you will study the intersections of sets of points. Remember that the intersection of two sets is the set that consists of only those elements that belong to both sets.

A Look at D4. Tabulate the set of points in ℓ_2 that are represented by dots. Tabulate the set of points in $\odot K$ that are represented by dots.

B How do you know that ℓ_2 and $\odot K$ intersect? Tabulate $\ell_2 \cap \odot K$.

The intersection of ℓ_2 and $\odot K$ contains exactly two points. ℓ_2 is a *secant* of $\odot K$. We will refer to ℓ_2 as "secant ℓ_2." A line is a secant of a circle if the intersection of the line and the circle contains exactly two points.

C Look again at D4. How do you know that $\overline{CD}$ is a chord of $\odot K$? Name another chord of $\odot K$.

D How many lines contain both C and D? What line is determined by C and D? Is ℓ_2 the same set of points as secant ℓ_2? How do you know that chord CD is a subset of secant ℓ_2?

E What other secant of $\odot K$ is represented in D4? Tabulate the intersection of this secant and $\odot K$. Name the chord of $\odot K$ that is a subset of this secant.

F Chord EF contains point K. How do you know that $\overline{EF}$ is a diameter of $\odot K$? How do you know that $\overline{EF}$ is a subset of ℓ_3?

G How many diameters of $\odot K$ contain E? Contain C? Explain your answers.

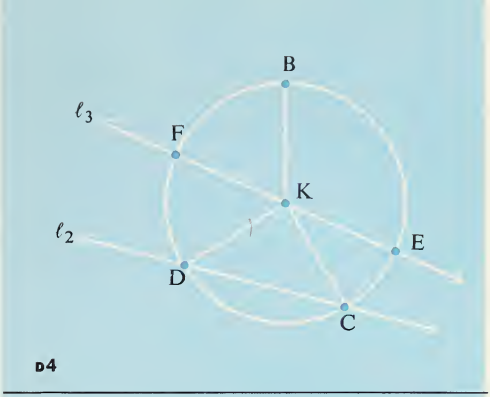
H Does every secant of a given circle include a chord of that circle? Does it include a diameter of that circle? Explain your answers.

I Look at D5. The geometric figures represented in D5 are subsets of the same plane. That is, they are coplanar. Tabulate the set of points in ℓ_4 that are represented by dots. Tabulate the set of points in $\odot A$ that are represented by dots.

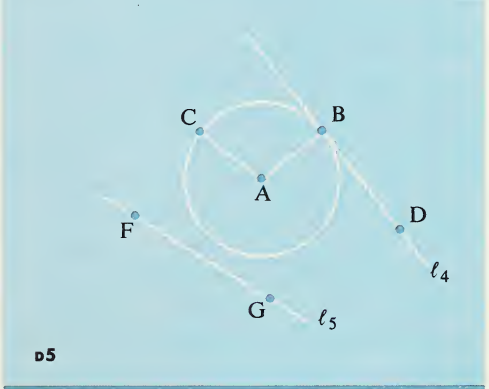
J How do you know that ℓ_4 and $\odot A$ intersect? Tabulate $\ell_4 \cap \odot A$.

The intersection of ℓ_4 and $\odot A$ is {B}. ℓ_4 is *tangent* to $\odot A$. Point B is the *point of tangency*. We say that ℓ_4 is tangent to $\odot A$ and $\odot A$ is tangent to ℓ_4 at the point of tangency. A line is a tangent to a circle if the intersection of the line and the circle contains exactly one point. The line and the circle must be coplanar.

K Look again at D5. Explain why ℓ_5 is neither a tangent to $\odot A$ nor a secant of $\odot A$. Tabulate $\ell_5 \cap \odot A$.



D4



D5

se cant of a circle (sē′kənt or sē′kant). A line whose intersection with the circle contains exactly two points.

tan gent to a circle (tan′jənt). A line included in the plane of the circle so that its intersection with the circle contains exactly one point. The point contained in the intersection is the **point of tangency** (tan′jən sē). The line is tangent to the circle and the circle is tangent to the line at the point of tangency.

In this lesson you have reviewed certain ideas that concern sets of points and geometric conditions. You have also studied chords, diameters, secants, and tangents of circles.

On your own

Use D6 as you answer the questions in exercises 1 through 9.

1 Name the centre of ⊙N.

2 Name a diameter of ⊙N.

3 Name four radii of ⊙N.

4 Name six chords of ⊙N.

5 Name three points in the interior of ⊙N.

6 Name five points in the exterior of ⊙N.

7 Name a secant of ⊙N.

8 Name a line tangent to ⊙N.

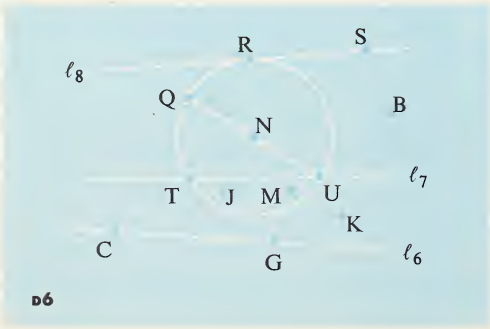
9 Tabulate the intersection of ℓ_6 and the interior of ⊙N. Tabulate the intersection of ℓ_8 and the interior of ⊙N. Describe the intersection of ℓ_7 and the interior of ⊙N.

10 Make a sketch of a circle and its centre. Name the circle by using the letter W. Use your sketch for exercises 11, 12, and 13.

11 Make a sketch of radius WZ of ⊙W.

12 Make a sketch of chord ZH that contains the centre of ⊙W. Then make a sketch of ℓ_9 that is tangent to ⊙W at point H.

13 Make a sketch of ℓ_{10} so that ℓ_{10} is a secant of ⊙W. Make a sketch of ℓ_{11} so that the intersection of ℓ_{11} and ⊙W is the empty set.



D6

Central angles and arcs

In the last lesson, you studied the intersections of lines and circles. In this lesson, you will study certain subsets of circles.

A Look at the picture of ⊙A in D1. The ray that contains point B and whose endpoint is A is named by the symbol $\overrightarrow{AB}$. Name the ray that contains C and whose endpoint is A. Name the common endpoint of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

B The angle that is the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is named by the symbol ∠BAC. Name the vertex of ∠BAC. Name the sides of ∠BAC.

> **REMINDER**
> An angle is the union of two rays that have a common endpoint and do not form a line. The common endpoint is the vertex of the angle. Each of the rays is a side of the angle.
> *See lesson 27, page 105, Book 1.*

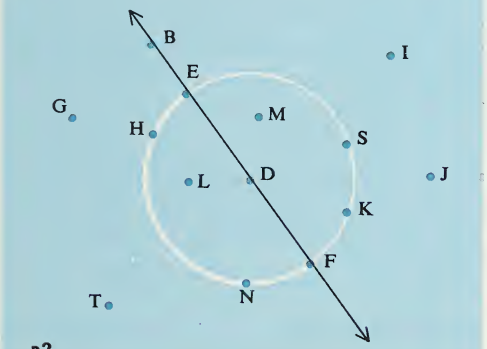
Notice that the vertex of ∠BAC is the centre of ⊙A. ∠BAC is a *central angle* of ⊙A. A central angle of a given circle is an angle whose vertex is the centre of the circle and whose sides intersect the circle.

C Look again at D1. Name the points of ∠BAC that are also in ⊙A. How many central angles of ⊙A contain both of these points?

Points B and C *determine* central angle BAC of ⊙A.

D Look at the picture of ⊙D in D2. Chord EF contains the centre of ⊙D. How do you

D1



D2

central angle of a circle (sen′trəl). An angle whose vertex is the centre of the circle and whose sides intersect the circle.

minor arc of a circle (mī′nər ärk). The subset of a circle that is included in the union of a central angle of the circle and the interior of the central angle. The points in this subset that are also contained in the central angle are the **endpoints** of the minor arc.

major arc of a circle (mā′jər). The subset of a circle that is included in the union of a central angle of the circle and the exterior of the central angle. The points in this subset that are also contained in the central angle are the **endpoints** of the major arc.

know that $\overline{EF}$ is a diameter of $\odot D$? How do you know that E and F do not determine a central angle of $\odot D$?

E Do any two points in a circle determine a central angle of the circle? Explain your answer.

F Which points represented in D1 are in $\angle BAC$? In the interior of $\angle BAC$? In the union of $\angle BAC$ and the interior of $\angle BAC$?

G Now think of the subset of $\odot A$ that is included in the union of $\angle BAC$ and the interior of $\angle BAC$. Which points represented by dots are in this subset?

The subset of $\odot A$ that is included in the union of $\angle BAC$ and the interior of $\angle BAC$ is a *minor arc* of $\odot A$. Points B and C are the *endpoints* of this minor arc.

H Look again at D1. Which points represented by dots are in the exterior of $\angle BAC$? Which points are in the union of $\angle BAC$ and the exterior of $\angle BAC$?

I Now think of the subset of $\odot A$ that is included in the union of $\angle BAC$ and the exterior of $\angle BAC$. Which points represented by dots are in this subset?

The subset of $\odot A$ that is included in the union of $\angle BAC$ and the exterior of $\angle BAC$ is a *major arc* of $\odot A$. Points B and C are the *endpoints* of this major arc.

J Look again at D2. The line determined by points E and F is named by the symbol $\overleftrightarrow{EF}$. Into how many half-planes does $\overleftrightarrow{EF}$ separate the plane that includes $\odot D$? Name the boundary of each of these half-planes.

K Think of the half-planes that are bounded by $\overleftrightarrow{EF}$ and that are subsets of the plane that includes $\odot D$. Which points represented by dots in D2 are in the same half-plane as L? Which points are in $\overleftrightarrow{EF}$? Which points are in the union of $\overleftrightarrow{EF}$ and the half-plane that contains L?

D2

D3

sem i cir cle (sem′ē sėr′kəl). If a line includes a diameter of a circle, then the subset of the circle included in the union of the line and a half-plane bounded by that line is a semicircle. The endpoints of the diameter are the **endpoints** of the semicircle.

L Now think of the subset of ⊙D that is included in the union of $\overleftrightarrow{EF}$ and the half-plane that contains L. Which points represented by dots are in this subset?

M Name the diameter of ⊙D that is included in $\overleftrightarrow{EF}$.

$\overleftrightarrow{EF}$ includes a diameter of ⊙D. The subset of ⊙D that is included in the union of $\overleftrightarrow{EF}$ and a half-plane bounded by $\overleftrightarrow{EF}$ is a *semicircle* of ⊙D. Points E and F are the *endpoints* of the semicircle.

N Look again at D2. How many semicircles of ⊙D have E and F as endpoints? Point K is in one of these semicircles. This semicircle contains what other points represented by dots? Name the endpoints of this semicircle.

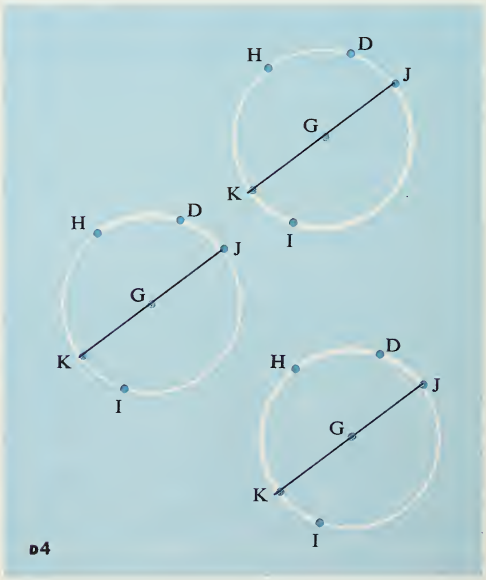
A subset of a circle that is a minor arc, a major arc, or a semicircle is an *arc of the circle*. D3 shows how to write and read a name of one of the arcs of ⊙D represented in D2. The name of a point in the arc that is not an endpoint of the arc is written between the names of the two endpoints. Notice the symbol for arc above the letters.

O Look at the three pictures of ⊙G in D4. Segment KJ is a diameter of ⊙G. Is $\overset{\frown}{KIJ}$ a semicircle? How do you know? How do you know that $\overset{\frown}{HDJ}$ is a minor arc? How do you know that $\overset{\frown}{IHJ}$ is a major arc?

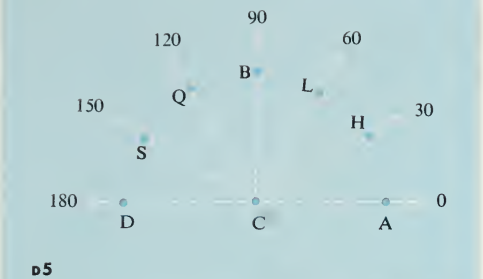
P Is $D \in \overset{\frown}{IHJ}$? Use the letter D in another name for $\overset{\frown}{IHJ}$.

Q Any two points in a circle are the endpoints of how many arcs of the circle?

R Suppose that $\overset{\frown}{BCD}$ and $\overset{\frown}{BLD}$ are two different arcs of a circle. If $\overline{BD}$ is a diameter of the



D4



D5

D6

"The measure in degrees of minor $\overset{\frown}{PR}$ ° = 70. minor arc PR is equal to 70."

D7

circle, what do you know about $\overset{\frown}{BCD}$ and $\overset{\frown}{BLD}$?

S Suppose that $\overset{\frown}{MNK}$ and $\overset{\frown}{MFK}$ are two different arcs of a circle, and that $\overset{\frown}{MNK}$ is a minor arc. What kind of arc is $\overset{\frown}{MFK}$?

Another way to name a major or a minor arc is to use the names of the endpoints and the symbol for arc, and then to indicate whether the arc is major or minor. For example, the minor arc whose endpoints are R and S can be named "minor $\overset{\frown}{RS}$." The major arc whose endpoints are R and S can be named "major $\overset{\frown}{RS}$." The semicircle whose endpoints are R and T and that contains V can be named "semicircle RVT."

In lesson 101 you learned how to find the measure of an angle in degrees. You can use what you know about angle measure to find the measures of arcs in degrees.

A Look at D5. What number is associated with $\overrightarrow{CL}$? What number is associated with $\overrightarrow{CQ}$? How can you use these two numbers to find the measure of ∠LCQ?

B The standard unit of angle measure called the degree was used to determine the numbers associated with the rays represented in D5. What is the measure in degrees of ∠LCB? What is ∠QCH°? Remember that the symbol ∠QCH° is read "the measure in degrees of ∠QCH." How do you know that ∠QCH is a right angle?

C Look at D6. Name the arcs of ⊙K whose endpoints are P and R. Name the central angle of ⊙K determined by P and R.

D In our work, all angles will be measured in degrees. What is ∠PKR°?

∠PKR° = 70. The measure in degrees of minor $\overset{\frown}{PR}$ is 70. The *measure in degrees of a minor arc of a circle* is the same as the measure in degrees of the central angle that is determined by the endpoints of the arc.

Look at D7. The display shows how to write and read a sentence that expresses a true statement about minor $\overset{\frown}{PR}$.

measure in degrees of a minor arc of a circle. The measure in degrees of the central angle that is determined by the endpoints of the arc.

E Look again at D6. What is $\overset{\frown}{LMR}°$?

F How many points are in the intersection of minor $\overset{\frown}{LR}$ and minor $\overset{\frown}{PR}$? Tabulate the intersection of minor $\overset{\frown}{LR}$ and minor $\overset{\frown}{PR}$.

Two minor arcs of the same circle that intersect in exactly one point are *adjacent arcs.*

G Look again at D6. How do you know that $\overset{\frown}{SLM}$ and $\overset{\frown}{MRT}$ are adjacent arcs? Are $\overset{\frown}{SLR}$ and $\overset{\frown}{MRT}$ adjacent arcs? Are $\overset{\frown}{SLM}$ and $\overset{\frown}{PTR}$ adjacent arcs? Explain your answers.

H You know that $\overset{\frown}{LMR}$ and $\overset{\frown}{PTR}$ are adjacent arcs. Name the arc that is the union of these two arcs. Is $\overset{\frown}{LMR} \cup \overset{\frown}{PTR} = \overset{\frown}{LRP}$?

I What is $\overset{\frown}{LMR}°$? What is $\overset{\frown}{PTR}°$? What is $\overset{\frown}{LMR}° + \overset{\frown}{PTR}°$?

J What is $\angle LKP°$? Is this also the measure of $\overset{\frown}{LRP}$? Is $\overset{\frown}{LRP}° = \overset{\frown}{LMR}° + \overset{\frown}{PTR}°$?

An arc of a circle that is the union of two adjacent arcs of that circle has a measure in degrees equal to the sum of the measures in degrees of the adjacent arcs. This property is expressed below.

The measure in degrees of the union of two adjacent arcs of a circle is the sum of the measures in degrees of the adjacent arcs.

K Look at D8. $\overline{BD}$ and $\overline{EC}$ are diameters of ⊙A. $\overleftrightarrow{BD}$ is perpendicular to $\overleftrightarrow{EC}$. What is the measure in degrees of each of the central angles of ⊙A that are determined by B, C, D, and E? How do you know?

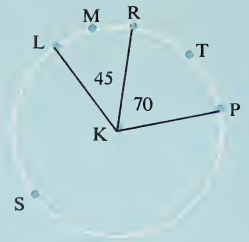
L What is semicircle BCD°? What is $\overset{\frown}{CDE}°$? Explain your answers.

The property of the measures in degrees of a semicircle is expressed below.

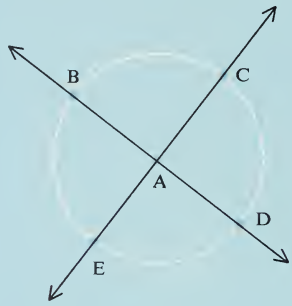
The measure in degrees of a semicircle is 180.

M Look again at D8. How many points are in the intersection of semicircle BCD and semicircle DEB? Tabulate $\overset{\frown}{BCD} \cap \overset{\frown}{DEB}$.

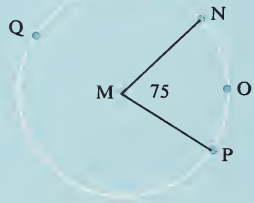
The measure in degrees of the union of $\overset{\frown}{BCD}$ and $\overset{\frown}{DEB}$ is 180 + 180, or 360. For any circle, the measure in degrees of the union of



D6



D8



D9

adjacent arcs. Two minor arcs of the same circle that intersect in exactly one point.

measure in degrees of a major arc of a circle. The difference of 360 and the measure in degrees of the minor arc of the circle that has the same endpoints as the major arc.

two of its semicircles that have exactly two points in common is 360. This property is expressed below.

If two semicircles of the same circle have exactly two points in common, then the union of the two semicircles has a measure in degrees of 360.

N Is ⊙A the union of $\overset{\frown}{CDE}$ and $\overset{\frown}{CBE}$? How do you know that the measure in degrees of ⊙A is 360?

The property of the measure in degrees of a circle is expressed below.

The measure in degrees of a circle is 360.

O Look at D9. Does minor $\overset{\frown}{NP}$ have the same endpoints as major $\overset{\frown}{NP}$? Tabulate the intersection of $\overset{\frown}{NOP}$ and $\overset{\frown}{NQP}$. What is the common name of $\overset{\frown}{NOP} \cup \overset{\frown}{NQP}$?

You know that major $\overset{\frown}{NP}$ has the same endpoints as minor $\overset{\frown}{NP}$. The union of these two arcs is ⊙M, and the measure in degrees of a circle is 360. Major $\overset{\frown}{NP}° = 360 -$ minor $\overset{\frown}{NP}°$. The measure in degrees of a major arc is the difference of 360 and the measure in degrees of the minor arc that has the same endpoints.

P Look again at D9. What is $\overset{\frown}{NOP}°$? What is $\overset{\frown}{NQP}°$?

Q If, for a given circle, minor $\overset{\frown}{AB}° = 27$, what is major $\overset{\frown}{AB}°$?

In this lesson you studied minor arcs, major arcs, semicircles, and central angles. You also learned how to find the measure in degrees of an arc of a circle.

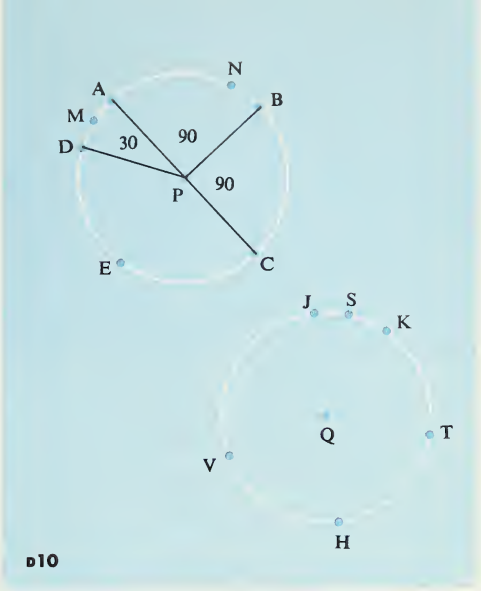
On your own

Use the picture of ⊙P in D10 for exercises 1 through 10. You will find it helpful to make a sketch of your own.

1 Name the central angle of ⊙P that is determined by points A and D.

2 What is ∠APD°?

3 Use three letters to name minor $\overset{\frown}{AD}$.



D10

4 Use three letters to name major $\overset{\frown}{AD}$.

5 What is minor $\overset{\frown}{AD}°$?

6 What is $\overset{\frown}{ACD}°$?

7 Name a diameter of ⊙P.

8 Name two semicircles of ⊙P.

9 What is $\overset{\frown}{DEC}°$?

10 What is major $\overset{\frown}{DB}°$?

Use the picture of ⊙Q in D10 for exercises 11 through 20. $\overline{JH}$ is a diameter of ⊙Q. $\overset{\frown}{JSK}° = 40$. ∠JQS° = 18. Again, you will find it helpful to make a sketch of your own.

11 What is $\overset{\frown}{JKH}°$?

12 What is $\overset{\frown}{JVH}°$?

13 What is ∠JQK°?

14 What is minor $\overset{\frown}{JS}°$?

15 What is major $\overset{\frown}{JS}°$?

16 What is minor $\overset{\frown}{SK}°$?

17 What is ∠KQH°?

18 What is $\overset{\frown}{KTH}°$?

19 Name an arc of ⊙Q whose measure in degrees is 162.

20 Name an arc of ⊙Q whose measure in degrees is 338.

Polygons

You know that a simple polygon is a simple closed curve that is the union of segments. In this lesson you will study several special kinds of simple polygons.

A Four polygons are represented in D1. Name the sides and vertices of polygon UVWXYZ. How many sides does polygon UVWXYZ have?

A simple polygon that has six sides is a *hexagon.*

B Name the polygon represented in D1 that has five sides.

A simple polygon that has five sides is a *pentagon.*

C Name the quadrilateral represented in D1. ? Name the triangle.

D Look at D2. Name the side of quadrilateral WXYZ that is opposite side XY. Name the side of quadrilateral WXYZ that is opposite side YZ.

E Line XY is parallel to line WZ. Is segment ? XY parallel to segment WZ? Tabulate $\overleftrightarrow{XY} \cap \overleftrightarrow{WZ}$.

The symbol at the right below is often used to express the idea of "parallel to." D3 shows how to write and read a sentence that includes this symbol. Study D3.

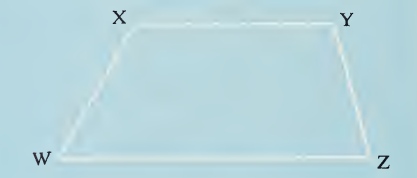
$\parallel$

F Look again at D2. You know that $\overline{XY} \parallel \overline{WZ}$. Is $\overline{WZ} \parallel \overline{XY}$?

G $\overleftrightarrow{WX}$ is not parallel to $\overleftrightarrow{YZ}$. Is $\overline{WX} \parallel \overline{YZ}$?

Notice that quadrilateral WXYZ has two, and only two, sides parallel. Such a quadri-

D1



D2

"Segment XY $\overline{XY} \parallel \overline{WZ}$. is parallel to segment WZ."

D3

hex a gon (hek′sə gon). A simple polygon that has six sides.

pen ta gon (pen′tə gon). A simple polygon that has five sides.

trap e zoid (trap′ə zoid). A quadrilateral in which two, and only two, sides are parallel.

lateral is a *trapezoid*. Any quadrilateral in which two, and only two, sides are parallel is a trapezoid.

H Look at D4. $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{AD}$. Is quadrilateral ABCD a trapezoid? Is quadrilateral ABCD a parallelogram? Explain your answers.

> REMINDER
> A parallelogram is a quadrilateral in which the opposite sides are parallel. The opposite sides of a parallelogram are congruent. The opposite angles are also congruent.
> *See lesson 106, pages 43 and 44.*

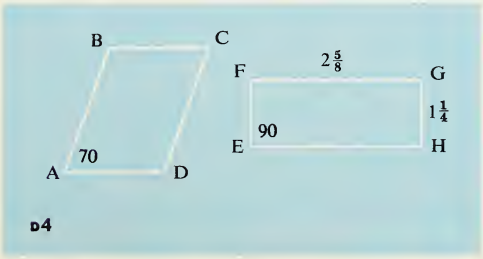
I Look again at D4. Name the side of parallelogram ABCD that is opposite $\overline{BC}$. Is $\overline{AD} \cong \overline{BC}$? Name the side of the parallelogram that is opposite $\overline{AB}$. Is $\overline{CD} \cong \overline{AB}$?

J Which angle of parallelogram ABCD is congruent to $\angle C$? Which angle of the parallelogram is congruent to $\angle B$?

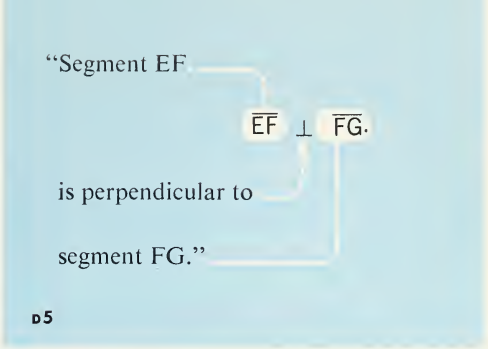
K How do you know that $m(\overline{BC}) = m(\overline{AD})$? Remember that you read the symbol $m(\overline{BC})$ as "the measure of segment BC." How do you know that $\angle D^\circ = \angle B^\circ$?

L What is the measure of $\angle A$? What is the measure of $\angle C$?

M Can a simple polygon be both a trapezoid and a parallelogram? Explain your answer.



D4



D5

In the following exercises, you will review some of the knowledge you have about quadrilaterals that are parallelograms. You will also study some new ideas that are related to parallelograms.

A Look again at D4. What is the measure of each side of parallelogram EFGH? The measure in degrees of $\angle E$ is 90. How do you know that 90 is the measure in degrees of each angle of parallelogram EFGH?

> REMINDER
> If one angle of a parallelogram is a right angle, then each of the angles of the parallelogram is a right angle.
> *See lesson 106, page 44.*

B How do you know that parallelogram EFGH represented in D4 is a rectangle? How do you know that $\overline{EF}$ is perpendicular to $\overline{FG}$?

The symbol at the right below is often used to express the idea of "perpendicular to." D5 shows how to write and read a sentence that includes this symbol.

C Look again at D4. How do you know that $\overline{EF} \perp \overline{EH}$? How do you know that $\overline{FG} \perp \overline{GH}$? How do you know that $\overline{EH} \perp \overline{GH}$?

D Now look at the picture of polygon IJKL in D6. $\overline{IJ} \parallel \overline{KL}$ and $\overline{JK} \parallel \overline{IL}$. How do you know that IJKL is a parallelogram?

E The small symbol printed in blue indicates that $\angle I$ is a right angle. You will often use this symbol in pictures of plane figures. What is the measure in degrees of each angle of IJKL?

F How do you know that IJKL is a rectangle? How do you know that IJKL is also a square?

G Name the two sides of IJKL that are adjacent to $\overline{JK}$. Remember that adjacent sides of a polygon are any two sides that intersect. Is $\overline{IJ} \perp \overline{JK}$? Is $\overline{LK} \perp \overline{JK}$?

H Polygon RSTU, represented in D6, is a parallelogram. How do you know that each side of RSTU is congruent to each of the other sides?

Any parallelogram in which each side is congruent to each of the other sides is a *rhombus*.

I Look again at D6. Is RSTU a rhombus? If $\angle R^\circ = 100$, how do you know that RSTU is not a square?

J Is IJKL a rhombus? How do you know?

K Is every rhombus a parallelogram? Is every parallelogram a rhombus? Is every square a rhombus? Is every rhombus a square?

In this lesson you studied different kinds of polygons. You also learned to use symbols to express the ideas of "parallel to" and "perpendicular to."

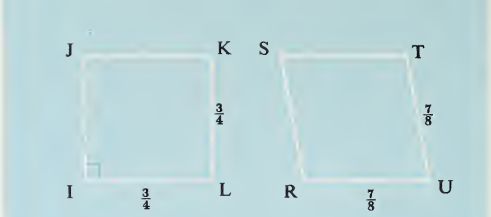
On your own

1 Make a sketch of a simple closed curve that is not a simple polygon.

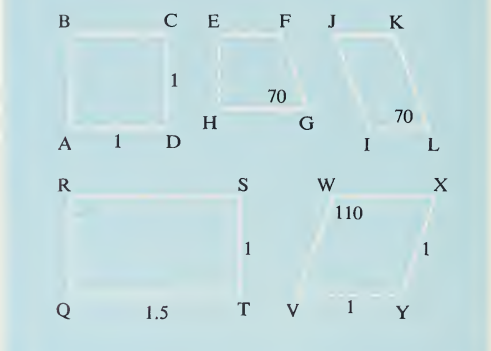
2 What is the least number of sides that a simple polygon can have?

3 Make a sketch of a pentagon.

4 Make a sketch of a hexagon.



D6



D7

rhom bus (rom′bəs). A parallelogram in which each of the sides is congruent to each of the other sides.

5 Make a sketch of a quadrilateral that is neither a trapezoid nor a parallelogram.

6 Make a sketch of a trapezoid.

7 Make a sketch of a parallelogram that is neither a rhombus nor a rectangle.

8 Make a sketch of a rhombus that is not a square. Then make a sketch of a rectangle that is not a square.

9 Make a sketch of a parallelogram that has a right angle and also has two adjacent sides that are congruent.

Use D7 in connection with exercises 10 through 17.

10 In quadrilateral EFGH, $\overline{EF} \parallel \overline{GH}$. How do you know that EFGH is not a parallelogram? What kind of quadrilateral is EFGH?

Each of the other quadrilaterals represented in D7 is a parallelogram.

11 Name each rhombus.
12 Name each rectangle.
13 Name each square.
14 Name each side whose measure is 1.
15 Name each side whose measure is 1.5.
16 Name each right angle.
17 What is $\angle J°$? What is $\angle Y°$?

KEEPING SKILFUL

Tabulate the solution set of each condition expressed in exercises 1 through 8. U = C.

1 $9/1 \sim 720/b$.
2 $152 - a > 25$.
3 $\sim (c < 12 \cdot 13)$.
4 x is 5% of 60.
5 $r + 16 < 45$.
6 345 is y% of 3450.
7 $20/21 \sim d/231$.
8 213 is 150% of z.

Tabulate the solution set of each condition expressed in exercises 9 through 14. U = C × C.

9 $c/180 \sim 4/720 \wedge c + d < 5$.
10 $x + y = 100 \wedge y > 95$.
11 $x + y < 7 \wedge x - y = 2$.
12 $s + t < 20 \wedge s = 3t$.
13 $m/n \sim 9/5 \wedge m + n < 30$.
14 $7/8 \sim r/s \wedge r + 4 > s$.

For each of exercises 15 through 23, first tabulate the solution set described. Then make a graph of each solution set. For exercises 15 through 18, U = N. For exercises 19 through 23, U = N × N.

15 $\{s \mid s + 8 < 104 \div 8\}$
16 $\{y \mid \sim (y + 2 > 14)\}$
17 $\{r \mid 25r > 150 \vee 16 - r = 9\}$
18 $\{x \mid x + 4 < 10 \vee x = 8\}$
19 $\{(x, y) \mid x + y = 10 \wedge x > y\}$
20 $\{(m, n) \mid m = 6 - n \vee n = 4 - m\}$
21 $\{(r, s) \mid r + s = 7 \vee r = 5 - s\}$
22 $\{(a, b) \mid a + 10 = 15 \wedge b + 1 < 7\}$
23 $\{(p, q) \mid p + 21 < 30 \wedge q = p + 3\}$

Triangles and related segments

You know that a triangle is the union of the three segments that are determined by any three noncollinear points. In this lesson, you will study certain segments that are related to triangles.

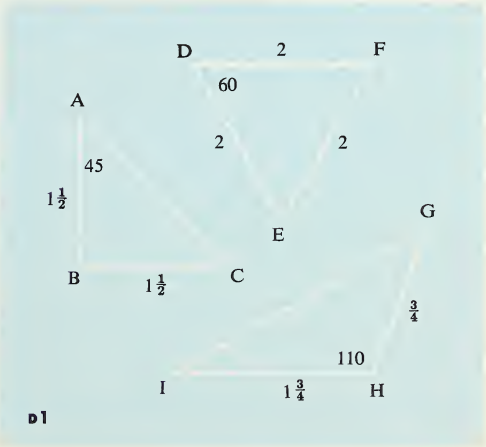
A Look at D1. Name the sides and vertices of $\triangle ABC$.

B Name the two sides of $\triangle ABC$ that are congruent to each other.

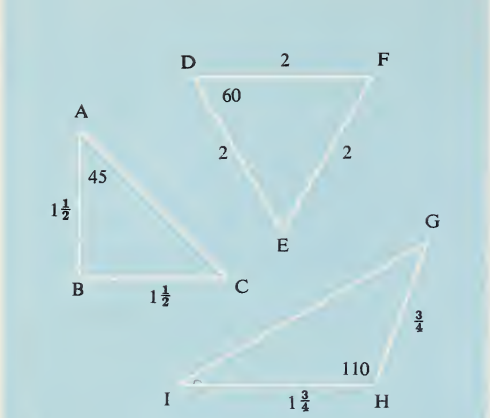
C The sentence $\overline{CA} > \overline{AB}$ expresses a true statement about sides CA and AB of $\triangle ABC$. Write a sentence that expresses a true statement about sides CA and BC.

D Which angle of $\triangle ABC$ is opposite $\overline{AB}$? Which angle is opposite $\overline{BC}$?

E What is $\angle A°$? What is $\angle C°$? What is $\angle A° + \angle B° + \angle C°$? What is $\angle B°$? Explain your answers.



D1



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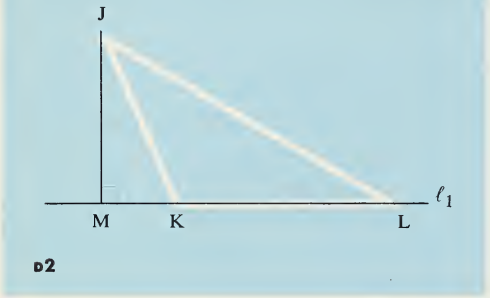


D2

F Now look at the picture of $\triangle DEF$ in D1. How many congruent sides does $\triangle DEF$ have?

G Which angle of $\triangle DEF$ is opposite $\overline{DE}$? Which angle is opposite $\overline{EF}$? Which angle is opposite $\overline{FD}$?

H What is $\angle F°$? What is $\angle E°$? Do all three angles of $\triangle DEF$ have the same measure?

Any triangle in which all three angles have the same measure is an *equiangular triangle.*

I How do you know that, in an equilateral triangle, each angle must have a measure in degrees of 60?

J Now look at the picture of $\triangle GHI$ in D1. $\overline{GI} > \overline{HI}$. How do you know that $\overline{HI} > \overline{GH}$? Is $\overline{GI} > \overline{GH}$? Explain your answer.

K How do you know that neither $\angle G$ nor $\angle I$ is congruent to $\angle H$?

L Which triangle represented in D1 is scalene? Which triangle is equilateral? Which triangles are isosceles?

M Which angles represented in D1 are acute angles? Which angle is a right angle? Which angle is an obtuse angle?

N Which triangle represented in D1 is an acute triangle? Which is a right triangle? Which is an obtuse triangle? Which triangle is equiangular?

Now you are ready to study certain segments that are related to triangles.

A Look at D2. Name the sides and angles of $\triangle JKL$. Use three letters to name each angle.

B How do you know that $\overline{JM}$ is an altitude of $\triangle JKL$? When $\overline{JM}$ is used as an altitude of $\triangle JKL$, which side of $\triangle JKL$ is used as the base?

C How many altitudes does a triangle have?

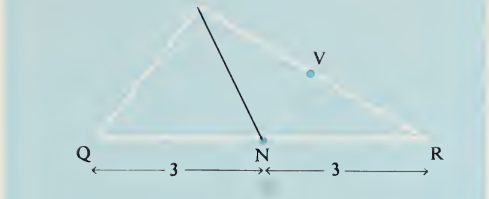
D Make a sketch of an obtuse triangle. In your sketch, show pictures of all the altitudes of the triangle.

E Look at D3. Points Q, N, and R are collinear. Is N between Q and R? How do you know that $\overline{QN} \cong \overline{NR}$?

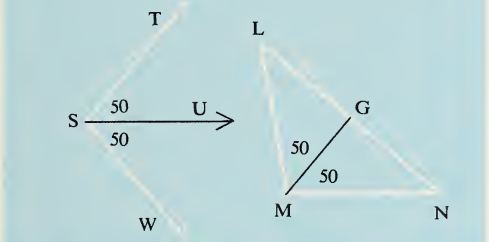
Point N is the *midpoint* of $\overline{QR}$. The midpoint of a segment is a point in the segment that, together with the endpoints of the segment, determines two congruent segments.

e qui an gu lar triangle (ē′kwē ang′gyü lər). A triangle all of whose angles have the same measure.

mid point of a segment (mid′point). A point contained in a segment so that the two segments that are determined by the point and the endpoints of the given segment are congruent. If point B is between A and C and $\overline{AB} \cong \overline{BC}$, then point B is the midpoint of $\overline{AC}$.



D3



D4

median of a triangle (mē′dē ən). A segment determined by a vertex of a triangle and the midpoint of the side opposite that vertex.

bisector of an angle (bī sek′tər). A ray determined by the vertex of an angle and a point in the interior of the angle so that the two angles formed by the ray and the sides of the original angle are congruent. If point J is in the interior of ∠ BAC and if ∠BAJ ≅ ∠ JAC, then ray AJ is the bisector of ∠ BAC.

bisector of an angle of a triangle. A segment that is included in the ray that bisects an angle of a triangle. One endpoint of the segment is the vertex of the angle, and the other endpoint is in the side of the triangle opposite the vertex.

F How many midpoints does a segment have?

The midpoint of a given segment *bisects* (bī sekts′) the segment. A geometric figure, such as a line or a circle, that intersects a segment in the midpoint of the segment bisects the segment. Such a geometric figure is a *bisector* (bī sek′tər) of the segment.

G Look again at D3. Describe the points that determine $\overline{PN}$.

$\overline{PN}$ is a *median* of △PQR. A median of a triangle is a segment that is determined by a vertex of the triangle and the midpoint of the side opposite that vertex.

H Look again at D3. $\overline{RV} \cong \overline{VP}$. Name the median of △PQR that is a bisector of $\overline{RP}$.

I How many medians does a triangle have?

J Make a sketch of an acute triangle. In your sketch, show pictures of all the medians of the triangle.

K Look at D4. Point S is the endpoint of three rays represented in the display. Name the three rays.

L Which of the rays that you named for exercise K is determined by the vertex of ∠TSW and a point in the interior of ∠TSW?

M Name the angle formed by the union of $\overrightarrow{SU}$ and $\overrightarrow{ST}$. By the union of $\overrightarrow{SU}$ and $\overrightarrow{SW}$.

N How do you know that ∠TSU ≅ ∠USW?

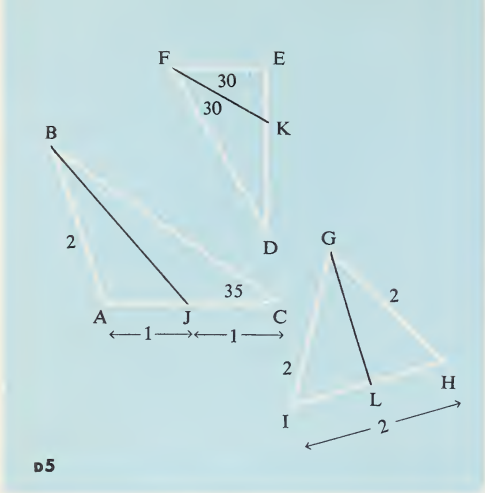
$\overrightarrow{SU}$ is the bisector of ∠TSW. The *bisector of an angle* is a ray. This ray is determined by the vertex of the angle and a point in the interior of the angle. The two angles formed by the ray and the sides of the original angle are congruent. A ray that is the bisector of an angle bisects the angle.

O How many bisectors does an angle have?

P Look again at D4. Name the sides and angles of △LMN. Use three letters to name each angle.

Q How do you know that ray MG is the bisector of ∠LMN? Is segment MG a subset of ray MG?

$\overline{MG}$ is the bisector of ∠LMN of △LMN. The *bisector of an angle of a triangle* is a segment. The segment is included in the ray that bisects the angle. One endpoint of the segment



D5

is the vertex of the angle, and the other endpoint is in the side opposite the vertex.

R Does any angle of a triangle have more than one bisector?

S Make a sketch of a right triangle. In your sketch, show pictures of all the bisectors of the angles of the triangle.

In this lesson you have reviewed the ways in which triangles can be identified. You have also studied the midpoint of a segment, a median of a triangle, the bisector of an angle, and the bisector of an angle of a triangle.

On your own

Use D5 as you answer exercises 1 through 18. When you name an angle, use three letters.

1 In $\triangle ABC$, which angle is opposite $\overline{AB}$?
2 Which side is opposite $\angle BAC$?
3 What is $\angle ACB°$?
4 What is $\angle ABC°$?
5 How is $\overline{BJ}$ related to $\triangle ABC$?
6 In $\triangle DFE$, what is $\angle DEF°$?
7 What is $\angle DFE°$?
8 What is $\angle EDF°$?
9 How is $\overline{FK}$ related to $\triangle DFE$?
10 What is $\angle DKF°$?
11 When $\overline{EF}$ is used as a base, which side of $\triangle DFE$ is the altitude?
12 In $\triangle HIG$, what is $\angle IHG°$?
13 In $\triangle HGL$, what is $\angle HLG°$? What is $\angle HGL°$?
14 Name the nine triangles that are represented in D5.
15 Name two isosceles triangles represented in D5.
16 Name three right triangles.
17 Name two obtuse triangles.
18 Which triangle is equiangular?
19 Make a sketch of $\triangle RST$ so that $\angle SRT$ is obtuse, $\overline{TR} < \overline{RS}$, and $\overline{RS} < \overline{ST}$. In your sketch, show pictures of altitude TM, angle bisector TN, and median TQ.

KEEPING SKILFUL

Using the sets described below, tabulate each of the sets named in exercises 1 through 6. U = N.

$A = \{x \mid x + 44 > 54\}$.
$B = \{x \mid x - 46 > 31\}$.
$C = \{x \mid 27 + x < 36\}$.
$D = \{x \mid 102 - 76 = x\}$.

1 $A \cap B$ 3 $A \cup D$ 5 $A \cap D$
2 $C \cap A$ 4 $D \cup C$ 6 $B \cup A$

Tabulate the solution set of each condition expressed in exercises 7 through 10. U = Z.

7 $x/\frac{1}{3} \sim 4/3$. 9 $5/6 \sim 7\frac{1}{2}/y$.
8 $30.5/g \sim 5/2.5$. 10 $16.2/100 \sim k/250$.

For each of the conditions expressed in exercises 11 through 15, give a standard description of the solution set. Then make a graph of each solution set. $U = R_a$.

11 $7\frac{1}{2} - 6 < a$.
12 $x + 13\frac{1}{2} \leqq 19\frac{3}{4}$.
13 $12 - y > 8.5 \wedge y + 1.75 > 2.25$.
14 $z + 10 \geqq 12\frac{1}{4} \wedge z + 6\frac{1}{2} < 9\frac{1}{2}$.
15 $c - .25 > 4.25 \vee .2 + .05 > c$.

SPECIAL CHALLENGE

You have studied both finite and infinite number systems. You have also worked with an infinite geometry. That is, you have thought about space as an infinite set of points. Now you will meet a *finite geometry* in which space is a set of exactly nine points. Picture A in the display shows the nine dots that represent the space of this geometry.

A Remember that the set of all points in a geometry is the space of that geometry. Tabulate the space of the geometry of nine points.

In an infinite geometry, space includes infinitely many lines, and each line contains infinitely many points. In the geometry of nine points, however, there are just twelve "lines," and each "line" contains exactly three points. Keep these two properties of the new geometry in mind as you work the rest of the exercises.

We have chosen twelve particular "lines" and represented them in the display.

B The three "lines" represented in picture B are named "ℓ_{KLM}," "ℓ_{NRS}," and "ℓ_{TVX}." Name the "lines" represented in picture C. Name each "line" represented in pictures D and E.

We do not use the word "straight" in the geometry of nine points. Look at picture F. ℓ_{KVS}, for example, is a "line" in this geometry.

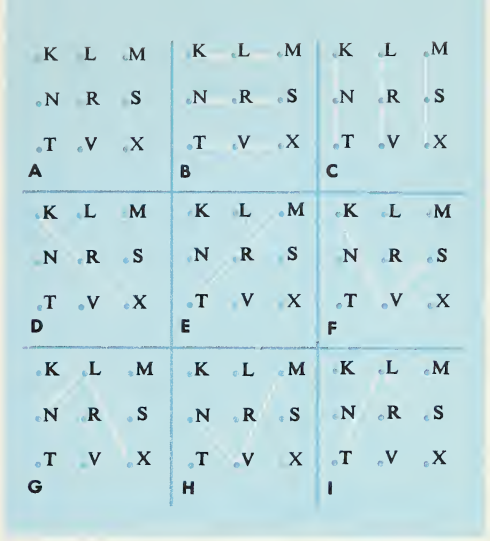
C Name each of the "lines" represented in pictures G, H, and I.

D Does $\ell_{TRM} = \{T, R, M\}$? Tabulate ℓ_{TLS}.

E In the geometry of nine points, as in the infinite geometry that you have studied, points in the same line are collinear. Which point is collinear with N and L? What property of the geometry of nine points tells you that four or more points cannot be collinear?

F Which of these sets is a set of collinear points, {N, R, M}, {L, S, T}, or {V, X, K}?

G Name the "lines" that contain point X.



H Another property of this geometry is that two given points can be contained in only one line. Which "line" contains both K and R? Both V and M? Why did we not choose both {K, R, X} and {K, M, X} for "lines"?

I Is any pair of points not in a "line"?

J Tabulate the intersection of ℓ_{NVM} and ℓ_{KVS}. Tabulate the intersection of ℓ_{NLX} and ℓ_{TRM}.

K Give a definition of two intersecting "lines."

L There are no skew lines in the geometry of nine points. Two parallel "lines" are two "lines" whose intersection is the empty set. What "lines" are parallel to ℓ_{TRM}? To ℓ_{TLS}?

For each of the following questions, first give an answer for infinite geometry. Then give an answer for the geometry of nine points.

M Space contains how many points?

N Space includes how many lines?

O A line contains how many points?

P Each point is in how many lines?

Q How many points determine a line?

R Two lines intersect in how many points?

S How many points are in the intersection of two parallel lines?

T How many lines are parallel to each line?

Geometric constructions

You have often been asked to make pictures of plane geometric figures. In this lesson, you will learn special methods of making pictures of certain plane figures.

In lesson 101, you learned how to use a protractor to measure a given angle and to make a picture of an angle with a given measure. Now you will learn to use a protractor and a straightedge to make a picture of a line that is parallel to a given line.

A Look at picture A in D1. $\overleftrightarrow{RS}$ is a given line. You will see how we used a protractor and a straightedge to make a picture of a line that is parallel to $\overleftrightarrow{RS}$. How many lines intersect any given line? How many lines intersect $\overleftrightarrow{RS}$?

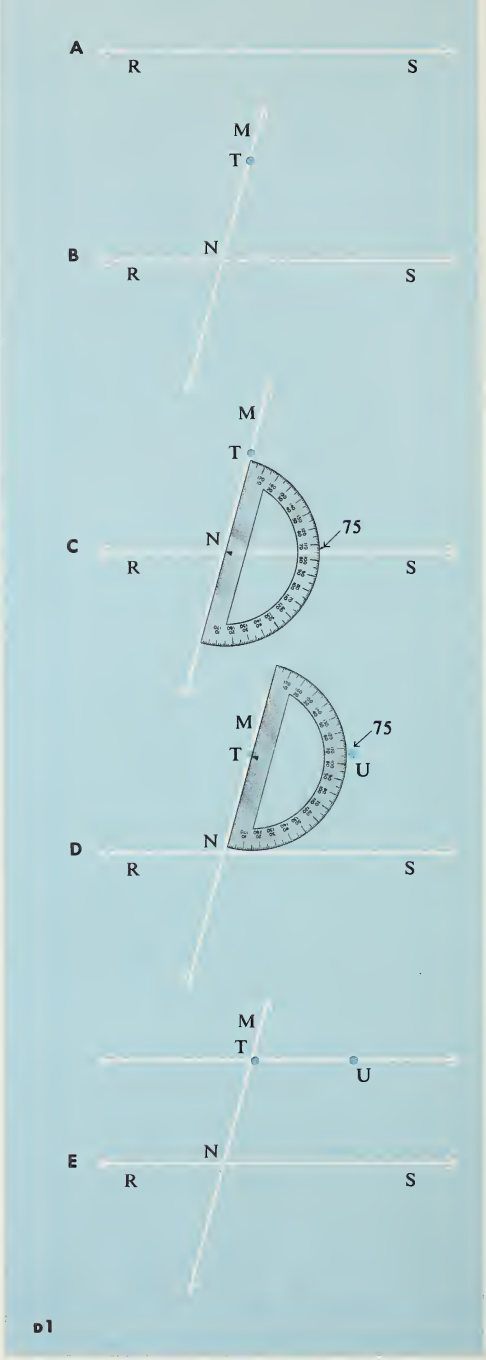
B Look at picture B. We used a straightedge to make a picture of one of the lines that intersect $\overleftrightarrow{RS}$. We say that $\overleftrightarrow{MN}$ is *any* line that intersects $\overleftrightarrow{RS}$. Tabulate $\overleftrightarrow{MN} \cap \overleftrightarrow{RS}$. How do you know that $\overleftrightarrow{MN}$ and $\overleftrightarrow{RS}$ are coplanar? Name the plane that includes $\overleftrightarrow{MN}$ and $\overleftrightarrow{RS}$.

C Look again at picture B. Next we located any point T in $\overleftrightarrow{MN}$. We will accept the fact that, for any point T in $\overleftrightarrow{MN}$, there is one, and only one, line parallel to $\overleftrightarrow{RS}$ that contains that point. How many lines that are parallel to $\overleftrightarrow{RS}$ intersect $\overleftrightarrow{MN}$?

D Look at picture C in D1. What is $\angle MNS°$?

E Look at picture D. Notice where we placed the protractor. Point U was located so that $\angle UTM° = \angle MNS°$. What is $\angle UTM°$?

F Look at picture E. We used a straightedge to make a picture of the line determined by



D1

points T and U. How do you know that $\overleftrightarrow{MN}$ is a transversal of $\overleftrightarrow{TU}$ and $\overleftrightarrow{RS}$?

G Are $\angle MNS$ and $\angle UTM$ corresponding angles? How do you know that $\angle MNS \cong \angle UTM$?

H How do you know that $\overleftrightarrow{TU} \parallel \overleftrightarrow{RS}$?

REMINDER

Corresponding angles are a pair of angles formed by two coplanar lines and a transversal. The interiors of both angles are subsets of the same half-plane bounded by the transversal. A side of one angle is a proper subset of a side of the other angle. If two corresponding angles are congruent, then the two lines are parallel.

See lesson 103, pages 30 and 31.

I On a sheet of paper, make a picture of any line AB. Then use the method shown in D1 to make a picture of any line CD parallel to $\overleftrightarrow{AB}$.

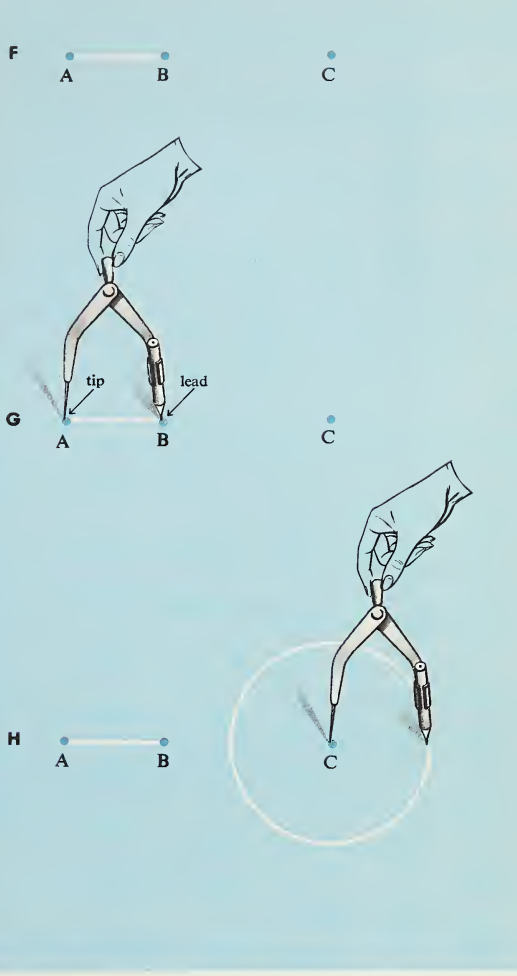
In the following exercises, you will learn how to use a new kind of instrument to help you draw pictures of geometric figures.

J Look at picture F in D2. Name the segment represented in the picture. What point is represented that is not in the segment?

Now look at picture G in D2. The instrument being used is called *compasses* (kum′pəs ez). Notice the parts of the compasses that we call the *tip* and the *lead* (led). Compasses are used to draw pictures of arcs and circles.

Picture G shows the first step in using the compasses to draw a picture of a circle whose center is point C and whose radii are congruent to $\overline{AB}$.

K Look again at picture G. The tip of the compasses is on dot A. Where is the lead of the compasses?



D2

The distance between the tip of the compasses and the lead is the same as $m(\overline{AB})$. We say that the compasses are *set* to the measure of radius AB, or, more simply, the *setting* of the compasses is radius AB.

L Look at picture H in D2. The setting of the compasses was not changed. Now where is the tip of the compasses?

M We used the lead to draw a picture of a simple closed curve. Explain why this simple closed curve is a circle. Name this circle.

Pictures of many geometric figures can be drawn by using only a straightedge and compasses. Pictures that are drawn with these two instruments represent *geometric constructions*. To construct a geometric figure actually means to think about certain relationships between sets of points. However, we will also use the word "construct" when we talk about drawing a picture that represents a geometric construction. That is, we will say "construct" instead of "draw a picture of." Measuring devices, such as rulers and protractors, are not permitted when geometric constructions are represented. Now you will learn to construct a segment that is congruent to a given segment.

Look at picture I in D3. $\overline{EF}$ is a given segment. First we used a straightedge to draw a picture of any line ℓ_1. Notice that we located any point G in ℓ_1.

N In picture J, what is the setting of the compasses?

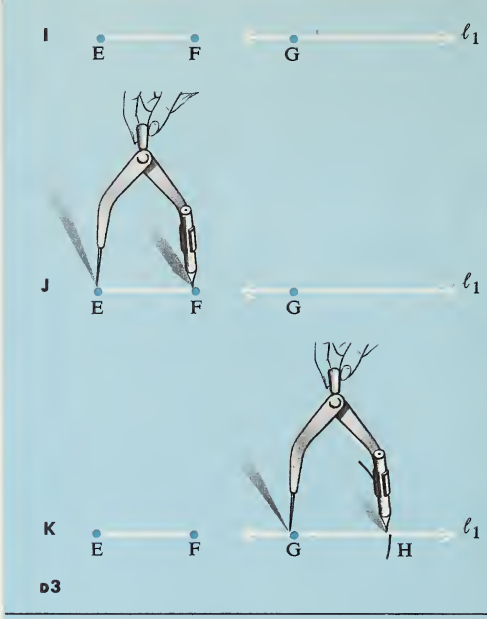
O Look at picture K. Without changing the setting of the compasses, and with the tip on dot G, we constructed an arc that intersects ℓ_1. How do segments GH and EF compare?

The picture of $\overline{GH}$ is a *copy* of $\overline{EF}$. When you are asked to copy a given geometric figure, you should construct a figure that is congruent to the given figure.

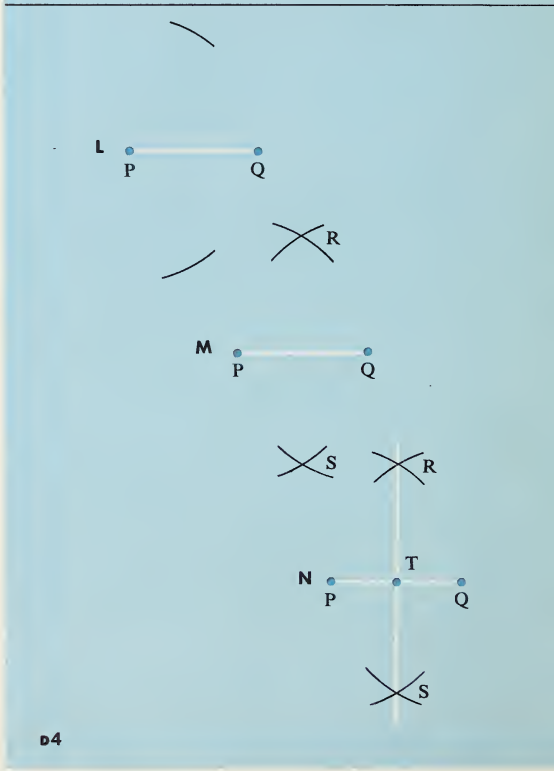
P Draw a picture of any segment MN on a sheet of paper. Use the steps shown in D3 to construct $\overline{OP}$ that is a copy of $\overline{MN}$.

Q You obtain an approximate check of your picture by measuring. Use a ruler to find the measure of $\overline{MN}$ and the measure of $\overline{OP}$. Are the two measures approximately equal? Remember that it is impossible to measure a physical object without a certain amount of error.

So far you have learned how to construct a circle that has a given radius and a given cen-



D3



D4

ter. You have also learned how to copy a given segment. Next you will learn how to construct a bisector of a given segment.

A What segment is represented in picture L? We set the compasses to a radius greater than $\frac{1}{2}$ of m($\overline{PQ}$). Then, with the tip on dot P, we constructed arcs both above and below $\overline{PQ}$.

B Look at picture M. Without changing the setting of the compasses, and with the tip on dot Q, we constructed two more arcs that intersect the first two arcs. Name the points of intersection.

C Now look at picture N. We used a straightedge to construct $\overleftrightarrow{RS}$. Tabulate $\overleftrightarrow{RS} \cap \overline{PQ}$.

D T is the midpoint of $\overline{PQ}$. Is $\overline{PT} \cong \overline{TQ}$?

$\overleftrightarrow{RS}$ bisects $\overline{PQ}$, and it is perpendicular to $\overline{PQ}$. Line RS is the *perpendicular bisector* of $\overline{PQ}$.

E Draw a picture of any segment EF. Then follow the steps shown in D4 to construct the perpendicular bisector of $\overline{EF}$.

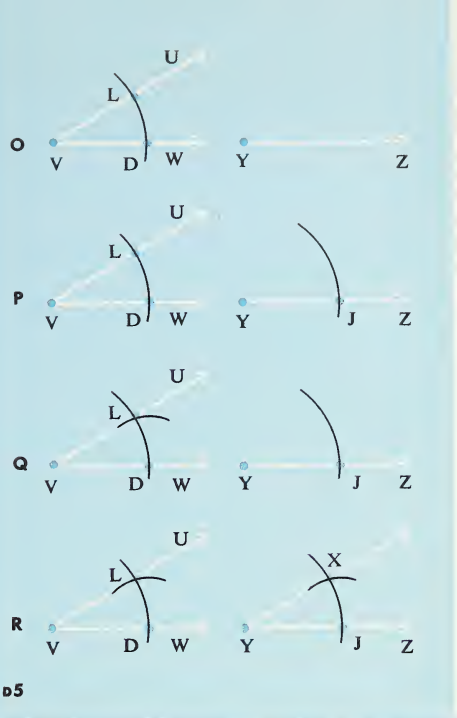
F How can you use a ruler and a protractor to check your picture?

Now you will learn how to copy a given angle.

G Look at picture O in D5. Name the angle represented in picture O. As the first step in copying the angle, we constructed any ray YZ as one side of the angle to be constructed. With the tip of the compasses on dot V, we constructed arc LD so that it intersects both sides of $\angle$UVW. Tabulate $\overset{\frown}{LD} \cap \angle UVW$.

In picture O, we set the compasses to a *convenient* radius to construct arc LD. This means that we chose a setting so that the construction would be accurate and yet not too large for the amount of space we had.

H Look at picture P in D5. With the setting of the compasses unchanged, and with the tip on dot Y, we constructed an arc that intersects $\overrightarrow{YZ}$. Name the point of intersection.



D5

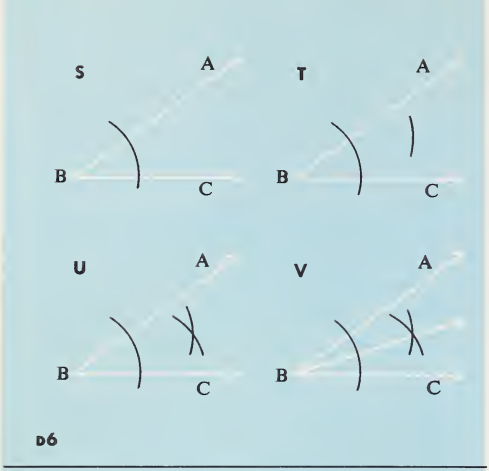
perpendicular bisector of a segment. A line that is perpendicular to a segment and that also contains the midpoint of the segment.

Look at picture Q. With the compasses set to radius DL, and with the tip on dot D, we constructed an arc that intersects $\overset{\frown}{LD}$.

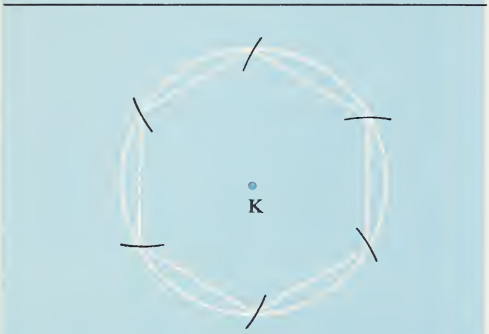
I Look at picture R. Without changing the setting of the compasses, and with the tip on dot J, we constructed an arc that intersects the first arc in point X. Then we constructed ray YX. Name the angle that is the union of $\overrightarrow{YZ}$ and $\overrightarrow{YX}$.

Angle XYZ is congruent to angle UVW; therefore, $\angle$XYZ is a copy of $\angle$UVW.

J On a sheet of paper, first draw a picture of any acute angle RST. Then follow the steps shown in D5 to copy $\angle$RST. Use a protractor to check your picture.



D6



D7

K The pictures in D6 show the steps used to construct the bisector of $\angle ABC$. Study the pictures and describe, in your own words, each step taken in representing the construction.

L On a sheet of paper, first draw a picture of any obtuse angle JKL. Then use the steps shown in D6 to construct the bisector of $\angle JKL$. Use a protractor to check your picture.

In this lesson you have learned how to construct circles, segments, angles, and bisectors of segments and angles.

On your own

For each of exercises 1 through 4, use a straightedge and a protractor to make a picture of the angle whose measure is expressed in the exercise.

1 $\angle CDE° = 35$.

2 $\angle FGH° = 116$.

3 $\angle PQR° = 90$.

4 $\angle STU° = 6$.

5 Construct a line that is parallel to a given line. Remember that you can use only a straightedge and compasses to construct a geometric figure.

6 Construct a circle whose radii are congruent to a given segment. Then, without changing the radius of the compasses, start at some point in the circle and construct arcs that intersect the circle as shown in D7.

7 Use a straightedge to construct the segments determined by the intersections of the arcs and the circle as shown in D7.

8 What is the special name of the simple polygon that you have constructed? How do you know?

9 Use a ruler to find the measure of each side of the polygon. Are the measures approximately the same?

10 Use a protractor to find the measure in degrees of each angle of the polygon. Are the measures approximately the same?

11 Use what you know about constructing the perpendicular bisector of a segment to construct a right angle.

12 Construct the bisector of the right angle that you constructed for exercise 11. Use a protractor to find the measure of each of the angles formed by the union of the bisector and a side of the right angle.

13 Draw a picture of an obtuse triangle. Use what you know about constructing a bisector of an angle to construct the bisector of the obtuse angle of the triangle.

14 Draw a picture of acute triangle DEF. Use what you know about constructing the perpendicular bisector of a segment to determine the midpoint of $\overline{EF}$. Then construct median DM.

Congruent triangles

In lesson 105 you learned how the vertices of two triangles can be put in one-to-one correspondence. You also learned what is meant by a congruence correspondence between the vertices of two triangles. In this lesson you will study some of the properties of congruence for triangles.

A Two scalene triangles are represented in D1. Name the angles and the sides of $\triangle RST$. Name the angles and the sides of $\triangle UVW$.

B Diagram A in D1 represents one way in which the vertices of $\triangle RST$ and $\triangle UVW$ can be put in one-to-one correspondence. The order of the letters in the names of the triangles shows that vertices R and U are mapped onto each other. Which other vertices are mapped onto each other in this correspondence?

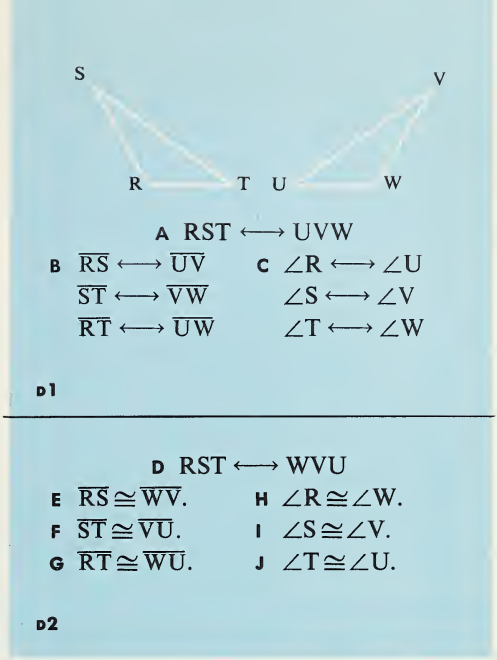
C In lesson 105 of unit 8, you learned that the correspondence between the vertices of two triangles determines a correspondence between the sides and the angles of the triangles. Use diagram B to help you name the pairs of corresponding sides determined by the correspondence given in diagram A in D1. Use diagram C to help you name the pairs of corresponding angles.

D Now look at D2. Diagram D represents another correspondence between the vertices of the two triangles represented in D1. Which vertices are mapped onto each other in this correspondence?

E Name the pairs of corresponding sides and the pairs of corresponding angles that are determined by the correspondence given in diagram D.

F Each sentence in D2 expresses a true statement about the corresponding parts of $\triangle RST$ and $\triangle WVU$. Does diagram D represent a congruence correspondence? How do you know that $\triangle RST \cong \triangle WVU$?

G From diagram A in D1 and diagram D in D2, you know two ways in which the vertices



REMINDER

In a congruence correspondence between the vertices of two triangles, the corresponding sides and the corresponding angles are congruent. If there is a congruence correspondence between the vertices of two triangles, then the triangles are congruent.

See lesson 105, page 39.

of △RST and △UVW can be put in one-to-one correspondence. There are six correspondences in all. Make diagrams to represent the four other correspondences.

H How many of the diagrams that you made for exercise G determine a congruence correspondence? How do you know?

I Suppose that △GHI ≅ △JKL. Name the corresponding sides and the corresponding angles of △GHI and △JKL. Remember that, in the sentence △GHI ≅ △JKL, the order in which the vertices are named indicates a congruence correspondence between the vertices.

J How do you know that the corresponding sides that you named for exercise I are congruent? How do you know that the corresponding angles are congruent?

You have learned one way to tell whether or not two triangles are congruent. In the work that follows, you will see that it is possible to decide if two triangles are congruent even though you do not know that the three pairs of corresponding sides are congruent, and the three pairs of corresponding angles are congruent.

A Look at D3. You are going to learn one method of constructing a triangle that is congruent to a given triangle. The given triangle is △FGH. We first constructed any segment JK included in ℓ_1 that is congruent to side FH. Describe how $\overline{JK}$ was constructed.

B Look at D4. We set the compasses to radius FG. Then, with the tip on dot J, we constructed an arc. How does any segment determined by a point in this arc and point J compare to $\overline{FG}$?

C Look at D5. We next set the compasses to radius GH. With the tip on dot K, we constructed a second arc that intersects the first arc. Name the point of intersection of the arcs.

D RST ⟷ WVU

E $\overline{RS} \cong \overline{WV}$.

F $\overline{ST} \cong \overline{VU}$.

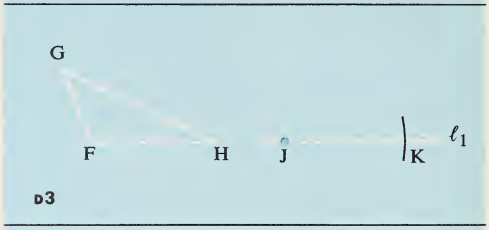
G $\overline{RT} \cong \overline{WU}$.

H $\angle R \cong \angle W$.

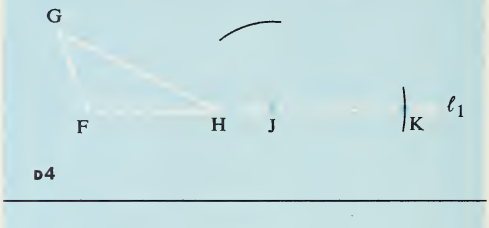
I $\angle S \cong \angle V$.

J $\angle T \cong \angle U$.

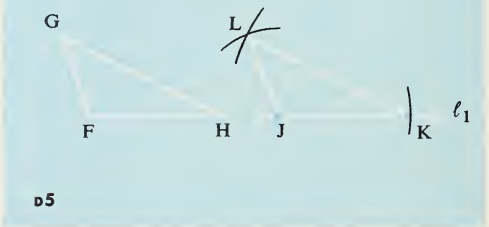
D2



D3



D4



D5

D Finally, we used a straightedge to construct $\overline{JL}$ and $\overline{LK}$. How do you know that $\overline{JL} \cong \overline{FG}$? That $\overline{LK} \cong \overline{GH}$?

From D3, D4, and D5, you have seen how to construct a triangle whose sides are congruent to the sides of a given triangle. Now you will decide if △JLK and △FGH are congruent. In other words, you will decide if the correspondence expressed at the right above is a congruence correspondence.

JLK ⟷ FGH

E Name the corresponding sides and the corresponding angles of △JLK and △FGH.

F You know that the corresponding sides of △JLK and △FGH are congruent. For each pair of corresponding sides, are the measures of the sides equal?

G You know that, if the measures of two angles are equal, the angles are congruent. To compare the measures of the corresponding angles of △JLK and △FGH, make a tracing of the picture of △JLK and place it over the picture of △FGH. For each pair of corresponding angles, do the measures of the angles appear to be equal?

H The corresponding sides and the corresponding angles of △JLK and △FGH are congruent. Is △JLK ≅ △FGH? How do you know?

The construction represented in D5 and the work in exercises E through H have suggested an important property of congruence for triangles. We will accept this property without proof. The property is expressed below.

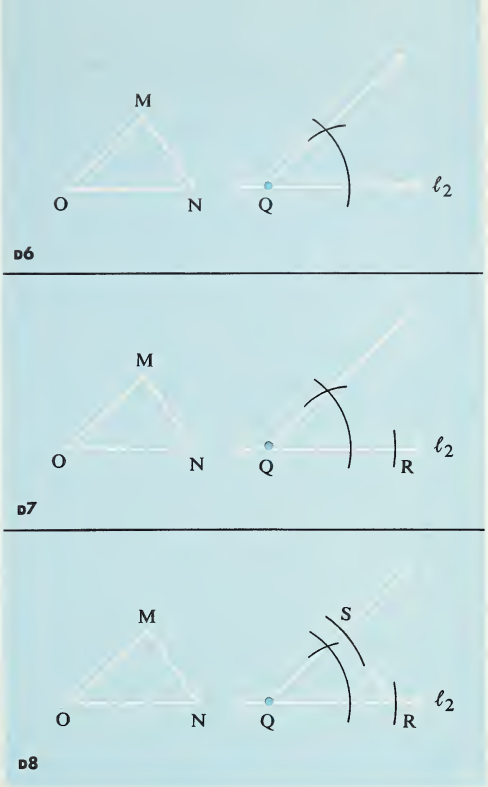
For any two triangles, if the corresponding sides are congruent, then the two triangles are congruent.

The property just expressed is often called the *s-s-s* (side, side, side) *property of congruence for triangles.*

I Use a straightedge to draw a picture of any triangle ABC. Then use the s-s-s property of congruence for triangles to construct a triangle DEF that is congruent to △ABC.

In the following exercises, you will learn a second method of constructing a triangle congruent to a given triangle. You will also study another property of congruence for triangles.

A Name the triangle represented in D6. To construct a triangle congruent to △OMN, we first drew a picture of any line ℓ_2 and chose any point Q in ℓ_2. We next constructed an angle with vertex Q that is congruent to ∠O of △OMN. How was this angle constructed?



B Look at D7. We constructed $\overline{QR}$ in ℓ_2 congruent to $\overline{ON}$. To what radius did we set the compasses for this construction?

C Look at D8. How did we construct $\overline{QS}$ congruent to $\overline{OM}$? What instrument did we use to construct $\overline{SR}$?

From D6, D7, and D8, you have seen how to construct a triangle in which one angle and the two sides of the triangle that are subsets of that angle are congruent to the corresponding parts of a given triangle. Now you will decide if △QSR ≅ △OMN.

D Name the corresponding sides and the corresponding angles of △QSR and △OMN.

E From the construction represented in D8, you know that two sides of $\triangle QSR$ are congruent to two sides of $\triangle OMN$. Name these pairs of congruent sides.

F Name the third pair of corresponding sides of the two triangles. Use a ruler to find the measure of each of these sides as represented in D8. Are the measures approximately equal?

G The corresponding sides of $\triangle QSR$ and $\triangle OMN$ are congruent. How do you know that $\triangle QSR$ and $\triangle OMN$ are congruent?

The construction represented in D8 and the work in exercises D through G have suggested another property of congruence for triangles. We will accept this property without proof. The property is expressed below.

For any two triangles, if an angle of one triangle and the two sides of the triangle that are subsets of that angle are congruent to the corresponding parts of the other triangle, then the two triangles are congruent.

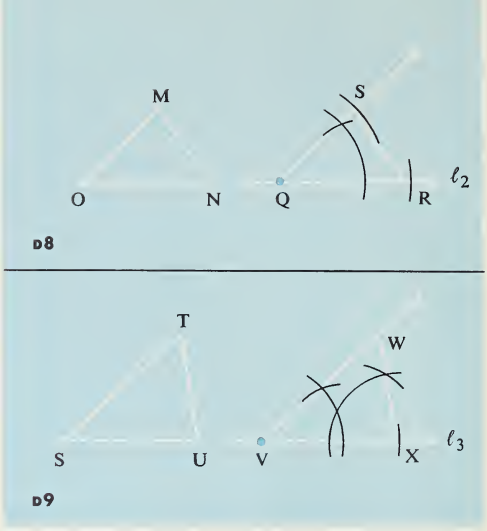
The property just expressed is often called the *s-a-s* (side, angle, side) *property of congruence for triangles.*

H Use a straightedge to draw a picture of any triangle GHI. Then use the s-a-s property of congruence for triangles to construct a triangle JKL that is congruent to $\triangle GHI$.

Now you will learn a third method of constructing a triangle that is congruent to a given triangle. You will also study a third property of congruence for triangles.

A Study the pictures in D9 carefully. To construct a triangle congruent to $\triangle STU$, we first drew a picture of any line ℓ_3 and chose any point V in the line. Next we constructed $\angle V$. $\angle V$ is a copy of $\angle S$ of $\triangle STU$. Is $\angle V \cong \angle S$?

B Then we constructed a segment VX that is congruent to $\overline{SU}$. Is $\overline{VX}$ a copy of $\overline{SU}$?



C Finally, we constructed $\angle X$ congruent to $\angle U$ of $\triangle STU$. Tabulate $\overrightarrow{VW} \cap \overrightarrow{XW}$. How was point W located?

D Is $\overline{VX}$ a subset of $\angle V$? Is it a subset of $\angle X$?

You have just seen how to construct a triangle in which two angles and the side of the triangle that is a subset of both those angles are congruent to the corresponding parts of a given triangle. Now you will decide whether or not $\triangle VWX \cong \triangle STU$.

E Name the corresponding sides and the corresponding angles of $\triangle VWX$ and $\triangle STU$.

F Use a ruler to find the measures of the corresponding sides VW and ST, as represented in D9. Are the measures approximately equal?

G $\overline{VW} \cong \overline{ST}$. Is $\triangle VWX \cong \triangle STU$? Upon what property does your answer depend?

The construction represented in D9 and the work in exercises D through G have suggested another property of congruence for triangles. We will accept this property without proof. The property is expressed below.

For any two triangles, if two angles of one triangle and the side of the triangle that is a

subset of each of those angles are congruent to the corresponding parts of the other triangle, then the two triangles are congruent.

The property just expressed is often called the *a-s-a* (angle, side, angle) *property of congruence for triangles.*

H Draw a picture of any triangle MNO. Use the a-s-a property of congruence for triangles to construct a triangle PQR that is congruent to $\triangle MNO$.

I Imagine two triangles ABC and DEF. ? $\angle A \cong \angle D$, and $\angle B \cong \angle E$. How do you know that $\angle C \cong \angle F$?

J Now imagine two triangles GHI and JKL. ? $\overline{GH} \cong \overline{JK}$, $\angle H \cong \angle K$, and $\angle I \cong \angle L$. Which property of congruence for triangles can you use to show that triangles GHI and JKL are congruent? Explain your answer.

K Does the following sentence express a true statement? "If two triangles are congruent, then the corresponding angles of the two triangles are congruent."

In lesson 103 of unit 8, you learned that the *converse* of a given statement may be obtained by interchanging the idea expressed by the "if" clause and the idea expressed by the "then" clause.

L Is the statement expressed by the following ? sentence the converse of the statement expressed in exercise K? "If the corresponding angles of two triangles are congruent, then the two triangles are congruent." Is the converse of the statement expressed in exercise K true? Make a sketch to help explain your answer.

M Does the following sentence express a true ? statement? "If the two sides of a right triangle that are subsets of the right angle are congruent to the corresponding parts of another right triangle, then the two right triangles are congruent." Make a sketch to help explain your answer.

N Does the following sentence express a true ? statement? "If two sides and an angle opposite one of the sides of one triangle are congruent to the corresponding parts of another triangle, then the triangles are congruent." Make a sketch to help explain whether or not the statement is true.

In this lesson you learned three methods of constructing a triangle congruent to a given triangle. You also studied the s-s-s, the s-a-s, and the a-s-a properties of congruence for triangles.

On your own

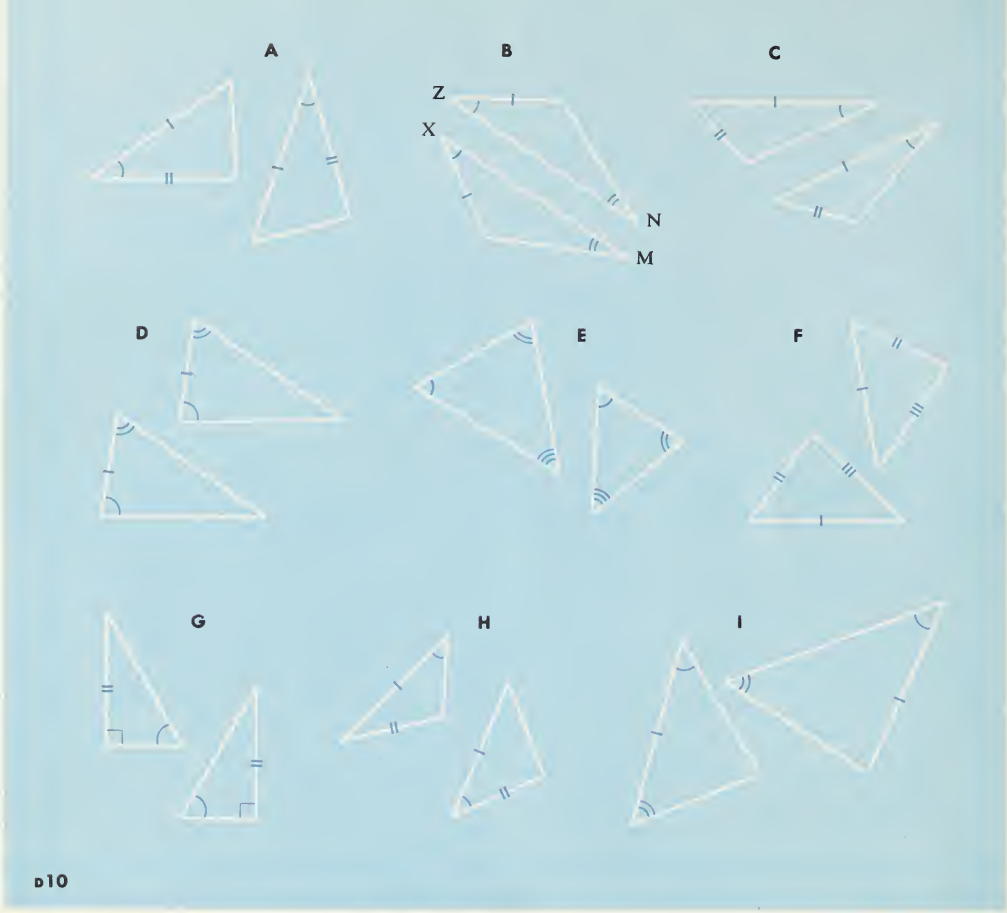
1 Use a straightedge to draw a picture of any obtuse triangle RST. Then use the s-s-s property of congruence for triangles to construct a triangle that is congruent to $\triangle RST$.

2 Draw a picture of any scalene triangle UVW. Then use the a-s-a property of congruence for triangles to construct a triangle that is congruent to $\triangle UVW$.

3 Draw a picture of any acute triangle XYZ. Then use the s-a-s property of congruence for triangles to construct a triangle that is congruent to $\triangle XYZ$.

4 Draw a picture of any acute triangle DEF. Use the s-s-s property of congruence for triangles to construct a triangle MNO that is congruent to triangle DEF. Then use what you know about constructing the bisector of an angle to construct the bisector of $\angle M$ of $\triangle MNO$.

5 Draw a picture of any scalene triangle GHI. Use the s-s-s property of congruence for triangles to construct a triangle JKL that is congruent to triangle GHI. Then use your knowledge of how to construct a perpendicular bisector of a segment to determine the midpoint of $\overline{JK}$. Finally, construct median KM.



D10

6 Each picture in D10 represents a pair of triangles. Study the pictures carefully to see how we have used marks to indicate pairs of congruent parts. In picture B, for example, $\angle X \cong \angle Z$, and $\angle M \cong \angle N$. Decide which pictures in D10 represent pairs of congruent triangles. In each case, tell which property you used to decide that the triangles are congruent.

For each of exercises 7 through 12, first make a sketch of two triangles that appear to be congruent. Name them ABC and DEF. Using the true statements expressed in each exercise to help you, next indicate in each picture the parts of the two triangles that are congruent. Mark each picture in the way shown in D10. Then tell whether you can use the s-s-s property, the s-a-s property, or the a-s-a property of congruence for triangles to show that the two triangles are congruent.

7 $\angle A \cong \angle D$. $\angle B \cong \angle E$. $\overline{BC} \cong \overline{EF}$.

8 $\overline{BC} \cong \overline{EF}$. $\angle B \cong \angle E$. $\angle C \cong \angle F$.

9 $\overline{AC} \cong \overline{DF}$. $\overline{BC} \cong \overline{EF}$. $\overline{AB} \cong \overline{DE}$.

10 $\overline{BC} \cong \overline{EF}$. $\overline{AB} \cong \overline{DE}$. $\angle C \cong \angle F$.

11 $\angle C^\circ = \angle F^\circ$. $\angle B^\circ = \angle E^\circ$. $\overline{AC} \cong \overline{DF}$.

12 $m(\overline{AB}) = m(\overline{DE})$. $\angle B^\circ = \angle E^\circ$.
$m(\overline{BC}) = m(\overline{EF})$.

Areas of geometric figures

In lesson 107 you learned that the union of a simple closed curve and its interior is a closed region. You know that you can assign to a closed region a number that is its area. In this lesson you will develop conditions for equivalence to find the areas of certain closed regions.

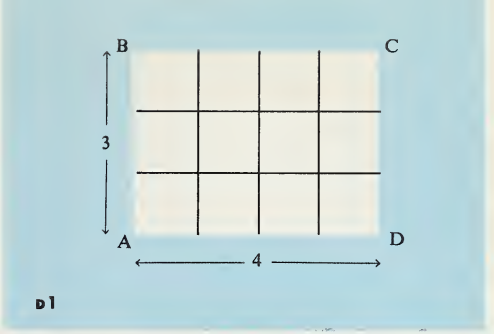
A The picture in D1 represents rectangle ABCD and its interior. Is the union of rectangle ABCD and its interior a closed region? Is the union a rectangular region? Explain your answers.

B Look again at D1. Rectangular region ABCD has been subdivided into square units. What is the measure of each side of a square unit?

> REMINDER
> A square unit is the union of a square and its interior. The measure of each side of the square is 1.
> *See lesson 107, page 48.*

You can use a condition for equivalence to find the area of rectangular region ABCD. Remember that the area of a rectangular region is the number of square units that can be included in the closed region without overlapping.

C What is $m(\overline{AD})$? The rate pair expressed at the right represents the fact that 4 square units are included in 1 row of rectangular region ABCD. $4/1$



D1

D What is the measure of $\overline{AB}$? How many rows are included in rectangular region ABCD?

E Use x as a variable for the number of square units in all the rows, or, in other words, for the area of rectangular region ABCD. Why can you use the rate pair expressed below at the right to represent the fact that x square units are included in 3 rows? $U = Z$. $x/3$

F Since the square units occur at the same rate in 3 rows as in 1 row, you can use a condition for equivalence to find the total number of square units in the rectangular region. Does the sentence at the right express a condition for equivalence for finding the area of rectangular region ABCD in square units? $4/1 \sim x/3$.

From now on in this book, we will say "area of a rectangle" whenever we mean "area of a rectangular region."

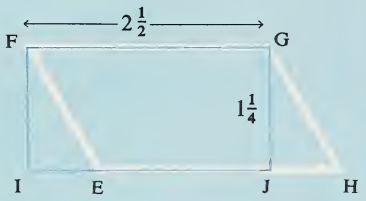
G What is the area of rectangle ABCD in square units?

H The measure of each side of rectangle WXYZ is 4.6. Is rectangle WXYZ a square? How do you know? What condition for equivalence can you use to find the area of rectangle WXYZ in square units? $U = Z$.

I What is the area of rectangle WXYZ in square units?

A rectangular driveway is 56.5 ft. long and 7.25 ft. wide. What is the area of the driveway in square feet? (Ones)

D2



D3

Now read the problem in D2. In the following exercises, you will develop a condition for the problem. Notice that the foot is the standard unit in which the measures are given. The universe for each variable is Z.

J Why can you use the rate pair expressed at the right in a condition for the problem? 56.5/1

K Use m as a variable for the area of the driveway in square feet. Is m a variable for the number of square feet in all the rows? Why can you use the rate pair expressed at the right in a condition for the problem? $m/7.25$

L Does the sentence at the right express a condition for the problem? $56.5/1 \sim m/7.25$.

M Tabulate $\{m|56.5/1 \sim m/7.25\}$.

Look again at D2. The word within parentheses at the end of the problem tells you that the number you use to get the answer should be rounded to the nearer one.

N Give the answer to the problem in D2.

Now that you know how to find the area of a rectangle, you can use what you have learned to find the area of a parallelogram that is not a rectangle.

Look at D3. Polygon EFGH is a parallelogram, but it is not a rectangle. Polygon IFGJ is a rectangle.

A Name the two altitudes that are common to parallelogram EFGH and rectangle IFGJ. Name the common base of EFGH and IFGJ.

B What is m($\overline{FG}$)? What is m($\overline{GJ}$)? Give a sentence that expresses a condition for equivalence that you can use to find the area of rectangle IFGJ. U = Z.

C Can you also find the area of parallelogram EFGH by using the condition that you obtained for exercise B? Explain your answer.

REMINDER

If a parallelogram that is a rectangle and a parallelogram that is not a rectangle have a common base and a common altitude, then the two parallelograms have the same area.

See lesson 108, page 53.

D What is the area of parallelogram EFGH in square units?

E You know that a rhombus is a parallelogram in which each side is congruent to each of the other sides. The measure of side ST of rhombus STUV is 3.5. What is the measure of each of the other sides of rhombus STUV?

F The measure of an altitude of rhombus STUV is 3. What is the measure of the base?

G You can use the condition expressed below at the right to find the area of rhombus STUV. For what is x a variable? U = Z. $3.5/1 \sim x/3$.

H What is the area of rhombus STUV in square units?

Read the problem in D4. The universe for each variable is Z.

Notice that the measure of an altitude and the measure of the base are given in different standard units. For this reason, it is helpful to develop a compound condition for the problem.

I What is the measure of the altitude in inches? First you will develop a simple condition for finding the number of feet that is the same length as 18 in. Why can you use the rate pair 12/1 in this simple condition? Why can you also use $18/x$ in the condition? For what is x a variable?

J Why is $12/1 \sim 18/x$ a simple condition for the problem?

K Next you will develop a simple condition to find the area of the piece of plywood in square feet. Why can you use the rate pair 4/1 in this simple condition? Use y as a variable for the area. Why can you use y/x in the condition?

L Is $4/1 \sim y/x$ another simple condition for the problem?

M What is the compound condition for the problem in D4? Why do you use "and" in the compound condition?

N Tabulate $\{(x, y) \mid 12/1 \sim 18/x \wedge 4/1 \sim y/x\}$.

? Then give the answer to the problem in D4.

Next you will use what you know about the area of a parallelogram to find the area of a triangle.

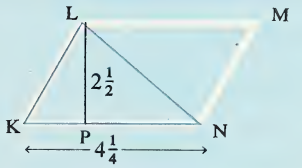
A Look at D5. $\overline{LP}$ is a common altitude of parallelogram KLMN and $\triangle KLN$. Name the common base of parallelogram KLMN and $\triangle KLN$.

B $\overline{LN}$ is a diagonal of parallelogram KLMN.

? Remember that a diagonal of a parallelogram is a segment that is determined by two opposite vertices. How do you know that

A piece of plywood is in the shape of a parallelogram. The length of an altitude of the piece is 18 in. The length of the base is 4 ft. What is the area of the piece in square feet?

D4



D5

$\triangle KLN \cong \triangle MNL$? Do $\triangle KLN$ and $\triangle MNL$ have the same area?

C How do you know that the area of parallelogram KLMN is 2 times the area of $\triangle KLN$?

REMINDER

If a parallelogram and a triangle have a common base and a common altitude, then the area of the parallelogram is 2 times the area of the triangle.

See lesson 108, page 55.

D What is $m(\overline{KN})$? What is $m(\overline{LP})$?

E Use x as a variable for the area of $\triangle KLN$. The universe for x is Z. How do you know that $2x$ refers to the area of parallelogram KLMN?

F Why can you use the condition expressed at the right to find the area of $\triangle KLN$?

$4\frac{1}{4}/1 \sim 2x/2\frac{1}{2}$.

G Tabulate $\{x \mid 4\frac{1}{4}/1 \sim 2x/2\frac{1}{2}\}$. What is the area of $\triangle KLN$ in square units?

A window in an old house is in the shape of a triangle. The length of an altitude is $\frac{3}{4}$ yd. The length of the base is $1\frac{1}{4}$ yd. What is the area of the window in square feet?

D6

Now read the problem in D6. The universe for each variable is Z.

You are to find the area of the window in square feet. Since the measure of an altitude and the measure of the base are given in yards, it is helpful to develop a compound condition for the problem.

H You will first develop a simple condition to find the area of the window in square yards. Why can you use the rate pair $1\frac{1}{4}/1$ in this simple condition? Why can you use $2x/\frac{3}{4}$ in the condition? For what is x a variable?

I Why is the condition expressed at the right one of the simple conditions for the problem?

$$1\tfrac{1}{4}/1 \sim 2x/\tfrac{3}{4}.$$

J The second simple condition for the problem will be used to find the area of the window in square feet. 1 square yard is the same as how many square feet? Use y as a variable for the area of the window in square feet.

K Why is the condition expressed at the right another simple condition for the problem?

$$9/1 \sim y/x.$$

L What is the compound condition for the problem in D6?

M Tabulate $\{(x, y) \mid 1\frac{1}{4}/1 \sim 2x/\frac{3}{4} \wedge 9/1 \sim y/x\}$.

? Then give the answer to the problem in D6.

In this lesson you used conditions for equivalence to find the areas of rectangles, parallelograms, and triangles.

On your own

For each problem, write a sentence that expresses a condition. Tabulate the solution set of the condition and give the answer to the problem. The universe for each variable is Z.

1 A rectangular lot is 75 ft. long and $50\frac{1}{2}$ ft. wide. What is the area of the lot in square feet?

2 A sign is in the shape of a parallelogram. The length of an altitude of the sign is 8 ft. The length of the base is $12\frac{1}{2}$ ft. What is the area of the sign in square feet?

3 Mr. Hammond made a shelf in the shape of a triangle. The length of an altitude of the shelf was $14\frac{1}{8}$ in. The length of the base was 28 in. What was the area of the shelf in square inches?

4 The shape of a baseball diamond is square. The length of a side is 90 ft. What is the area of a baseball diamond in square feet?

5 John has a piece of glass in the shape of a parallelogram. The area of the piece of glass is 4 sq. yd. The length of an altitude is $1\frac{1}{3}$ yd. What is the length of the base in yards?

THINK For what will you use x as a variable? Is $x/1 \sim 4/1\frac{1}{3}$ a condition for the problem?

6 The length of an altitude of rhombus WXYZ is 2.5 ft. The length of the base is 4.8 ft. What is the area in square inches?

THINK Is 1 sq. ft. the same area as 144 sq. in?

7 The length of an altitude of $\triangle$IJK is 8.5 in. The length of the base is 4 times the length of the altitude. What is the area of $\triangle$IJK in square inches?

8 The area of a rectangle is 162 sq. ft., and the length is 13.5 ft. What is the width in feet?

9 The length of an altitude of parallelogram JKLM is 2.75 yd. The length of the base is 6 yd. What is the area in square feet?

10 The area of $\triangle$DEF is 27 sq. in. The length of an altitude is $1\frac{1}{4}$ ft. What is the length of the base of $\triangle$DEF in inches?

APPLYING MATHEMATICS

For each problem, first write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. For problems 1 through 11, the universe for each variable is Z.

1 The rectangular ceiling of a television studio is 180 ft. long and $90\frac{1}{2}$ ft. wide. What is the area of the ceiling in square feet?

2 Mr. Evans built a rectangular patio. The length of the patio is $24\frac{1}{2}$ ft. The width of the patio is 15 ft. What is the perimeter of the patio?

THINK Is the perimeter of a simple polygon the sum of the measures of the sides of the polygon? For what will you use x as a variable? Is $x = 24\frac{1}{2} + 15 + 24\frac{1}{2} + 15$ a condition for the problem?

3 Allen made a triangular pennant with sides of 35 in., 18 in., and 35 in. What is the perimeter of the pennant in feet?

4 The area of the rectangular bulletin board in Mrs. Brown's classroom is 50 sq. ft. The bulletin board is 12 ft. long. How many feet wide is the bulletin board?

5 The sail on Mr. Nelson's boat is in the shape of a triangle. The length of an altitude of the sail is $10\frac{1}{2}$ ft. The length of the base is 6 ft. What is the area of the sail in square feet?

6 John made a tent out of a square piece of canvas. The length of each side of the piece of canvas is $3\frac{1}{2}$ yd. What is the area of the canvas in square feet?

7 Mr. Alessandro bought a field shaped like a parallelogram. The length of an altitude of the field is 40 rd. The length of the base is 75 rd. Mr. Alessandro paid \$310 per acre for the field. How much did he pay for the field?

THINK Is 1 acre the same area as 160 sq. rd?

8 The length of an altitude of rhombus PQRS is $1\frac{7}{8}$ ft. The area of the rhombus is $5\frac{1}{4}$ sq. ft. What is the length of the base of the rhombus in feet?

9 The area of triangle QRS is 39 sq. in. The length of an altitude of the triangle is $6\frac{1}{2}$ in. What is the length of the base of triangle QRS in inches?

10 The length of an altitude of parallelogram ABCD is $6\frac{2}{3}$ in. The length of the base is $4\frac{1}{2}$ in. The area of parallelogram ABCD is the same as the area of parallelogram LMNO. The length of an altitude of parallelogram LMNO is 4 in. What is the length of the base of parallelogram LMNO in inches?

11 The length of side AC of $\triangle ABC$ is $4\frac{1}{2}$ ft. The length of side AB is $1\frac{1}{3}$ times the length of side AC. The length of side BC is $1\frac{2}{3}$ times the length of side AC. What is the perimeter of $\triangle ABC$ in feet?

For problems 12, 13, and 14, the universe for each variable is the set of rational numbers between 0 and 180.

12 In triangle RST, the measure of angle R is 5 times the measure of angle S. The measure of $\angle R$ is 10 less than the measure of a right angle. What is the measure in degrees of each angle of triangle RST?

THINK Use x as a variable for $\angle R°$. Use y as a variable for $\angle S°$. Use z as a variable for $\angle T°$. How do you know that $(x + y + z = 180) \wedge (5/1 \sim x/y) \wedge (90 - x = 10)$ is a compound condition for the problem?

13 $\triangle KLM$ is a right triangle in which angle K is the right angle. $\angle L°$ is 29 less than $\angle K°$. What is the measure in degrees of each angle of triangle KLM?

14 In triangle XYZ, $\angle X°$ is 3 times $\angle Y°$. If $\angle X°$ were 17 greater, $\angle X°$ would be 132. What is the measure in degrees of each angle of triangle XYZ?

CHECKING UP

The small numerals within parentheses tell what pages to turn to for help if you need it.

Test 119

Nine sets are described below. Each set is a subset of R_a.

$\{x|x > 7\frac{7}{12}\}$ $\{x|x < \frac{5}{12}\}$ $\{x|x > 2.5\}$
$\{x|x \leqq \frac{5}{12}\}$ $\{x|x \leqq 1.7\}$ $\{x|8.5 < x\}$
$\{x|x > 1.7\}$ $\{x|1\frac{1}{12} > x\}$ $\{\ \}$

From the descriptions given above, select a description that is a name of each set expressed in exercises 1 through 6. $U = R_a$.

1 $\{x|\sim(\frac{5}{12} < x)\}$ (64, 80)
2 $\{x|4\frac{1}{3} - x > 3\frac{1}{4}\}$ (64, 67)
3 $\{x|2.5 < x\} \cap \{x|1.7 < x\}$ (63, 67)
4 $\{x|5.75 < x \wedge 8.8 > 7.1 + x\}$ (65, 67)
5 $\{x|x - 2 > 6\frac{1}{2} \vee x + 1\frac{5}{6} > 9\frac{5}{12}\}$ (70)
6 $\{x|4.2 < 2.5 + x\}$ (71, 79)

Test 120

For exercises 7 through 18, decide what words or symbols best complete each exercise.

7 (5, 9, 3) is an ordered ~~~~. (92)
8 The symbol ~~~~ expresses the ideas of "less than or equal to." (71)
9 The solution set of a compound condition that involves two simple conditions and the connective "or" is the ~~~~ of the solution sets of the two simple conditions. (71)
10 The measure in degrees of a circle is ~~~~. (121)
11 The number associated with a point in a number line is the ~~~~ of the point. (75)
12 A pentagon is a simple polygon that has ~~~~ sides. (122)
13 The symbol ~~~~ expresses the idea of "perpendicular to." (123)
14 The union of a given set and its complement is the ~~~~. (78, 79)
15 A ~~~~ is a quadrilateral in which two, and only two, sides are parallel. (122)
16 The solution set of $2\frac{1}{6} < m \leqq 7\frac{3}{4}$ is ~~~~ interval. $U = R_a$. (86)
17 In $w \approx_2 50$, the tolerance is ~~~~. $U = R_a$. (90)
18 A line that is perpendicular to a segment and contains the midpoint of the segment is the ~~~~ of the segment. (133)

Test 121

19 The measure in degrees of a minor arc of a circle is always less than what number? (119)
20 Tabulate $\{(x, y)|x + y \leqq 3\}$. $U = N \times N$. (76)
21 Tabulate $\{y|\sim(3 + y \geqq 9)\}$. $U = N$. (80)
22 $F = \{2, 3, 4\}$. $M = \{6, 7\}$. $P = \{0, 5\}$. Tabulate $\{(f, m, p)|f \in F \wedge m \in M \wedge p \in P\}$. (93)
23 Give a standard description of the solution set of $s \approx_{.8} 12$. Use the symbol for "less than or equal to." $U = R_a$. (90)
24 Set A has 5 members, and set B has 6 members. How many members are there in $A \times B$? (74)
25 Set C has 3 members, set D has 5, and set E has 5. How many members are there in $\{(c, d, e)|c \in C \wedge d \in D \wedge e \in E\}$? (93)
26 Make a graph of $\{x|x \leqq 2\frac{1}{2}\}$. $U = R_a$. (72)
27 Make a graph of $\{b|b \approx_{.6} 3\}$. $U = R_a$. (88)
28 Make a graph of $\{z|\frac{1}{4} - z \geqq 0 \wedge z + \frac{1}{3} \geqq \frac{8}{15}\}$. $U = R_a$. (85)

Test 122

For each problem, write a sentence that expresses a condition. Tabulate the solution set of the condition, if possible. Otherwise, give a standard description of the solution set. Then give the answer to the problem.

For problems 29 through 32, the universe for each variable is Z.

29 A rectangular parking lot is 128 ft. long and $48\frac{1}{4}$ ft. wide. What is the area of the parking lot in square feet? (142)

30 An object that is located near the equator moves about 24,900 mi. in 24 hr. because of the rotation of the earth upon its axis. About how many miles per hour does this object move because of the earth's rotation? (Ones) (102)

31 Last year 260 tokens for the Wilmot Beach were sold. This year 294 beach tokens were sold. What was the per cent of increase in the number of beach tokens sold? (Tenths) (103)

32 A window in a new bus is in the shape of a parallelogram. The length of an altitude of the parallelogram is $1\frac{1}{4}$ ft. The length of the base is $2\frac{2}{3}$ ft. What is the area of the window in square feet? (142)

For problems 33 and 34, the universe for each variable is N.

33 Brad's mother is 2 years younger than his father. His father is at least 41 years old, but less than 47 years old. How old is each of Brad's parents? (99)

34 The sum of a first number and a second number is less than 6. The difference of the first number and a third number is 1. The third number is greater than 2. What are the three numbers? (110)

For problems 35 and 36, the universe for each variable is C.

35 Certain rate pairs are equivalent to 8/3. The sum of the components of each of these rate pairs is not greater than 44. What are the rate pairs? (104)

36 A certain rate pair is equivalent to 18/12. The product of the components of the rate pair is 20. What is the rate pair? (104)

For problem 37, the universe for each variable is R_a.

37 What numbers are at least as great as $32\frac{1}{2}$ and not greater than the difference of $412\frac{1}{4}$ and $286\frac{3}{4}$? (100)

Unit 10

The rational number system

lesson page

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The natural numbers and the rational numbers of arithmetic

In your study of mathematics up to now, you have used only two sets of numbers, the natural numbers and the rational numbers of arithmetic. Much of mathematics makes use of other sets of numbers. In this unit you will learn why at least one other set of numbers is needed. First, however, you will review some of the properties of natural numbers and of rational numbers of arithmetic.

A Tabulate the set of natural numbers. Is $15.6 \in N$? Is $1\frac{2}{3} \in N$? Is $349 \in N$?

In *Book 1* you learned that the sum of any two members of N is also a member of N, and that the product of any two members of N is also a member of N. You can express these two properties in another way: N is closed under the operations of addition and multiplication.

B How do you know, without computing, that $648 + 5984 \in N$? How do you know, without computing, that $648 \times 5984 \in N$?

You can use the relationship between subtraction and addition to decide whether or not N is closed under subtraction also.

C Think about $8 - 3 = 5$. When you add 5 to 3, you obtain the sum 8. Is the difference $8 - 3$ a natural number?

D How do you know that the difference $13 - 28$ is not a natural number?

As you know, the closure property of addition of natural numbers can be expressed as follows.

The universe for a *and* b *is N. For each replacement of* a *and* b, $a + b \in N$.

However, since you know that you replace a variable only by members of the universe, the property also can be expressed in the way given below. Notice that the words "replacement of" are omitted. From now on in this book, we will usually omit the words "replacement of."

The universe for a *and* b *is N. For each* a *and* b, $a + b \in N$.

E The universe for a and b is N. For each a and b, is $a \times b \in N$? How do you know?

F The universe for a and b is N. For each a and b, is $a - b \in N$? Is N closed under subtraction? Explain your answers.

You can use the relationship between division and multiplication to decide whether or not N is closed under division.

G Think about $35 \div 7 = 5$. By what natural number can you multiply 7 to get 35? Is the quotient $35 \div 7$ a natural number?

H How do you know that the quotient $18 \div 10$ is not a natural number?

I The universe for a and b is N. For each a and b, is $a \div b \in N$? Is N closed under division? Explain your answers.

Because N is not closed under either subtraction or division, there are many problems that cannot be answered when the set of natural numbers is used as a universe. The following exercises will help you see why this is so.

J Read the problem in D1. Suppose that the universe is N. Is $19/3 \sim x/1$ a condition for the problem? How can you obtain $19 = 3x$ from $19/3 \sim x/1$?

To find the solution of $19 = 3x$, you must find a natural number by which you can multiply 3 to obtain the product 19. This means that you must find a natural number that is the quotient of 19 and 3.

The distance by water between Pirates' Point and Walton Harbor is 19 mi. One day, Jerry sailed from Pirates' Point to Walton Harbor in 3 hr. He sailed an average of how many miles per hour?

D1

At noon on Tuesday, the temperature in Lake City was 10 degrees above zero. By 7 P.M., the temperature had dropped 15 degrees. What was the temperature at 7 P.M.?

D2

$\{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \ldots\}$

D3

K Is there a natural number that is the quotient of 19 and 3? Is there a natural number that satisfies $19 = 3x$? Is there a natural number that satisfies $19/3 \sim x/1$? Is there a natural number that provides an answer to the problem in D1?

L Next read the problem in D2. Again suppose that the universe is N. Is $10 - 15 = a$ the condition for the problem? Is there a natural number that satisfies $10 - 15 = a$? Is there a natural number that provides an answer to the problem?

There are no natural numbers that satisfy the conditions $19/3 \sim x/1$ and $10 - 15 = a$. To find answers to problems like the ones in D1 and D2, you must use other sets of numbers.

Another set of numbers you might use is the set of rational numbers of arithmetic, which you studied in *Book 1*. In the following exercises you will review some of the properties of the rational numbers of arithmetic. Then you will decide if this set of numbers provides answers to the problems in D1 and D2.

Look at D3. Each member of the set tabulated in D3 is a fraction. A fraction is an ordered pair of natural numbers in which the second component is not zero. The first component of such an ordered pair is the numerator of the fraction, and the second component is the denominator.

A What is the product of the numerator of $\frac{3}{5}$ and the denominator of $\frac{6}{10}$? What is the product of the numerator of $\frac{6}{10}$ and the denominator of $\frac{3}{5}$? Is $3 \times 10 = 6 \times 5$?

In *Book 1* you learned that, since $3 \times 10 = 6 \times 5$, the fractions $\frac{3}{5}$ and $\frac{6}{10}$ are equivalent.

B Show that $\frac{3}{5}$ and $\frac{9}{15}$ are equivalent fractions.

Show that $\frac{6}{10}$ and $\frac{9}{15}$ are equivalent fractions.

You will recall that numbers whose only common factor is 1 are relatively prime numbers, and that a fraction whose numerator and denominator are relatively prime is a basic fraction.

C Look again at D3. Explain why $\frac{3}{5}$ is a basic fraction. Is either $\frac{6}{10}$ or $\frac{9}{15}$ a basic fraction? How do you know?

D The set tabulated in D3 is the set whose members are $\frac{3}{5}$ and all fractions equivalent to $\frac{3}{5}$. How do you know that this set is a rational number of arithmetic?

For each rational number of arithmetic expressed in exercises E through H, name two more fractions that indicate the number.

E $\{\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \ldots\}$

F $\{\frac{5}{8}, \frac{10}{16}, \frac{15}{24}, \ldots\}$

G $\{\frac{9}{8}, \frac{18}{16}, \frac{27}{24}, \ldots\}$

H $\{\frac{10}{7}, \frac{20}{14}, \frac{30}{21}, \ldots\}$

In the following exercises you will see whether R_a is closed under the operations of addition and multiplication. First you will review how to find the product of two members of R_a.

Universe for a and c = N.
Universe for b and d = C.

A $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

B $\frac{a}{b} + \frac{c}{d} \doteq \frac{ad + bc}{bd}$.

D4

I Look at sentence A in D4. What is the universe for the variables? Sentence A expresses the definition of the product of two members of R_a. For each a, b, c, and d, $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

J Use the definition to find the product of $\frac{1}{2}$ and $\frac{3}{5}$. To find the product of $\frac{7}{3}$ and $\frac{9}{14}$. To find the product of $\frac{0}{4}$ and $\frac{11}{8}$. Is each of these products a member of R_a?

K Is the product of any two members of R_a ? also a member of R_a? Use the closure property of multiplication of natural numbers to explain your answer.

Next you will review how to find the sum of two members of R_a.

L Look at sentence B in D4. What is the universe for the variables? Sentence B expresses the sum property of members of R_a. For each a, b, c, and d, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

M Use the sum property to find the sum of $\frac{3}{4}$ and $\frac{2}{5}$. To find the sum of $\frac{7}{6}$ and $\frac{8}{9}$. To find the sum of $\frac{13}{4}$ and $\frac{5}{16}$. Is each of these sums a member of R_a?

N Is the sum of any two members of R_a also a ? member of R_a? Use the closure properties of N to explain your answer.

Your answers for exercises K and N should remind you that both the product and the sum of any two members of R_a are also members of R_a. The set of rational numbers of arithmetic is closed under addition and multiplication.

Before you decide whether or not R_a is closed under division, you will review some of the other properties of R_a. In *Book 1* you learned that, if the product of two members of R_a is 1, then each number is the reciprocal of the other.

O What is the product of $1\frac{2}{3}$ and $\frac{3}{5}$? Are $1\frac{2}{3}$ and $\frac{3}{5}$ reciprocals? How do you know that $\frac{12}{16}$ and $\frac{4}{3}$ are reciprocals?

The universe for x is Z. For each x, the reciprocal of x is $1 \div x$, or $1/x$. This is the reciprocal property of the rational numbers of arithmetic.

P What is the reciprocal of $\frac{9}{10}$? Of $3\frac{2}{3}$? Of 7?

To find the quotient of any two members of R_a, you can use the quotient property of the rational numbers of arithmetic and the reciprocal property. The quotient property is: For each x and y, $x \div y = x \cdot 1/y$. The universe for x is R_a, and the universe for y is Z. From the quotient property you know that, to find the quotient $\frac{5}{2} \div \frac{1}{4}$, for example, you find the product of $\frac{5}{2}$ and the reciprocal of $\frac{1}{4}$.

Q Is $1/\frac{1}{4}$ the reciprocal of $\frac{1}{4}$? Is $1/\frac{1}{4} = \frac{4}{1}$? Use the quotient property to find the quotient of $\frac{5}{2}$ and $\frac{1}{4}$. Is the number you obtain a member of R_a?

R What is the quotient of $\frac{11}{9}$ and $\frac{8}{15}$? Of $\frac{13}{4}$ and $\frac{25}{26}$? Is each quotient a member of R_a?

Remember that, if the divisor is a *non-zero* rational number of arithmetic, the quotient of any two members of R_a is also a member of R_a. Consequently, the set of non-zero rational numbers of arithmetic is closed under division.

S Read the problem in D1 again. Now suppose that the universe is R_a. Is $19/3 \sim x/1$ a condition for the problem? Is there a member of R_a that satisfies this condition? Can you use the solution set of the condition to get an answer to the problem? What is the answer?

Although you cannot find an answer to the problem in D1 when you use natural numbers, you can find an answer when you use rational numbers of arithmetic. This is possible because

the set of non-zero rational numbers of arithmetic is closed under division.

To decide whether or not R_a is closed under subtraction, you can use the relationship between subtraction and addition.

T How do you know that the difference $\frac{3}{8} - \frac{7}{8}$ is not a member of R_a?

U Is the difference of any two members of R_a always a member of R_a? Is R_a closed under subtraction?

V Read the problem in D2 again. Suppose that the universe is R_a. Is $10 - 15 = a$ the condition for the problem? Is there a member of R_a that satisfies the condition? Explain your answer. Can you get an answer to the problem in D2 by using rational numbers of arithmetic?

The distance by water between Pirates' Point and Walton Harbor is 19 mi. One day, Jerry sailed from Pirates' Point to Walton Harbor in 3 hr. He sailed an average of how many miles per hour?

D1

At noon on Tuesday, the temperature in Lake City was 10 degrees above zero. By 7 P.M., the temperature had dropped 15 degrees. What was the temperature at 7 P.M.?

D2

You have seen that neither the natural numbers nor the rational numbers of arithmetic provide answers to problems like the one in D2. This is because neither N nor R_a is closed under subtraction. In the lessons that follow, you will study a set of numbers that can be used to obtain answers to such problems. Before you begin to work with this set of numbers, you will review the idea of a number system.

In *Book 1* you learned that a number system consists of a set of objects called numbers, along with two operations. Each operation must be closed, commutative, and associative. One of the operations must distribute over the other.

As you answer the questions below, remember that addition and multiplication are binary operations. "Binary" means that you work with only two numbers at a time.

A Look at D5. Suppose that S is any set of objects. What is the universe for the variables in each condition expressed in D5?

B What two operations are named in D5? Which conditions tell you that S is closed under the two operations?

Universe for x, y, and z = S.

Operations ⟶	Addition	Multiplication
Closure properties ⟶	$x + y \in$ S.	$xy \in$ S.
Commutative properties ⟶	$x + y = y + x$.	$xy = yx$.
Associative properties ⟶	$(x + y) + z = x + (y + z)$.	$(xy)z = x(yz)$.
Distributive property ⟶	$x(y + z) = xy + xz$.	

D5

Universe for x, y, and z = S.		
Operations ⟶	Addition	Multiplication
Closure properties ⟶	$x + y \in$ S.	$xy \in$ S.
Commutative properties ⟶	$x + y = y + x$.	$xy = yx$.
Associative properties ⟶	$(x + y) + z = x + (y + z)$.	$(xy)z = x(yz)$.
Distributive property ⟶	$x(y + z) = xy + xz$.	

D5

C Which condition tells you that the order in which you add any two members of S does not affect the sum? Which condition tells you that the order in which you multiply any two members of S does not affect the product?

D Which conditions tell you that the way in which you group the members of S does not affect either the sum or the product?

E Which property relates the two operations?

F Does set S, together with the operations of addition and multiplication, form a number system?

G Now suppose that N is the universe for the variables in the conditions expressed in D5. How do you know that N, together with the operations of addition and multiplication, forms a number system?

H Next suppose that R_a is the universe for the variables in the conditions expressed in D5. How do you know that R_a, together with the operations of addition and multiplication, forms a number system?

In this lesson you learned that N and R_a do not contain solutions to conditions for certain problems. You also reviewed some of the properties of addition and multiplication of natural numbers and rational numbers of arithmetic.

On your own

For each condition expressed in exercises 1 through 10, first use N as the universe and tell whether or not there is a natural number that satisfies the condition. If not, explain why. Then use R_a as the universe and tell whether or not there is a rational number of arithmetic that satisfies the condition. If not, explain why.

1 $1095 + 8 = y$.
2 $752 - 753 = x$.
3 $1300 + 0 = n$.
4 $269 \div 2 = n$.
5 $25 \times 1 = q$.
6 $555 \div 3 = m$.
7 $0 - 1 = b$.
8 $10 \times 100 = c$.
9 $250 \div 75 = p$.
10 $692 - 659 = m$.

Name the basic fraction that indicates each rational number of arithmetic that is expressed in exercises 11 through 15.

11 $\frac{24}{27}$ **12** $\frac{15}{14}$ **13** $\frac{33}{18}$ **14** $\frac{30}{35}$ **15** $\frac{12}{30}$

For each of exercises 16 through 27, tell whether the sentence expresses a true statement or a false statement. For each true statement, name the property that you used to make your decision.

16 $\frac{2}{7}(\frac{1}{9} + \frac{3}{5}) = \frac{2}{7}(\frac{3}{5} + \frac{1}{9})$.
17 $\frac{2}{11} - \frac{5}{11} = \frac{5}{11} - \frac{2}{11}$.
18 $(\frac{1}{2} \times 6) \times \frac{13}{8} = (6 \times \frac{1}{2}) \times \frac{13}{8}$.
19 $17 \div 15.2 \in R_a$.
20 $100 - 320 \in R_a$.
21 $(1.09 + 2.1)8.5 = (1.09)8.5 + (2.1)8.5$.
22 $255 \div 25 = 25 \div 255$.
23 $.1(8 \times 10) = (.1 \times 8)10$.
24 $\frac{3}{4} + (8 \times \frac{1}{4}) = (\frac{3}{4} + 8)(\frac{3}{4} + \frac{1}{4})$.
25 $(\frac{6}{13} + \frac{4}{7}) + 1 = \frac{6}{13} + (\frac{4}{7} + 1)$.
26 $16(3 + \frac{4}{5}) = 16(\frac{4}{5} + 3)$.
27 $.5 + 1.3 \in R_a$.

The positive rational numbers

In lesson 126 you reviewed the system of natural numbers and the system of rational numbers of arithmetic. In this lesson you will begin the study of another system of numbers. The set of numbers in this system contains solutions of conditions like $10 - 15 = x$.

To introduce you to the new set of numbers, we will use a special kind of segment. Look at the picture of a number line in D1. The coördinates of the points in this line are rational numbers of arithmetic.

A What is the coördinate of point B? Of point F? Of point A?

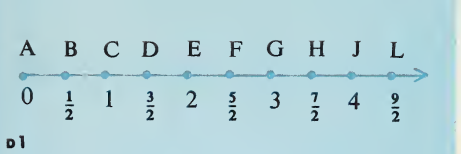
B Name the coördinates of the endpoints of $\overline{DG}$. What is the measure of $\overline{DG}$?

> REMINDER
>
> The measure of a segment is the absolute difference of the coördinates of the endpoints of the segment. If the coördinate of A is 2 and the coördinate of B is $4\frac{1}{5}$, then $m(\overline{AB}) = 2\frac{1}{5}$.
>
> *See lesson 98, page 5.*

C What is the measure of $\overline{CF}$? What is $m(\overline{FL})$?

D Which segments, whose endpoints are named in D1, have a measure of 3?

Next look at the picture of a number line in D2. You can use this number line to determine the measures of segments MN, RS, and TU, which are represented in D2.



D1

D2

directed segment (di rekt′əd). A segment in which one endpoint is chosen as the starting point and the other endpoint is chosen as the terminal point. A directed segment is directed from its starting point to its terminal point.

E Which endpoint of $\overline{MN}$ has the coördinate $\frac{1}{2}$? What is the coördinate of N? What is $m(\overline{MN})$?

F Is $\overline{NM}$ the same segment as $\overline{MN}$? What is $m(\overline{NM})$?

G How do you know that $m(\overline{RS}) = \frac{1}{2}$? What is $m(\overline{TU})$?

You can think about segments MN, RS, and TU in another way. Imagine that each segment represents a trip from one point to another point. If, for example, you start at point M and travel to point N, you travel a certain distance to the right. But if you start at point N and travel to point M, you travel a certain distance to the left.

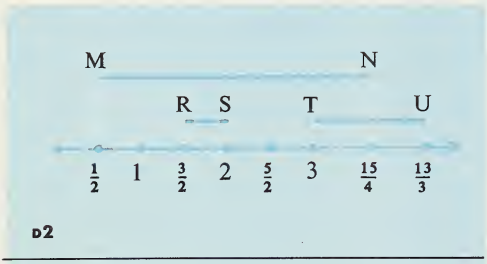
H In which direction are you traveling if you start at point U and travel to point T? If you start at R and travel to S? If you start at T and travel to U?

When you choose one endpoint of a segment as a *starting point* and the other endpoint as a *terminal* (tėr′mə nəl) *point*, the segment is a *directed segment*.

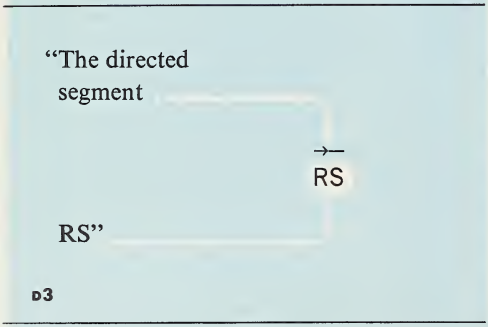
I D3 shows how to write and read a name of the directed segment that has R as the starting point and S as the terminal point. How does the symbol for a directed segment differ from the symbol for a segment?

J In the name of a directed segment, the first letter indicates the starting point and the second letter indicates the terminal point. Name the starting and terminal points of $\overrightarrow{DL}$.

K The arrow in a symbol like $\overrightarrow{DL}$ is not intended to show whether the segment is directed to the right or to the left. Look again at D2. How do you know that $\overrightarrow{SR}$ is directed to the left? How do you know that $\overrightarrow{TU}$ is directed to the right?



D2



D3

L Is $\overrightarrow{GW}$ the same as $\overrightarrow{WG}$? How are the two directed segments alike? How are they different?

M Write the name of the directed segment whose starting point is K and whose terminal point is A. Write the name of the directed segment whose terminal point is L and whose starting point is Z.

Each directed segment is directed either to the left or to the right. The endpoint that has the greater coördinate is always to the right of the endpoint that has the lesser coördinate.

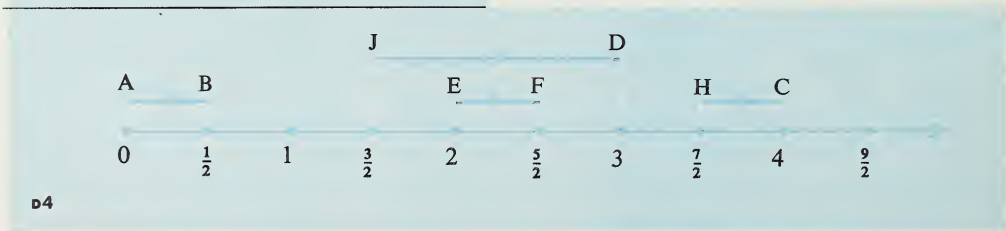
N Four directed segments are represented in D4. The arrows in the pictures of these segments show that you are to think of the segments as directed to the right. Name each of these directed segments.

From now on in this lesson, you will work only with directed segments that are directed to the right.

O How do you know that $m(\overrightarrow{AB}) = \frac{1}{2}$? What is $m(\overrightarrow{EF})$? Is $m(\overrightarrow{AB}) = m(\overrightarrow{EF})$? Do $\overrightarrow{AB}$ and $\overrightarrow{EF}$ have the same direction?

$\overrightarrow{AB}$ and $\overrightarrow{EF}$ have the same direction and the same measure. Directed segments that have the same direction and the same measure are *equivalent directed segments*. $\overrightarrow{AB} \sim \overrightarrow{EF}$ is a true statement. If you think of directed segments as representing trips, then equivalent directed segments represent trips of the same distance in the same direction.

P Look again at D4. How do you know that $\overrightarrow{HC}$ is equivalent to $\overrightarrow{AB}$? That $\overrightarrow{HC} \sim \overrightarrow{EF}$?



D4

Q Using the number line represented in D4, imagine a directed segment WV. Point V has a coördinate of $\frac{51}{2}$, and point W has a coordinate of 25. What is the direction of $\overrightarrow{WV}$? What is $m(\overrightarrow{WV})$? Explain your answers.

R Name the directed segments represented in D4 that are equivalent to $\overrightarrow{WV}$.

The directed segments AB, EF, and HC, represented in D4, are members of an infinite set of equivalent directed segments. Each member of this set is a directed segment that has a measure of $\frac{1}{2}$ and is directed to the right. Now you will learn about the number that is associated with this set.

A For each directed segment represented in D4, the coördinates of the endpoints form an ordered pair of rational numbers of arithmetic. Which component of $(\frac{5}{2}, 2)$ is the coördinate of the starting point of $\overrightarrow{EF}$? Which component is the coördinate of the terminal point?

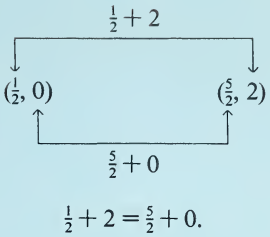
We say that $(\frac{5}{2}, 2)$ *determines* $\overrightarrow{EF}$. Each ordered pair of rational numbers of arithmetic determines exactly one directed segment. The first component of the ordered pair that determines the directed segment is the coördinate of the terminal point. The second component is the coördinate of the starting point.

B Which ordered pair, $(0, \frac{1}{2})$ or $(\frac{1}{2}, 0)$, determines $\overrightarrow{AB}$, represented in D4? How do you know?

You know that $\overrightarrow{AB} \sim \overrightarrow{EF}$. Now you will learn how $(\frac{1}{2}, 0)$ and $(\frac{5}{2}, 2)$, which determine $\overrightarrow{AB}$ and $\overrightarrow{EF}$, are related.

C Look at D5. What is the sum of the first component of $(\frac{1}{2}, 0)$ and the second component of $(\frac{5}{2}, 2)$? What is the sum of the first component of $(\frac{5}{2}, 2)$ and the second component of $(\frac{1}{2}, 0)$? Are these sums equal?

D Does the sentence in D5 express a true statement, or does it express a false statement?



D5

equivalent ordered pairs for directed segments. The universe for a, b, c, and d is R_a. For each a, b, c, and d, if $a + d = c + b$, then $(a, b) \sim (c, d)$. Also, if $(a, b) \sim (c, d)$, then $a + d = c + b$.

The ordered pairs that determine the equivalent directed segments AB and EF are *equivalent ordered pairs for directed segments*. You can use variables to define equivalent ordered pairs for directed segments. Suppose that (a, b) and (c, d) are any two ordered pairs for directed segments. The universe for a, b, c, and d is R_a. For each a, b, c, and d, if $a + d = c + b$, then $(a, b) \sim (c, d)$. Also, if $(a, b) \sim (c, d)$, then $a + d = c + b$.

Notice that this definition of equivalent ordered pairs is different from the definition you use when the ordered pairs are rate pairs or fractions. "Equivalent" has several meanings in mathematics. The definition given here applies to the ordered pairs that determine directed segments.

E Look again at D4. You know that $\overrightarrow{HC} \sim \overrightarrow{AB}$ and that $\overrightarrow{HC} \sim \overrightarrow{EF}$. Name the ordered pair that determines $\overrightarrow{HC}$. Use the definition of equivalent ordered pairs for directed segments to show that $(4, \frac{7}{2}) \sim (\frac{1}{2}, 0)$ and that $(4, \frac{7}{2}) \sim (\frac{5}{2}, 2)$.

F Is $\overrightarrow{JD}$ equivalent to $\overrightarrow{AB}$? What ordered pair determines $\overrightarrow{JD}$? Use the definition to find out if $(3, \frac{3}{2})$ is equivalent to $(\frac{1}{2}, 0)$. Is $3 + 0 = \frac{1}{2} + \frac{3}{2}$? Is $(3, \frac{3}{2}) \sim (\frac{1}{2}, 0)$ a true statement?

Universe for $(x, y) = R_a \times R_a$.
$(x, y) \sim (\frac{1}{2}, 0)$.

D6

G Suppose that two ordered pairs for directed segments are equivalent. What do you know about the directed segments determined by these ordered pairs?

H Suppose that two directed segments are not equivalent. What do you know about the ordered pairs that determine these directed segments?

I The directed segments that you have been studying are directed to the right. For each ordered pair that determines one of these segments, which component is the greater?

J Look at D6. What condition is expressed? Notice that the universe for (x, y) is $R_a \times R_a$. The Cartesian set $R_a \times R_a$ is an infinite set of ordered pairs of rational numbers of arithmetic. Both components of each member of $R_a \times R_a$ are rational numbers of arithmetic.

For any infinite set of equivalent directed segments, there is an infinite set of equivalent ordered pairs of rational numbers of arithmetic that determine these segments. $(\frac{1}{2}, 0)$, $(\frac{5}{2}, 2)$, and $(4, \frac{7}{2})$, which determine $\overrightarrow{AB}$, $\overrightarrow{EF}$, and $\overrightarrow{HC}$, are members of an infinite set of ordered pairs that satisfy the condition expressed in D6.

K Is $(3, \frac{5}{2})$ a member of the solution set of $(x, y) \sim (\frac{1}{2}, 0)$? Is $(\frac{11}{2}, 5)$ a member? Explain your answers. How do you know that $(\frac{1}{4}, \frac{3}{4})$ is not a member of $\{(x, y) | (x, y) \sim (\frac{1}{2}, 0)\}$?

L Does $(\frac{1}{2}, 0)$ satisfy $(x, y) \sim (\frac{1}{2}, 0)$? Is $(\frac{2}{4}, 0)$ the same ordered pair of rational numbers of arithmetic as $(\frac{1}{2}, 0)$? Is $(x, y) \sim (\frac{1}{2}, 0)$ satisfied by any other ordered pair with 0 as the second component? Explain your answers. Remember that the universe for (x, y) is $R_a \times R_a$.

In each set of equivalent ordered pairs of rational numbers of arithmetic, there is one pair that has 0 as at least one of its components. This pair is the *basic pair* for the set. For $\{(x, y) | (x, y) \sim (\frac{1}{2}, 0)\}$, the basic pair is $(\frac{1}{2}, 0)$. In this lesson, every basic pair has 0 as its *second* component. In the next lesson, you will study basic pairs that have 0 as the first component.

M Is $\{(x, y) | (x, y) \sim (\frac{1}{2}, 0)\}$ a finite set, or is it an infinite set?

$\{(x, y) | (x, y) \sim (\frac{1}{2}, 0)\}$ is a *positive rational number*. The members of a positive rational number determine equivalent directed segments that are directed to the right. The number *describes* each of these directed segments. Therefore, a positive rational number describes an infinite set of equivalent directed segments that are directed to the right.

A standard name of the positive rational number $\{(x, y) | (x, y) \sim (\frac{1}{2}, 0)\}$ is shown at the right. This standard name is read "positive one half." The symbol above and to the left of the symbol $\frac{1}{2}$ is read "positive."

$+\frac{1}{2}$

To obtain a standard name of a positive rational number from the basic pair for the number, you can use the name of the non-zero component. If, for example, the basic pair for a

basic pair. An ordered pair of rational numbers of arithmetic that has 0 as at least one of its components.

positive rational number (poz′ə tiv). An infinite set of equivalent ordered pairs of rational numbers of arithmetic, each of whose members determines a directed segment that is directed to the right. A positive rational number contains a basic pair whose second component is 0, along with all other ordered pairs that are equivalent to the basic pair.

positive rational number is (3, 0), the non-zero component is 3. A standard name of this positive rational number is the numeral $^{+}3$.

N What is a standard name of the positive rational number whose basic pair is $(\frac{11}{4}, 0)$?

O Name the basic pair for $^{+}\frac{29}{5}$.

P Is $^{+}\frac{1}{4}$ the same positive rational number as $^{+}\frac{2}{8}$? Explain your answer.

Q You know that $^{+}\frac{1}{2}$ describes each directed segment that has a measure of $\frac{1}{2}$ and is directed to the right. What is the measure of each directed segment described by $^{+}\frac{37}{8}$? What is the direction of these directed segments?

R What positive rational number describes a directed segment that has a measure of $\frac{7}{8}$ and is directed to the right?

You can use any member of a positive rational number to *indicate* the number. It is often convenient, however, to use the basic pair that is a member of the set.

S What basic pair indicates $^{+}\frac{2}{3}$? How do you know that $(\frac{5}{3}, 1)$ also indicates $^{+}\frac{2}{3}$?

T What is a standard name of the positive rational number that is indicated by (6, 0)?

In the following exercises, you will study the set of all positive rational numbers.

A Look at D7. What ordered pair determines $\overrightarrow{TR}$? What ordered pair determines $\overrightarrow{NK}$? How do you know that $\overrightarrow{TR} \sim \overrightarrow{NK}$?

B How do you know that both $(\frac{1}{3}, 0)$ and $(\frac{4}{3}, 1)$ are members of the positive rational number $\{(x, y) | (x, y) \sim (\frac{1}{3}, 0)\}$? The universe for (x, y) is $R_a \times R_a$.

C Which ordered pair, $(\frac{1}{3}, 0)$ or $(\frac{4}{3}, 1)$, is the basic pair for $\{(x, y) | (x, y) \sim (\frac{1}{3}, 0)\}$? What is a standard name of $\{(x, y) | (x, y) \sim (\frac{1}{3}, 0)\}$?

D Does $(\frac{1}{3}, 0)$ indicate $^{+}\frac{1}{3}$? Does $(\frac{4}{3}, 1)$ also indicate $^{+}\frac{1}{3}$?

E Look again at D7. Name the ordered pair that determines $\overrightarrow{SL}$. Is this ordered pair a basic pair? How do you know?

You can find the basic pair that is equivalent to (3, 2) by using a condition for equivalence. You know from the definition of equivalent ordered pairs that, if $(a, b) \sim (c, d)$, then $a + d = c + b$. The universe for each variable is R_a.

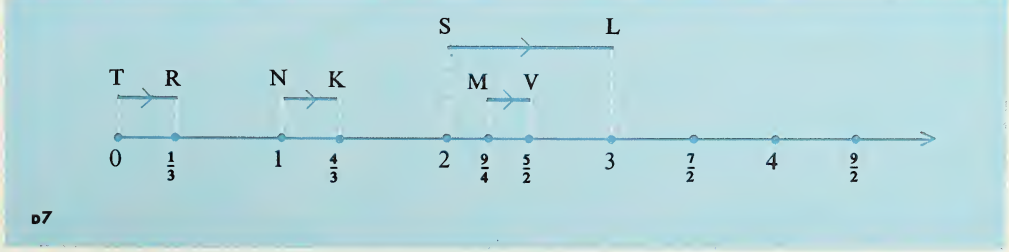
F How do you obtain $(a, 0) \sim (3, 2)$ from $(a, b) \sim (c, d)$?

G How do you know that $a + 2 = 3 + 0$ has the same solution as $(a, 0) \sim (3, 2)$? What is the solution?

H Name the basic pair that is equivalent to (3, 2). Is $\{(x, y) | (x, y) \sim (3, 2)\}$ the same positive rational number as $\{(x, y) | (x, y) \sim (1, 0)\}$? The universe for $(x, y) = R_a \times R_a$.

I What is a standard name of the positive rational number indicated by (3, 2)?

You can also find the basic pair that indicates the same positive rational number as (3, 2) by finding the absolute difference of the components of (3, 2). Remember that the absolute difference of two numbers is the number



D7

obtained when the lesser number is subtracted from the greater number.

J What is the absolute difference of the components of (3, 2)? Is the absolute difference the first component, or is it the second component, of the basic pair (1, 0)?

K The ordered pair $(\frac{5}{2}, \frac{9}{4})$ determines $\overrightarrow{MV}$, represented in D7. What is the absolute difference of the components of $(\frac{5}{2}, \frac{9}{4})$? What basic pair indicates the same positive rational number as $(\frac{5}{2}, \frac{9}{4})$?

L In an ordered pair that indicates a positive rational number, which component is the greater?

M Think about the set of all ordered pairs of rational numbers of arithmetic, in which each second component is 0 and each first component is not zero. Is this set a finite set, or is it an infinite set? Does each of these basic pairs indicate a different positive rational number?

N Now think about the set of all positive rational numbers. Is this set a finite set, or is it an infinite set?

We will use the symbol at the right below to name the set of positive rational numbers. This symbol is read "the set of positive rational numbers."

R_p

Give a standard name of each member of R_p that is described below. The universe for (x, y) is $R_a \times R_a$.

O $\{(x, y) | (x, y) \sim (\frac{7}{9}, 0)\}$

P $\{(x, y) | (x, y) \sim (\frac{2}{5}, 0)\}$

Q $\{(x, y) | (x, y) \sim (25, 0)\}$

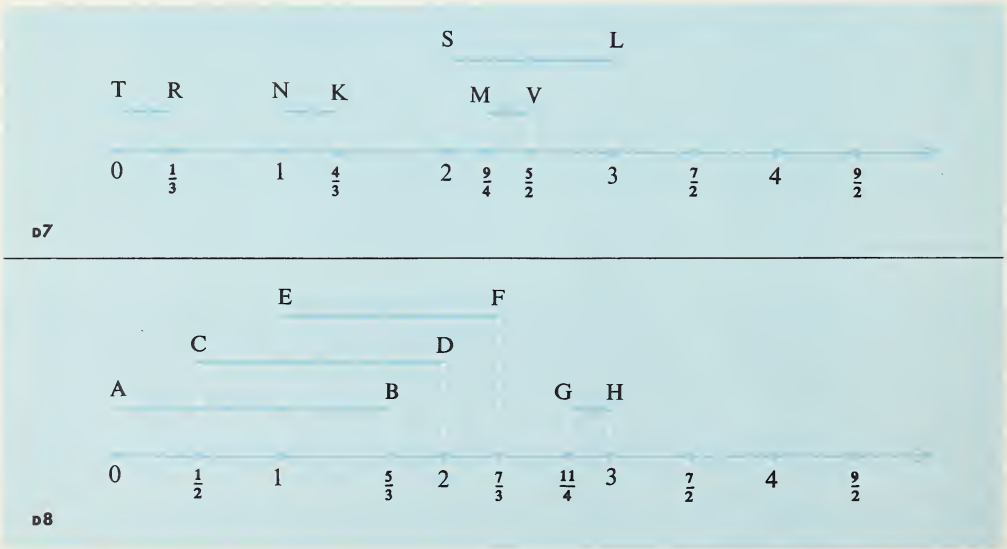
R $\{(x, y) | (x, y) \sim (8, \frac{3}{4})\}$

S $\{(x, y) | (x, y) \sim (100, 15)\}$

In this lesson you learned that directed segments are determined by ordered pairs of rational numbers of arithmetic. You also learned that a positive rational number is an infinite set of equivalent ordered pairs of rational numbers of arithmetic.

On your own

Use the number line and the directed segments represented in D8 in connection with



D7

D8

exercises 1 through 6. Each of the directed segments represented is directed to the right.

1 Name the ordered pair of rational numbers of arithmetic that determines $\overrightarrow{AB}$.

2 Give a standard description of the set of ordered pairs equivalent to the ordered pair you named for exercise 1. The universe for (x, y) is $R_a \times R_a$.

3 Give a standard name of the positive rational number that you described for exercise 2.

4 For each of the other directed segments represented in D8, name the ordered pair of rational numbers of arithmetic that determines the directed segment.

5 For each of the ordered pairs that you named for exercise 4, give a standard description of the set of ordered pairs equivalent to the given ordered pair. The universe for (x, y) is $R_a \times R_a$.

6 Give a standard name of each positive rational number that you described for exercise 5.

For exercises 7 through 11, first make one picture of a number line. Use rational numbers of arithmetic as the coördinates of the points in the line. Then, for each positive rational number named in exercises 7 through 11, make a picture of a directed segment described by the number.

7 $^{+}\frac{12}{5}$ 8 $^{+}\frac{5}{4}$ 9 $^{+}2$ 10 $^{+}\frac{2}{3}$ 11 $^{+}\frac{7}{2}$

Give a standard name of each member of R_p described below. The universe for (x, y) is $R_a \times R_a$.

12 $\{(x, y) | (x, y) \sim (\frac{4}{3}, 0)\}$

13 $\{(x, y) | (x, y) \sim (\frac{1}{12}, 0)\}$

14 $\{(x, y) | (x, y) \sim (\frac{5}{6}, 0)\}$

15 $\{(x, y) | (x, y) \sim (\frac{2}{3}, \frac{1}{6})\}$

16 $\{(x, y) | (x, y) \sim (\frac{13}{12}, 0)\}$

17 $\{(x, y) | (x, y) \sim (2, \frac{1}{2})\}$

Find the basic pair equivalent to each of the ordered pairs named below. Then give a standard name of the positive rational number indicated by each pair.

18 $(\frac{13}{2}, 3)$ 21 $(\frac{1}{5}, \frac{1}{8})$ 24 $(175, 16)$

19 $(\frac{1}{2}, \frac{3}{20})$ 22 $(7, \frac{2}{3})$ 25 $(\frac{23}{10}, 2)$

20 $(32, \frac{25}{2})$ 23 $(\frac{17}{2}, \frac{14}{3})$ 26 $(7, \frac{11}{2})$

KEEPING SKILFUL

Make one sketch for exercises 1 through 8. Then use this sketch as you answer the questions in exercises 9 through 18. The universe is the plane that includes circle A.

1 Make a sketch of ⊙A.

2 Make a sketch of diameter BC of ⊙A.

3 Locate D in ⊙A. Make a sketch of $\overline{AD}$.

4 Make a sketch of secant ℓ_1 whose intersection with ⊙A contains points B and D.

5 In ⊙A locate point E that is not in semicircle BDC.

6 Make a sketch of ℓ_2 that is tangent to circle A at point E.

7 Locate point F between A and B.

8 Locate point G in the exterior of ⊙A.

9 Name two chords of ⊙A.

10 Name three radii of ⊙A.

11 Write a standard description of ⊙A.

12 Write a standard description of the interior of ⊙A. Of the exterior of ⊙A.

13 Name the central angle of ⊙A that is determined by points B and D.

14 Tabulate the intersection of ℓ_2 and the interior of ⊙A.

15 Describe in words the intersection of ℓ_1 and the interior of ⊙A.

16 Describe the union of ⊙A and $\overline{AG}$.

17 Write a sentence that expresses a true statement comparing $\overline{AF}$ and $\overline{AB}$.

18 Write a sentence that expresses a true statement comparing $\overline{FC}$ and $\overline{AC}$.

The negative rational numbers; the set of rational numbers

In the last lesson you learned that a positive rational number describes each member of a set of equivalent directed segments that are directed to the right. In this lesson you will study sets of equivalent directed segments that are directed to the left.

Look at D1. The coördinates of the points in the number line represented in D1 are rational numbers of arithmetic. The arrows in the pictures in D1 indicate that you are to think of the directed segments QP, SR, UT, and WV as segments directed to the left.

A Which point is the starting point of $\overrightarrow{QP}$? Which point is the terminal point of $\overrightarrow{QP}$? Why is the symbol $\overrightarrow{QP}$, rather than the symbol $\overrightarrow{PQ}$, used to name this directed segment?

B Name the coördinate of the starting point of $\overrightarrow{QP}$. Name the coördinate of the terminal point. What is $m(\overrightarrow{QP})$?

C For each directed segment represented in D1, which is greater, the coördinate of the starting point or the coördinate of the terminal point?

D How do you know that $m(\overrightarrow{UT}) = \frac{3}{5}$?

E How do you know that $\overrightarrow{QP}$ and $\overrightarrow{UT}$ are equivalent directed segments?

F What is $m(\overrightarrow{SR})$? Is $\overrightarrow{SR} \sim \overrightarrow{QP}$? Is $\overrightarrow{SR} \sim \overrightarrow{UT}$?

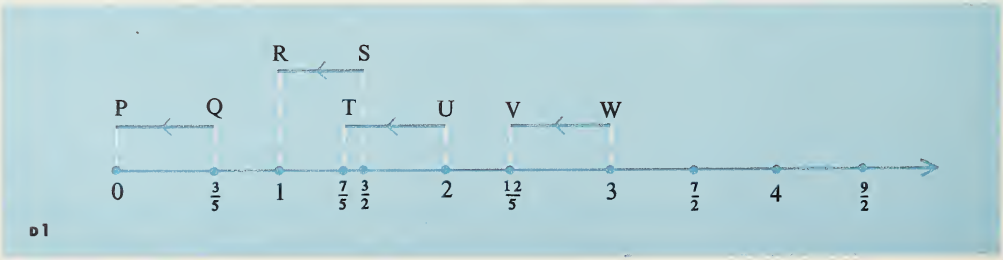
G What is $m(\overrightarrow{WV})$? Is $\overrightarrow{WV} \sim \overrightarrow{QP}$? Is $\overrightarrow{WV} \sim \overrightarrow{UT}$?

H Think about the set of directed segments, each of which is directed to the left and has a measure of $\frac{3}{5}$. Is this set a finite set, or is it an infinite set?

In the last lesson, you studied certain ordered pairs of rational numbers of arithmetic that determine directed segments that are directed to the right. In this lesson, you will study the ordered pairs of rational numbers of arithmetic that determine directed segments that are directed to the left.

I Look again at D1. $\overrightarrow{QP}$ is the directed segment that is determined by $(0, \frac{3}{5})$. Which component of $(0, \frac{3}{5})$ is the coördinate of the terminal point of $\overrightarrow{QP}$? Which component is the coördinate of the starting point of $\overrightarrow{QP}$? Notice that, just as for directed segments that are directed to the right, the first component of the ordered pair is the coördinate of the terminal point, and the second component is the coördinate of the starting point.

J Name the ordered pair of rational numbers of arithmetic that determines $\overrightarrow{UT}$. Use the definition of equivalent ordered pairs for directed segments to show that $(\frac{7}{5}, 2) \sim (0, \frac{3}{5})$.



D1

Universe for $(x, y) = R_a \times R_a$.
$\{(x, y) | (x, y) \sim (0, \frac{3}{5})\}$

D2

negative rational number (neg′ə tiv). An infinite set of equivalent ordered pairs of rational numbers of arithmetic, each of whose members determines a directed segment that is directed to the left. A negative rational number contains a basic pair whose first component is 0, along with all other ordered pairs that are equivalent to the basic pair.

K Name the ordered pair of rational numbers of arithmetic that determines $\overrightarrow{WV}$. Is the ordered pair that you just named equivalent to $(0, \frac{3}{5})$? Is it equivalent to $(\frac{7}{5}, 2)$? How do you know?

L Name the ordered pair of rational numbers of arithmetic that determines $\overrightarrow{SR}$. Show that this ordered pair is not equivalent to $(0, \frac{3}{5})$.

M Now read the standard description in D2. $(0, \frac{3}{5})$, $(\frac{7}{5}, 2)$, and $(\frac{12}{5}, 3)$ are all members of $\{(x, y) | (x, y) \sim (0, \frac{3}{5})\}$. Is $(1, \frac{3}{2})$ a member of this set? Is $(\frac{3}{5}, 0)$ a member? Is $(5, \frac{28}{5})$ a member? Explain your answers.

N Is $\{(x, y) | (x, y) \sim (0, \frac{3}{5})\}$ a finite set, or is it an infinite set?

O How do you know that $(0, \frac{3}{5})$ is a basic pair? How do you know that $(0, \frac{9}{15})$ is the same ordered pair as $(0, \frac{3}{5})$?

P How many basic pairs are members of the solution set of $(x, y) \sim (0, \frac{3}{5})$? How do you know?

$\{(x, y) | (x, y) \sim (0, \frac{3}{5})\}$ is a *negative rational number*. The members of a negative rational number determine equivalent directed segments that are directed to the left. The number describes each of these directed segments. Therefore, a negative rational number describes an infinite set of equivalent directed segments that are directed to the left.

A standard name of the negative rational number $\{(x, y) | (x, y) \sim (0, \frac{3}{5})\}$ is shown at the right. This standard name is read "negative three fifths." The symbol that is above and to the left of the symbol $\frac{3}{5}$ is read "negative."

$$\frac{-3}{5}$$

You can obtain a standard name of a negative rational number from the basic pair for the number by using the name of the non-zero component. If, for example, the basic pair for a negative rational number is (0, 6), the non-zero component is 6. A standard name of this negative rational number is the numeral $^-6$.

Q What is a standard name of the negative rational number whose basic pair is $(0, \frac{9}{8})$?

R Name the basic pair of $^-11$.

S How do you know that $\frac{-2}{3}$ is the same negative rational number as $\frac{-4}{6}$?

T You know that $\frac{-3}{5}$ describes each directed segment that has a measure of $\frac{3}{5}$ and is directed to the left. What is the measure of each directed segment described by $\frac{-17}{10}$? What is the direction of each directed segment described by $\frac{-17}{10}$?

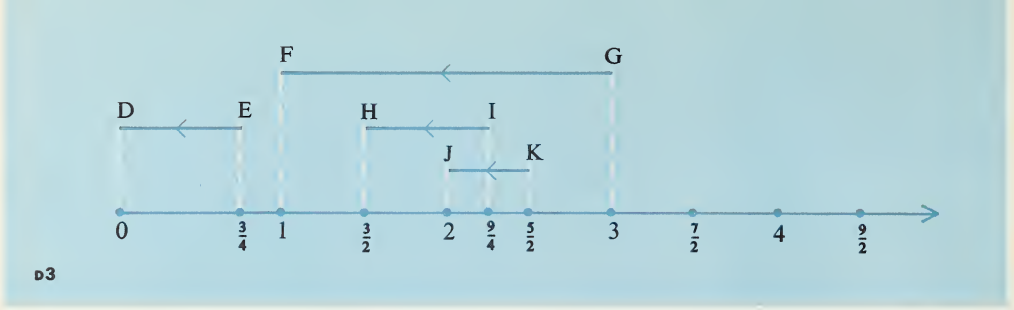
U What negative rational number describes each directed segment that has a measure of $\frac{7}{4}$ and is directed to the left?

You can use any member of a negative rational number to indicate the number. It is often convenient, however, to use the basic pair.

V What basic pair indicates $\frac{-1}{3}$? Explain how you know that $(2, \frac{7}{3})$ also indicates $\frac{-1}{3}$.

W What is a standard name of the negative rational number indicated by $(0, \frac{11}{12})$?

In the preceding exercises, you learned what a negative rational number is. Now you will examine the set of all negative rational numbers.



D3

A Look at D3. Name the ordered pair that determines $\overrightarrow{ED}$. Name the ordered pair that determines $\overrightarrow{IH}$. How do you know that $\overrightarrow{IH} \sim \overrightarrow{ED}$?

B How do you know that both $(0, \frac{3}{4})$ and $(\frac{3}{2}, \frac{9}{4})$ are members of $\{(x, y) | (x, y) \sim (0, \frac{3}{4})\}$? The universe for (x, y) is $R_a \times R_a$. What is a standard name of $\{(x, y) | (x, y) \sim (0, \frac{3}{4})\}$?

C Name the ordered pair that determines $\overrightarrow{GF}$. Is this ordered pair a basic pair? How do you know?

In lesson 127 you learned how to find the basic pair for a positive rational number when you know another ordered pair that is a member of the same number. Now you will learn how to find the basic pair for a negative rational number.

D How do you know that (1, 3) indicates a negative rational number rather than a positive rational number? Does the basic pair that is equivalent to (1, 3) indicate the same negative rational number as (1, 3)?

You can find the basic pair that is equivalent to (1, 3) by using the definition of equivalent ordered pairs for directed segments. The universe for each variable in $(a, b) \sim (c, d)$ is R_a.

E How do you obtain $(0, b) \sim (1, 3)$ from $(a, b) \sim (c, d)$? What does $(0, b)$ refer to? Why do you use $(0, b)$, and not $(b, 0)$?

F How do you know that $0 + 3 = 1 + b$ has the same solution as $(0, b) \sim (1, 3)$? What is the solution?

G What basic pair is equivalent to (1, 3)? Is $\{(x, y) | (x, y) \sim (1, 3)\}$ the same negative rational number as $\{(x, y) | (x, y) \sim (0, 2)\}$? The universe for (x, y) is $R_a \times R_a$.

H What is a standard name of the negative rational number indicated by (1, 3)?

You can also find the basic pair that indicates the same negative rational number as (1, 3) by finding the absolute difference of the components of (1, 3).

I What is the absolute difference of the components of (1, 3)? Is the absolute difference the first component, or is it the second component, of (0, 2)?

J Look again at D3. Name the ordered pair that determines $\overrightarrow{KJ}$.

K How can you use the absolute difference of the components of $(2, \frac{5}{2})$ to find the basic pair that is equivalent to $(2, \frac{5}{2})$?

L Which number does $(\frac{3}{16}, \frac{5}{16})$ indicate, a positive rational number or a negative rational number? How do you know?

M In an ordered pair that indicates a negative rational number, which component is the greater?

N Think about the set of all ordered pairs of rational numbers of arithmetic that have 0 as the first component and that have a number that is not zero as the second component. Which is this set, a finite set or an infinite set? Does each of these basic pairs indicate a different negative rational number?

We will use the symbol at the right below to name the set of all negative rational numbers. This symbol is read "the set of negative rational numbers."

Give a standard name of each member of R_n that is described below. The universe for (x, y) is $R_a \times R_a$.

O $\{(x, y) | (x, y) \sim (0, \frac{5}{6})\}$

P $\{(x, y) | (x, y) \sim (0, 15)\}$

Q $\{(x, y) | (x, y) \sim (0, \frac{1}{4})\}$

R $\{(x, y) | (x, y) \sim (5, 9)\}$

S $\{(x, y) | (x, y) \sim (\frac{5}{12}, \frac{1}{2})\}$

You have observed that, in an ordered pair that indicates either a positive rational number or a negative rational number, the components are different numbers. For positive numbers, the first components are greater. For negative numbers, the second components are greater. There is also a number that is indicated by an ordered pair in which both components are the same number. Now you will study the number indicated by such a pair.

A Read the standard description in D4. Use the definition of equivalent ordered pairs for directed segments to show that $(\frac{1}{2}, \frac{1}{2})$ is a member of $\{(x, y) | (x, y) \sim (0, 0)\}$. Show that (3, 3) is a member of the set. How do you know that $(\frac{6}{7}, \frac{6}{7})$ is a member?

B What do you know about the first and second components of each member of the solution set of $(x, y) \sim (0, 0)$?

C Is $\{(x, y) | (x, y) \sim (0, 0)\}$ a finite set, or is it an infinite set? How do you know?

D How do you know that (0, 0) is the basic pair for $\{(x, y) | (x, y) \sim (0, 0)\}$?

The *rational number zero* is the infinite set of equivalent ordered pairs of rational numbers of arithmetic whose first and second components are the same number. That is, the rational number zero is $\{(x, y) | (x, y) \sim (0, 0)\}$. You may use the symbol 0 to name this number.

Universe for $(x, y) = R_a \times R_a$.
$\{(x, y) | (x, y) \sim (0, 0)\}$

D4

rational number. An infinite set of equivalent ordered pairs of rational numbers of arithmetic. A rational number contains a basic pair and all other ordered pairs that are equivalent to the basic pair.

E Is $0 \in R_p$? Is $0 \in R_n$? Explain your answers.

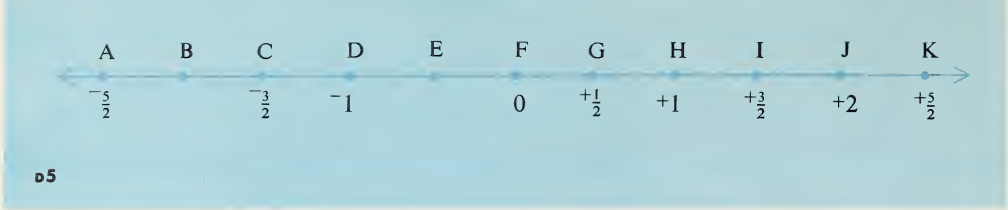
F Think about the set of all possible basic pairs. Is (0, 5) a member of this set? Is $(\frac{11}{3}, 0)$ a member of this set? Is (0, 0) a member? Is this set of basic pairs a finite set, or is it an infinite set?

G Now think about the set that consists of all the negative rational numbers, all the positive rational numbers, and the rational number 0. The set just described is the set of *rational numbers*. Each rational number is an infinite set of ordered pairs. Is the set of rational numbers a finite set, or is it an infinite set?

We will use the symbol at the right below to name the set of rational numbers. This symbol is read "the set of rational numbers."

H Is $R = R_p \cup R_n \cup \{0\}$? Tabulate the intersection of R_p, R_n, and $\{0\}$.

In *Book 1* we developed the set of rational numbers of arithmetic by forming infinite sets of equivalent ordered pairs of natural numbers. Now we have developed the set of rational numbers by forming infinite sets of equivalent ordered pairs of rational numbers of arithmetic. In each case, we developed a new set of numbers from a familiar set.



D5

Now look at D5. The number line represented is a *rational-number line*. The coördinates of the points in this line are rational numbers, not rational numbers of arithmetic. Notice that, in a rational-number line, coördinates are assigned to points to the left of point 0 as well as to points to the right of point 0.

I How do the coördinates of points to the right of point 0 differ from the coördinates of points to the left of point 0?

J Is point $^{+}\frac{8}{3}$ to the right of point 0, or is it to the left of point 0? Is point $^{-}2$ to the left of point 0, or is it to the right of point 0?

K Each point in a rational-number line is associated with a different rational number. Is a rational-number line a finite set of points? An infinite set of points?

L In D5, what do the two arrows tell you about the points in a rational-number line?

M What is the coördinate of point G? What is the coördinate of point F?

N Now suppose that you want to find the coordinate of point E. If $\overline{FG}$ is congruent to $\overline{FE}$, what do you think is the coördinate of point E?

O Look again at D5. $\overline{FB} \cong \overline{FJ}$. What is the coördinate of point B?

You know that, in the rational-number line represented in D5, $^{+}1$ is associated with point H. $^{+}1$ is also associated with $\overrightarrow{FH}$ and with each directed segment that is equivalent to $\overrightarrow{FH}$.

P What rational number is associated with each directed segment that is equivalent to $\overrightarrow{FA}$?

Q Name a directed segment whose starting point is F and that is associated with $^{+}\frac{5}{2}$.

R In a rational-number line like the one represented in D5, how do the directed segments associated with $^{-}3$ differ from the directed segments associated with $^{+}3$?

In this lesson you have studied the set of negative rational numbers, the rational number zero, and the set of all rational numbers. You have also learned that the rational numbers can be associated with both points and directed segments in a number line.

On your own

Use the number line and the directed segments represented in D6 in connection with exercises 1 through 6. Each of the directed segments is directed to the left. The coördinates of the points in the number line are rational numbers of arithmetic.

1 Name the ordered pair that determines $\overrightarrow{BA}$.

2 Give a standard description of the set of all ordered pairs equivalent to the ordered pair you named for exercise 1. $U = R_a \times R_a$.

3 Give a standard name of the rational number that you described for exercise 2.

4 Name the ordered pair that determines each of the other directed segments represented in D6.

5 For each of the ordered pairs that you named for exercise 4, give a standard description of the set of all ordered pairs equivalent to the given ordered pair. $U = R_a \times R_a$.

6 Give a standard name of each rational number that you described for exercise 5.

Give a standard name of each rational number that is described in exercises 7 through 12. The universe for (x, y) is $R_a \times R_a$.

7 $\{(x, y) | (x, y) \sim (0, 10)\}$

8 $\{(x, y) | (x, y) \sim (0, \frac{4}{9})\}$

9 $\{(x, y) | (x, y) \sim (\frac{6}{5}, \frac{6}{5})\}$

10 $\{(x, y) | (x, y) \sim (104, 83)\}$

11 $\{(x, y) | (x, y) \sim (\frac{13}{7}, 0)\}$

12 $\{(x, y) | (x, y) \sim (12, 20)\}$

Find the basic pair that is equivalent to each of the ordered pairs named in exercises 13 through 20. Then give a standard name of the rational number indicated by each pair.

13 $(\frac{3}{4}, 1)$ **15** $(1, 8)$ **17** $(1, 1)$ **19** $(\frac{1}{8}, \frac{1}{5})$

14 $(12, 9)$ **16** $(\frac{1}{2}, \frac{9}{10})$ **18** $(9, 2)$ **20** $(\frac{9}{16}, \frac{9}{16})$

Use the rational-number line represented in D7 when you work exercises 21 through 26.

21 What is the coördinate of point B? Of point E? Of point F?

22 $^+\frac{3}{2}$ is the coördinate of which point? 0 is the coördinate of which point? $^-\frac{5}{4}$ is the coördinate of which point?

23 $\overline{GI} \cong \overline{DG}$. Name the coördinate of point I.

24 $\overline{DE} \cong \overline{DC}$. Name the coördinate of point C.

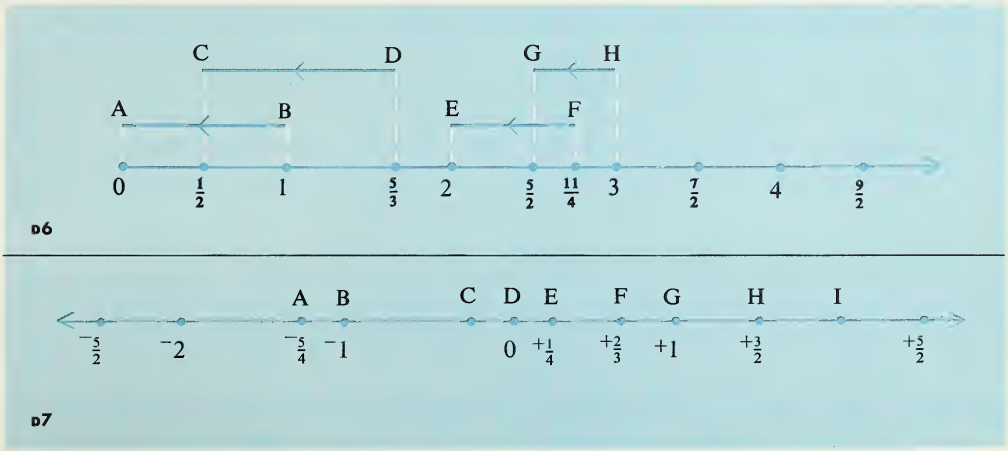
25 What rational number is associated with $\overrightarrow{DF}$? With $\overrightarrow{DA}$? With $\overrightarrow{DI}$?

26 If D is the starting point, name the directed segment associated with $^+\frac{3}{2}$. Name the directed segment associated with $^-\frac{1}{4}$. Name the directed segment associated with $^+1$.

27 The positive and negative rational numbers are often used to represent vertical distances above and below sea level. Lake Victoria in Africa is 3720 ft. above sea level. Lake Eyre in Australia is 39 ft. below sea level. What rational numbers could you use to represent these distances?

28 The positive and negative rational numbers are often used to represent temperatures above and below zero. What rational number could you use to represent a temperature of 15 degrees below zero? A temperature of 212 degrees above zero?

29 Suppose that the negative rational numbers are used to represent the number of seconds before a missile launching and that the positive rational numbers are used to represent the number of seconds after a missile launching. What rational number could you use to represent 5 seconds before the launching? What rational number could you use to represent 10 seconds after the launching?



D6

D7

10 129 Exploring ideas

Ordering of the rational numbers

In *Book 1* you learned how to order the natural numbers and also the rational numbers of arithmetic according to the idea of "less than." In this lesson you will learn how to order the rational numbers.

A Look at the picture of a rational-number line in D1. Is each point whose coördinate is a positive rational number located to the right of point 0? To the left of point 0? Notice the dots for the points whose coördinates are negative rational numbers. Where are these points located with respect to point 0?

B Is point $^{+}\frac{1}{4}$ to the right of point $^{+}1$? Is it to the left of point $^{+}1$?

Point $^{+}\frac{1}{4}$ is to the left of point $^{+}1$. Therefore, $^{+}\frac{1}{4} < {}^{+}1$ is a true statement. In a rational-number line, if the point associated with one rational number is to the left of the point associated with a second rational number, then the first number is less than the second number.

C What basic pair indicates the number $^{-}\frac{3}{4}$? How do you know that $(\frac{1}{2}, \frac{5}{4})$ also indicates $^{-}\frac{3}{4}$?

A rational number may be expressed by the name of any ordered pair that indicates the rational number.

D Is $(0, \frac{3}{4}) = {}^{-}\frac{3}{4}$? Is $(\frac{1}{2}, \frac{5}{4}) = {}^{-}\frac{3}{4}$? Is $(0, \frac{3}{4}) = (\frac{1}{2}, \frac{5}{4})$? Explain your answers.

E Look again at D1. Is $^{-}1 < {}^{-}\frac{1}{4}$? Is $(0, \frac{1}{2}) < (1, 0)$? Explain your answers.

F How do you know that 0 is less than any positive rational number? How do you know that any negative rational number is less than 0?

Now look at D2. Sentence A expresses a true statement. $^{-}\frac{7}{8} < {}^{+}\frac{1}{4}$ because point $^{-}\frac{7}{8}$ is to the left of point $^{+}\frac{1}{4}$. You do not need a rational-number line, however, to tell whether or not $^{-}\frac{7}{8}$ is less than $^{+}\frac{1}{4}$. The work that follows explains one way to order rational numbers without using a rational-number line.

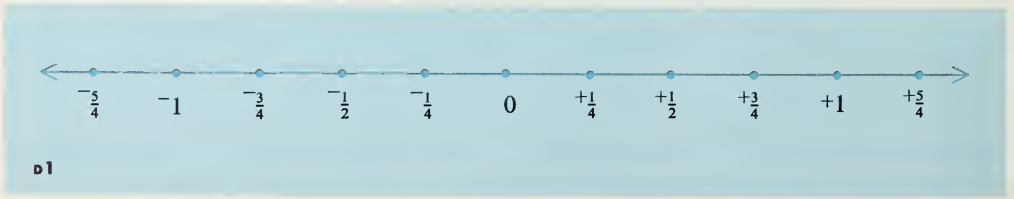
G Does sentence B in D2 express the same true statement as sentence A?

H Look at diagram C in D2. The arrows show that the first component of $(0, \frac{7}{8})$ has been added to the second component of $(\frac{1}{4}, 0)$. What is the sum of these components?

I Look at diagram D in D2. Which components of the two ordered pairs have been added? What is the sum of $\frac{1}{4}$ and $\frac{7}{8}$?

J Now look at sentence E. How do you know that $0 + 0$ and $\frac{1}{4} + \frac{7}{8}$ are rational numbers of arithmetic? Does sentence E express a true statement?

The sum of the first component of $(0, \frac{7}{8})$ and the second component of $(\frac{1}{4}, 0)$ is less than the sum of the first component of $(\frac{1}{4}, 0)$ and the second component of $(0, \frac{7}{8})$. Therefore, the rational number $(0, \frac{7}{8})$, or $^{-}\frac{7}{8}$, is less than the rational number $(\frac{1}{4}, 0)$, or $^{+}\frac{1}{4}$.



D1

A $\frac{-7}{8} < \frac{+1}{4}$.

B $(0, \frac{7}{8}) < (\frac{1}{4}, 0)$.

C $0 + 0$

$(0, \frac{7}{8})$ $(\frac{1}{4}, 0)$

D $(0, \frac{7}{8})$ $(\frac{1}{4}, 0)$

$\frac{1}{4} + \frac{7}{8}$

E $0 + 0 < \frac{1}{4} + \frac{7}{8}$.

D2

F $\frac{+2}{5} < \frac{+4}{15}$.

G $(\frac{2}{5}, 0) < (\frac{4}{15}, 0)$.

H $(\frac{2}{5}, 0)$ $(\frac{4}{15}, 0)$

I $\frac{2}{5} + 0 < \frac{4}{15} + 0$.

D3

Universe for a, b, c, and $d = R_a$.

J $(a, b) < (c, d)$.

K $(0, \frac{2}{3}) < (0, \frac{1}{2})$.

L $a + d < c + b$.

M $0 + \frac{1}{2} < 0 + \frac{2}{3}$.

D4

"less than" for rational numbers. The universe for a, b, c, and d is R_a. For each a, b, c, and d, if $a + d < c + b$, then $(a, b) < (c, d)$.

K Look at D3. Do sentence F and sentence G express the same statement? Explain your answer.

L Look at diagram H and sentence I in D3. Which components of the basic pairs were added to obtain $\frac{2}{5} + 0$? To obtain $\frac{4}{15} + 0$? Does sentence I express a true statement?

Because statement I is false, you know that the statement expressed by sentences F and G is also false.

You can use variables to help you develop the definition of "less than" for rational numbers.

A Look at D4. What is the universe for a, b, c, and d?

B What replacements were made in condition J to obtain statement K? In condition L to obtain statement M?

C Is statement M true? How do you know that $\frac{-2}{3} < \frac{-1}{2}$?

Now we can say exactly what we mean by the idea of "*less than*" *for rational numbers.* For each a, b, c, and d, if $a + d$ is less than $c + b$, then (a, b) is less than (c, d). The universe for each variable is R_a.

D How can you obtain $(0, \frac{3}{4}) < (0, \frac{5}{8})$ from $(a, b) < (c, d)$? When you make the same replacements in $a + d < c + b$, do you obtain a true statement?

E Is $\frac{-3}{4} < \frac{-5}{8}$ a true statement?

Tell whether each sentence below expresses a true statement or a false statement. Explain your answers.

F $+4 < +3$.

H $-9 < \frac{+1}{2}$.

J $\frac{+3}{4} < \frac{-3}{2}$.

G $\frac{-5}{6} < \frac{-2}{3}$.

I $-4 < -8$.

K $\frac{+8}{3} < +5$.

L The inequality $+3 > +2$ gives you the same information as $+2 < +3$. What inequality that involves the idea of "greater than" gives you the same information as $\frac{+1}{3} < \frac{+2}{5}$?

M What inequality that involves the idea of "less than" gives you the same information as $(0, \frac{1}{4}) > (0, \frac{1}{2})$? Is $\frac{-1}{2} < \frac{-1}{4}$? Is $\frac{-1}{4} > \frac{-1}{2}$? Explain your answers.

You can see that the idea of "greater than" can be explained by using the idea of "less than." Thus $(c, d) > (a, b)$ means $(a, b) < (c, d)$.

Tell whether each sentence below expresses a true statement or a false statement. Explain your answers.

N $\frac{+3}{4} > \frac{+1}{2}$.

O $\frac{-5}{8} > \frac{-1}{4}$.

P $\frac{-7}{8} > \frac{+5}{6}$.

Q Use the definition of the idea of "less than" ? that has just been developed to show that any negative rational number is less than any positive rational number.

Now you know how to order the rational numbers according to both the idea of "greater than" and the idea of "less than." Next you will study three important subsets of the set of rational numbers.

A In *Book 1* you learned that $\{\frac{0}{1}, \frac{1}{1}, \frac{2}{1}, \ldots\}$ is a particular subset of the set of rational numbers of arithmetic. Each member of this subset is a *mate* of a natural number. The natural number 0 is the mate of $\frac{0}{1}$, which is a rational number of arithmetic. What member of N is the mate of $\frac{5}{1}$? Use a fraction numeral to name the mate of the natural number 7.

The set of mates of the natural numbers behaves in the same way as the set of natural numbers with respect to the operations of addition and multiplication. For this reason, you can use the numerals 0, 1, 2, and so on to name the mates that are members of R_a.

B Now think about the set of rational numbers whose basic pairs have first and second components that are mates of natural numbers. Is $^-5$ a member of this set? Is (16, 0) a member of this set? Is $\frac{^-7}{8}$ a member? Is (0, 0) a member? Explain your answers.

The set described in exercise B is the *set of integers* (in′tə jərz). We will use the symbol shown at the right as a name of the set of integers.

C Is $^-15 \in I$? Is $^+23 \in I$? Is $(0, 118) \in I$? Name three rational numbers that are not members of I.

D How do you know that I is a subset of R?

E Look at the tabulation in D5. Is each member of the set a member of R_p? Is each member a member of I?

$\{^+1, ^+2, ^+3, \ldots\}$

D5

$\{^-1, ^-2, ^-3, \ldots\}$

D6

"The set whose members are

$\{\ldots, ^-3, ^-2, ^-1, 0, ^+1, ^+2, ^+3, \ldots\}$

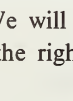
the negative integers,

zero,

and the positive integers"

D7

The set tabulated in D5 is the *set of positive integers*. We will use the symbol shown at the right as a name of this set.



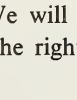
F Is $^-52 \in I_p$? Is $(\frac{4}{5}, 0) \in I_p$? Is $^+\frac{7}{1} \in I_p$?

G Does I_p contain all the positive rational numbers? How do you know that I_p is a subset of R? A subset of R_p? A subset of I?

H Name the least member of I_p. Does I_p have a greatest member?

I Look at the tabulation in D6. Is each member of the set a negative rational number? Is each member an integer?

The set tabulated in D6 is the *set of negative integers*. We will use the symbol shown at the right as a name of this set.



J Is $^-83 \in I_n$? Is $^+6 \in I_n$? Is $\frac{^-7}{2} \in I_n$? Explain your answers.

K How do you know that I_n is a subset of R? A subset of R_n? A subset of I?

L Does I_n have a least member? What is the greatest member of I_n?

M Name the integer that is neither a member of I_p nor a member of I_n.

N What set is the same as $I_p \cup I_n \cup \{0\}$? Tabulate $I_p \cap I_n \cap \{0\}$.

D7 shows how to write and read a tabulation of the set of all integers. Study D7.

O In D7, what do the three dots to the right of the numeral $^+3$ indicate? What do the three dots to the left of the numeral $^-3$ indicate?

P Is there a least integer? A greatest integer?

In this lesson you learned that you can use the idea of "less than" to order the rational numbers. You also studied the set of integers.

On your own

Give a standard name of each rational number expressed in exercises 1 through 4.

1 (0, 2) **2** $(\frac{15}{16}, 0)$ **3** (8, 5) **4** $(1, \frac{5}{8})$

Name the basic pair for each rational number expressed in exercises 5 through 8.

5 $^-3$ **6** 0 **7** $^+\frac{2}{3}$ **8** $^-\frac{3}{4}$

For each of exercises 9 through 12, list the names of the rational numbers in order according to the idea of "less than." Begin with the name of the least number.

9 $^+\frac{5}{6}, ^+\frac{7}{8}, ^+\frac{2}{3}, ^-\frac{5}{8}$ **11** $^+\frac{4}{3}$, 0, $^-\frac{5}{3}, ^+\frac{1}{3}, ^+1, ^-\frac{4}{3}$

10 $^-\frac{1}{2}, ^-\frac{3}{2}$, 0, $^-1, ^-2$ **12** $^+1, ^-1, ^-2, ^+3, ^+2, ^-3$

Tell whether each sentence in exercises 13 through 18 expresses a true statement or a false statement.

13 $0 < ^-\frac{1}{2}$. **15** $^-\frac{2}{5} > ^-\frac{3}{5}$. **17** $^-\frac{1}{2} > ^-\frac{3}{2}$.

14 $^+\frac{5}{8} < ^+1$. **16** $^-\frac{3}{4} > ^+\frac{1}{4}$. **18** $^+\frac{1}{3} < ^+\frac{1}{4}$.

Use set S, tabulated below, for exercises 19 through 24.

$S = \{^-2, ^+\frac{8}{9}, ^+3, ^-\frac{2}{3}, ^-\frac{3}{4}, 0, ^-\frac{6}{2}\}$.

19 Tabulate $S \cap I_n$.
20 Tabulate $I \cap S$.
21 Tabulate $I_p \cap S$.
22 Tabulate $S \cap R$.
23 Tabulate $R_p \cap S$.
24 Tabulate $R_n \cap S$.

SPECIAL CHALLENGE

In lesson 67 of *Book 1*, you studied a finite number system that was formed from a finite set together with two operations, addivision and muldivision. Now you will see how conditions may be used with a finite number system.

The display below shows a part of an addivision and of a muldivision table for a set of numbers. Use the display for the work below.

A Set F is tabulated in the display. When you divide a natural number by 3, is the remainder always one of the numbers that belong to set F? Explain your answer.

B We will represent the operation of *addivision* by the symbol $\oplus$. The symbol is read "addivided by." The *addiv* of 1 and 2, which is symbolized by $(1 \oplus 2)$, is the remainder you obtain when you divide the sum of 1 and 2 by 3. What is $1 \oplus 2$? What is $1 \oplus 1$? What is $2 \oplus 2$? What is $1 \oplus 0$?

C Copy and complete the addivision table shown at the left in the display.

D We will represent the operation of *muldivision* by the symbol $\otimes$. The symbol is read "muldivided by." The *muldiv* of 1 and 2, which is symbolized by $(1 \otimes 2)$, is the remainder you obtain when you divide the product of 1 and 2 by 3. What is $1 \otimes 2$? What is $2 \otimes 2$? What is $2 \otimes 1$? What is $0 \otimes 0$?

E Copy and complete the muldivision table shown at the right in the display.

$F = \{0, 1, 2\}$.

$\oplus$	0	1	2
0			
1	1	2	0
2			1

$\otimes$	0	1	2
0	0		
1			2
2		2	1

Tabulate the solution set of each condition expressed in exercises F through K. The universe for x is F. F = {0, 1, 2}.

F $2 \oplus 0 = x$.

G $\sim (1 \otimes x = 2)$.

H $1 = x \otimes 2$.

I $1 \oplus x = 2$.

J $x \otimes 2 = 0$.

K $x = x \oplus 2$.

L Tabulate F × F.

Tabulate the solution set of each condition expressed in exercises M through R. The universe for (x, y) is F × F.

M $x = y$.

N $2 \otimes y = x$.

O $x \oplus 2 = y$.

P $\sim (y = 0 \oplus x)$.

Q $x = 0$.

R $2 \otimes x = y \oplus 1$.

S $\{(x, y, z) | x \in F \wedge y \in F \wedge z \in F\}$ has how many members?

Tabulate the solution set of each condition expressed in exercises T through Y. The universe is $\{(x, y, z) | x \in F \wedge y \in F \wedge z \in F\}$.

T $x \oplus y = z$.

U $y \otimes z = x \oplus 2$.

V $x \oplus y \oplus z = 1$.

W $x \oplus (1 \otimes z) = 1 \otimes y$.

X $x = y \wedge y = z$.

Y $x \otimes y = 2 \wedge z = 0$.

KEEPING SKILFUL

For each exercise below, find the sum, difference, product, or quotient.

1 $2500 - 112$

2 25.12×9.05

3 $7\frac{2}{3} + 25\frac{3}{4} + \frac{19}{24}$

4 $16 + 17 + 925$

5 $1752 - 968$

6 1200×999

7 $29.001 - 1.008$

8 $795 + 342 + 18$

9 807×78

10 $3\frac{1}{7} \div 5\frac{4}{9}$

11 $.004 \times .09$

12 $20\frac{3}{4} - 8\frac{1}{8}$

13 $112\frac{1}{2} + 120\frac{3}{9}$

14 $65\frac{5}{8} - 13\frac{15}{16}$

15 $10\frac{5}{9} \div 1\frac{2}{3}$

16 $21\frac{2}{3} \times 3\frac{1}{5}$

17 $.012 + .104 + .187$

18 $24.651 - 1.495$

19 $6\frac{1}{2} \times 6\frac{1}{2}$

20 $175.2 + 85.8$

21 $850 \div 32$ (Ones)

22 $100.76 \div .85$ (Tenths)

23 $85.12 \div 21$ (Hundredths)

24 $654.3013 \div 5.1$ (Thousandths)

25 $1000 \div 130$ (Tenths)

CHECKING UP

If you have trouble with this test, you can find help in lessons 126 through 129.

Test 123

Each statement expressed in exercises 1 through 4 is an example of a property of R_a. For each statement, name the property.

1 $\frac{3}{4}(\frac{1}{2} + \frac{1}{3}) = \frac{3}{4}(\frac{1}{2}) + \frac{3}{4}(\frac{1}{3})$.

2 $4.6 \times 9.7 = 9.7 \times 4.6$.

3 $1 \cdot \frac{5}{6} = \frac{5}{6}$.

4 $\frac{2}{9} + (\frac{5}{7} + 2\frac{3}{8}) = (\frac{2}{9} + \frac{5}{7}) + 2\frac{3}{8}$.

For each of exercises 5 and 6, list the names of the rational numbers in order according to the idea of "less than." Begin with the name of the least number.

5 $^{+}\frac{1}{2}$ $^{-}2$ 0 $^{-}\frac{3}{2}$ $^{+}\frac{1}{4}$ $^{-}\frac{7}{4}$ $^{-}\frac{1}{2}$

6 $^{-}\frac{2}{3}$ $^{+}\frac{2}{5}$ $^{+}\frac{1}{10}$ 0 $^{-}\frac{3}{5}$ $^{-}5$ $^{+}\frac{5}{2}$

For each of exercises 7 through 10, give a standard name of the rational number expressed. $U = R_a \times R_a$.

7 $\{(x, y) | (x, y) \sim (\frac{13}{2}, 2)\}$

8 $\{(x, y) | (x, y) \sim (\frac{16}{5}, \frac{19}{4})\}$

9 $\{(x, y) | (x, y) \sim (12, 12)\}$

10 $\{(x, y) | (x, y) \sim (5, 6)\}$

11 Which numbers that you named for exercises 7 through 10 are members of the set of integers?

Use the display below in connection with exercises 12 through 16.

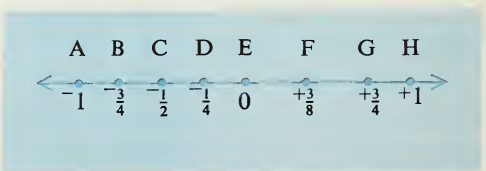
12 Name the coördinate of point F.

13 $^{-}\frac{3}{4}$ is the coördinate of which point?

14 What number is associated with $\overrightarrow{EG}$?

15 What number is associated with $\overrightarrow{EC}$?

16 What number is associated with $\overrightarrow{EA}$?



End-of-block test on the rational numbers of arithmetic; the rational numbers; ordering of the rational numbers; integers

Preparation for addition

You know how to find the sum of two natural numbers and the sum of two rational numbers of arithmetic. Now you will use trips to suggest the sum of two rational numbers.

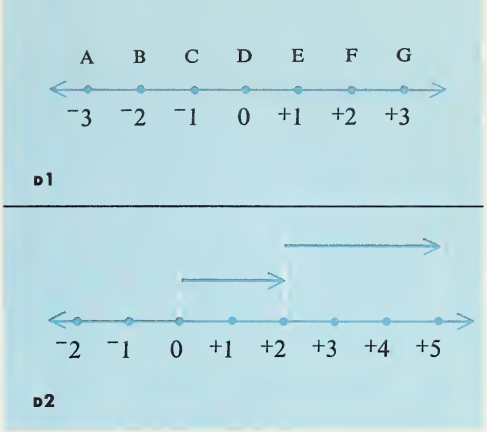
Earlier in this unit, you learned that you can think of trips as representing rational numbers. You can think of the result of two successive trips as representing the sum of two rational numbers. A rational-number line will help establish where the trips begin and end.

Look at the picture of a rational-number line in D1. When we use trips to represent a sum, we will agree to begin the first trip at point 0. The second trip will begin where the first trip ends. A trip to the right will represent a positive rational number. A trip to the left will represent a negative rational number.

A You can think of $+2$ as a trip of 2 units to the right, that is, from D to F. If you start at D, to what point does a trip of $^{-}3$ take you? If you start at D, to what point does a trip of $+3$ take you? Explain your answers.

B Suppose that you take a trip of $+1$, followed by a second trip of $+2$. The trip of $+1$ is a trip from D to what point? If you start where the first trip ended and take the second trip, why do you go 2 units to the right from E? Do you go from E to G for the trip of $+2$?

C Suppose that you take two successive trips, $+2$ and $^{-}3$. You go from D to what point for the trip of $+2$? From this point, how many units to the left do you go for the trip of $^{-}3$? Do you go from F to C for the trip of $^{-}3$?



D Now look at D2. If you start at point 0, to what point does a trip of $+2$ take you? If you follow the trip of $+2$ with a trip of $+3$, at what point do you start? To what point do you go?

E A trip of $+5$ takes you from point 0 to what point? Does a trip of $+5$ take you to the same point as two successive trips of $+2$ and $+3$?

Since the result of a trip of $+2$ followed by a trip of $+3$ is the same as a single trip of $+5$, it appears that $+5$ should be the sum of $+2$ and $+3$.

F Suppose that you take a trip of $^{+}\frac{1}{4}$. If you follow the trip of $^{+}\frac{1}{4}$ with a trip of $^{+}\frac{5}{4}$, at what point do you start the second trip? Why would you go to point $\frac{6}{4}$? Draw a picture of a rational-number line if you need to.

G A trip of $^{+}\frac{6}{4}$ takes you from point 0 to what point? Does a trip of $^{+}\frac{6}{4}$ take you to the same point as two successive trips of $^{+}\frac{1}{4}$ and $^{+}\frac{5}{4}$?

H Do your answers for exercise G suggest that $^{+}\frac{6}{4}$ should be the sum of $^{+}\frac{1}{4}$ and $^{+}\frac{5}{4}$?

I ? Use the idea of successive trips to answer the following questions: What number should be the sum of $+5$ and $+7$? The sum of $+13$ and $+9$? The sum of $^{+}\frac{3}{2}$ and $^{+}\frac{3}{4}$?

J ? Use the idea of successive trips to explain why the sum of two positive numbers should be a positive number.

K The picture in D3 will help you see what number should be the sum of $^-1$ and $^-2$. If you start at point 0, to what point does a trip of $^-1$ take you? If you follow the trip of $^-1$ with a trip of $^-2$, at what point do you start? To what point do you go?

L A trip of $^-3$ takes you from point 0 to what point? Is a trip of $^-3$ the same as the result of two successive trips of $^-1$ and $^-2$?

M Do your answers for exercise L suggest that $^-3$ should be the sum of $^-1$ and $^-2$?

N Use the idea of successive trips to answer the following questions: What number should be the sum of $^-5$ and $^-6$? The sum of $^-\frac{4}{9}$ and $^-\frac{1}{3}$? The sum of $^-13$ and $^-7$?

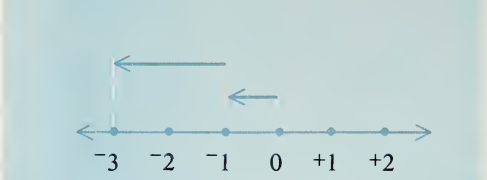
O Use the idea of successive trips to explain why the sum of two negative numbers should be a negative number.

So far, you have used the idea of successive trips to suggest that the sum of two positive numbers should be positive and that the sum of two negative numbers should be negative. When we refer to a sum as negative or positive, we mean that the sum should be a negative rational number or a positive rational number. Now you will use trips to suggest what the sum of a positive number and a negative number should be.

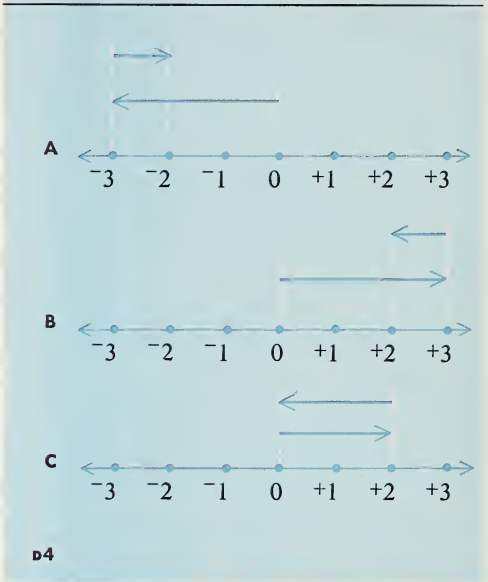
A Look at picture A in D4. If you start at point 0, to what point does a trip of $^-3$ take you? If you follow the trip of $^-3$ with a trip of $^+1$, at what point do you start the second trip? From this point, why should you go 1 unit to the right? To what point does the trip of $^+1$ take you?

B A trip of $^-2$ takes you from point 0 to what point? Is a trip of $^-2$ the same as the result of two successive trips of $^-3$ and $^+1$?

C What number should be the sum of $^-3$ and $^+1$?



D3



D4

D Look again at picture A in D4. Which is longer, the trip of $^+1$ or the trip of $^-3$? Is the result of taking a trip of $^-3$, followed by a trip of $^+1$, the same as a trip to the right from point 0, or is it the same as a trip to the left from point 0?

E Use the idea of successive trips to answer the following questions: What number should be the sum of $^-5$ and $^+2$? The sum of $^-14$ and $^+6$? The sum of $^-\frac{5}{2}$ and $^+\frac{1}{2}$?

F Next look at picture B in D4. If you start at point 0, to what point does a trip of $^+3$ take you? If you follow a trip of $^+3$ with a trip of $^-1$, you go from point $^+3$ to what point?

G Does a trip of $^+2$ take you to the same point as two successive trips of $^+3$ and $^-1$?

H What number do you think should be the sum of $^{+}3$ and $^{-}1$?

I Look again at picture B. Which is longer, the trip of $^{+}3$ or the trip of $^{-}1$? Is the result of taking a trip of $^{+}3$, followed by a trip of $^{-}1$, the same as a trip to the right from point 0, or is it the same as a trip to the left from point 0?

J Use the idea of successive trips to answer the following questions: What number should be the sum of $^{+}9$ and $^{-}3$? The sum of $^{+}16$ and $^{-}10$? The sum of $^{+}\frac{7}{4}$ and $^{-}\frac{3}{4}$?

K Use the idea of successive trips to explain when the sum of a positive number and a negative number should be positive, and when the sum should be negative.

L Use picture C in D4 to answer the following questions: When does a trip to the right, followed by a trip to the left, take you to point 0? When should 0 be the sum of a positive number and a negative number?

M Is it also possible for a trip to the left, followed by a trip to the right, to take you to point 0? How do you know that $^{-}3 + {}^{+}3$ should be 0?

N Are there two positive rational numbers whose sum should be 0? Are there two negative rational numbers whose sum should be 0?

O Use the idea of successive trips to decide what number should be the sum of 0 and any given rational number.

In this lesson you used the idea of trips to suggest what the sum of two given rational numbers should be. In the next lesson you will use ordered pairs and variables to develop a definition of the sum of two rational numbers.

On your own

Use the rational-number line represented in D5 to help you answer the questions in the exercises below.

1 If you start at point 0, to what point does a trip of $^{+}3$ take you? A second trip of $^{+}4$ takes you from the point you just named to what point?

2 What number should be the sum of $^{+}3$ and $^{+}4$?

3 At what point would you be if you took a trip of $^{-}5$, followed by a trip of $^{+}7$?

4 What number should be the sum of $^{-}5$ and $^{+}7$?

5 To what point do you go for a trip of $^{-}2$? If you follow this trip with another trip of $^{-}2$, to what point do you go?

6 What number should be the sum of $^{-}2$ and $^{-}2$?

7 Suppose that you follow a trip of $^{+}5$ with a trip of $^{-}7$. The result is the same as the result of a trip from point 0 to what point?

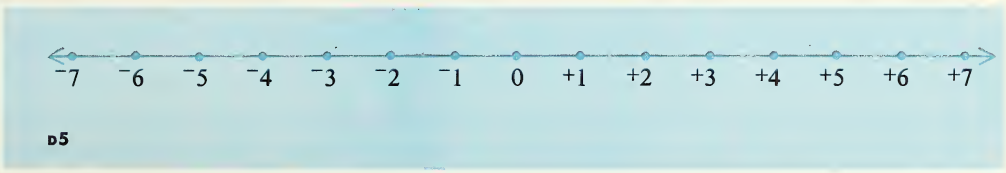
8 What number should be the sum of $^{+}5$ and $^{-}7$?

9 Use trips to explain why $^{+}6 + {}^{-}2$ should be the same number as $^{-}2 + {}^{+}6$.

10 Use trips to explain why $^{-}5 + {}^{+}5$ should be 0.

11 Use trips to explain why $0 + {}^{-}7$ should be $^{-}7$.

12 Use trips to explain why the sum of $^{+}5$ and $^{-}6$ should be the same number as the sum of $^{-}4$ and $^{+}3$.



D5

Sum of two rational numbers

In this lesson you will see how you can use the ordered pairs for two rational numbers to find an ordered pair for their sum. Then you will use variables to develop a definition of the sum of any two rational numbers.

In the last lesson you used the idea of successive trips to suggest that the sum of two positive rational numbers should be positive and that the sum of two negative rational numbers should be negative. Now you will see how you can obtain the sum of two such numbers by using their basic pairs.

A Look at D1. Use trips to explain why statement A should be true. Do sentence A and sentence B express the same statement about the sum of $^+2$ and $^+3$? How do you know?

B Compare sentences C and D. How do trips suggest that sentence C should express a true statement? How do you know that sentence C and sentence D express the same statement about the sum of $^-1$ and $^-2$?

C Look again at sentence B. What is the non-zero component of the basic pair for $^+5$? Is this number the sum of the non-zero components of (2, 0) and (3, 0)?

D Look again at sentence D. What is the non-zero component of the basic pair for $^-3$? Is this number the sum of the non-zero components of (0, 1) and (0, 2)?

From exercises C and D, you can see how to use the non-zero components of basic pairs to help you get the sum of two rational numbers. To obtain the non-zero component of the basic pair for the sum of two positive or two negative rational numbers, you can add the non-zero components of the basic pairs for the two given numbers. If you are finding the sum of two positive rational numbers, the component you obtain is a first component, and the sum is positive. If you are finding the sum of two negative rational numbers, the component you obtain is a second component, and the sum is negative.

A $^+2 + {}^+3 = {}^+5$.
B $(2, 0) + (3, 0) = (5, 0)$.
C $^-1 + {}^-2 = {}^-3$.
D $(0, 1) + (0, 2) = (0, 3)$.

D1

E Suppose that you want to find the sum of ? $^+\frac{3}{4}$ and $^+\frac{1}{2}$. Are $\frac{3}{4}$ and $\frac{1}{2}$ the non-zero components of the basic pairs for $^+\frac{3}{4}$ and $^+\frac{1}{2}$? What is the non-zero component of the basic pair for the sum of $^+\frac{3}{4}$ and $^+\frac{1}{2}$? Is the sum that you obtain $^+\frac{5}{4}$, or is it $^-\frac{5}{4}$?

F Suppose that you want to find the sum of $^-2$? and $^-\frac{2}{3}$. Is $\frac{8}{3}$ the non-zero component of the basic pair for their sum? Is the non-zero component of the basic pair for their sum a first component, or is it a second component? What is the sum of $^-2$ and $^-\frac{2}{3}$?

Your work with trips in lesson 130 also suggested what the sum of two rational numbers should be when one of the numbers is positive and the other number is negative. Next you will work with basic pairs to obtain the sum of two such numbers.

G Look at D2. From your work with trips, you know that sentences E, G, and I should express true statements. Compare sentence F with sentence E. How do you know that both sentences express the same statement? Next, compare sentence H with sentence G. How do

you know that both sentences express the same statement? Finally, compare sentence J with sentence I. How do you know that both sentences express the same statement?

H Look again at sentence F. What is the non-zero component of the basic pair for $^{-}3$? Is this number the difference of the non-zero components of (0, 4) and (1, 0)?

I Study sentence H. What is the non-zero component of the basic pair for $^{+}3$? Is this number the difference of the non-zero components of (4, 0) and (0, 1)?

J Study sentence J. Does the basic pair for 0 have a non-zero component? What is the difference of the non-zero components of (0, 2) and (2, 0)?

From exercises H through J, you see how the non-zero components of basic pairs can be used to help you get the sum of two rational numbers. To find one of the components of the basic pair for the sum of a positive number and a negative number, you can find the difference of the non-zero components of the basic pairs for the two given numbers. The other component of the basic pair for the sum will, of course, be 0.

If the greater of the non-zero components of two basic pairs is a second component, then the difference of these components is the second component of the basic pair for their sum. The sum of the rational numbers is negative. (See sentence F in D2.)

If the greater of the non-zero components of two basic pairs is a first component, then the difference of these components is the first component of the basic pair for their sum. The sum of the two rational numbers is positive. (See sentence H in D2.)

If the non-zero components of two basic pairs are equal, then the difference of these components is 0, and the components of the basic pair for their sum are also equal. The sum of the two rational numbers is 0. (See sentence J in D2.)

E $^{-}4 + {}^{+}1 = {}^{-}3.$
F $(0, 4) + (1, 0) = (0, 3).$
G $^{+}4 + {}^{-}1 = {}^{+}3.$
H $(4, 0) + (0, 1) = (3, 0).$
I $^{-}2 + {}^{+}2 = 0.$
J $(0, 2) + (2, 0) = (0, 0).$

D2

K Suppose that you want to find the sum of $^{-}2$ and $^{+}\frac{3}{2}$. Can you find the difference of 2 and $\frac{3}{2}$ to obtain the non-zero component of the basic pair for their sum? Which number is the sum, $^{+}\frac{1}{2}$ or $^{-}\frac{1}{2}$?

L Suppose that you want to find the sum of $^{-}\frac{1}{3}$ and $^{+}\frac{5}{6}$. Can you find the difference of $\frac{5}{6}$ and $\frac{1}{3}$ to obtain the non-zero component of the basic pair for their sum? Which number is the sum, $^{+}\frac{1}{2}$ or $^{-}\frac{1}{2}$?

M Suppose that you want to find the sum of ? $^{-}\frac{4}{3}$ and $^{+}\frac{4}{3}$. Explain how you can use the non-zero components of the basic pairs for $^{-}\frac{4}{3}$ and $^{+}\frac{4}{3}$ to help you find the sum. What is the sum of $^{-}\frac{4}{3}$ and $^{+}\frac{4}{3}$?

For each pair of rational numbers named below, use the non-zero components of their basic pairs to help you find their sum.

N $^{-}3, {}^{+}3$	**Q** $^{-}3, {}^{+}2$	**T** $^{-}\frac{5}{2}, {}^{+}8$	**W** $^{+}1, {}^{+}\frac{7}{9}$
O $^{-}12, {}^{-}9$	**R** $^{+}6, {}^{-}6$	**U** $^{-}7, {}^{+}\frac{5}{3}$	**X** $^{-}\frac{11}{12}, {}^{+}\frac{5}{4}$
P $^{+}\frac{8}{7}, {}^{-}2$	**S** $^{-}\frac{2}{3}, {}^{-}\frac{5}{6}$	**V** $^{-}\frac{3}{5}, {}^{+}\frac{5}{8}$	**Y** $^{+}\frac{1}{9}, {}^{+}\frac{1}{9}$

From the exercises that follow, you will learn how to use the components of any ordered pairs that indicate two rational numbers to find the ordered pair that indicates their sum. Then you will use variables to develop a definition of the sum of any two rational numbers.

A Look at expression K in D3. Does (7, 5) indicate $^{+}2$? What rational number is indicated by (4, 1)? Expression K names the sum of what two rational numbers?

B Now look at diagram L and sentence M in D3. What is the sum of the first components of (7, 5) and (4, 1)? Is this number the first component of (11, 6)? What is the sum of the second components of (7, 5) and (4, 1)? Is this number the second component of (11, 6)?

C What rational number does (11, 6) indicate?

? From your work with trips, you know that $^{+}5$ should be the sum of $^{+}2$ and $^{+}3$. Should (11, 6) be the sum of (7, 5) and (4, 1)?

D Look at D4. Statement N concerns the sum of what two negative rational numbers? Give a standard name of each number.

E What is the sum of the first components of (1, 2) and (8, 10)? Is this number the first component of (9, 12)? What is the sum of the second components of (1, 2) and (8, 10)? Is this number the second component of (9, 12)?

F What number should be the sum of $^{-}1$ and $^{-}2$? Should (9, 12) be the sum of (1, 2) and (8, 10)?

G Statement O, expressed in D4, concerns the sum of a negative number and a positive number. What rational number is indicated by (2, 6)? By (5, 4)?

H What is the sum of the first components of (2, 6) and (5, 4)? Of the second components of (2, 6) and (5, 4)? Are the sums that you named the first and second components of (7, 10)?

I Explain why (7, 10) indicates the number

? that should be the sum of $^{-}4$ and $^{+}1$.

J Statement P concerns the sum of a positive

? number and a negative number. Explain how you can use the components of (8, 4) and (3, 4) to obtain the components of an ordered pair that indicates the number that should be the sum of $^{+}4$ and $^{-}1$.

K $(7, 5) + (4, 1)$

$$7 + 4$$

L $(7, 5) + (4, 1)$

$$5 + 1$$

M $(7, 5) + (4, 1) = (11, 6)$.

D3

N $(1, 2) + (8, 10) = (9, 12)$.
O $(2, 6) + (5, 4) = (7, 10)$.
P $(8, 4) + (3, 4) = (11, 8)$.

D4

Universe for a, b, c, and $d = R_a$.
$(a, b) + (c, d) = (a + c, b + d)$.
Q $(6, 1) + (4, \frac{1}{2}) = (6 + 4, 1 + \frac{1}{2})$.
R $(\frac{1}{3}, 8) + (2, \frac{7}{2}) = (\frac{1}{3} + 2, 8 + \frac{7}{2})$.
S $(2, 3) + (10, 5) = (2 + 10, 3 + 5)$.
T $(\frac{1}{2}, \frac{1}{3}) + (\frac{7}{8}, 6) = (\frac{1}{2} + \frac{7}{8}, \frac{1}{3} + 6)$.

D5

You can use variables to develop the definition of the sum of two rational numbers. Look at D5. Think of (a, b) and (c, d) as any two rational numbers. $(a + c, b + d)$ is the sum of (a, b) and (c, d).

K In $(a + c, b + d)$, is $a + c$ the sum of the first components of (a, b) and (c, d)? Is $b + d$ the sum of the second components of (a, b) and (c, d)?

Now we can define the sum of the rational numbers (a, b) and (c, d). The universe for a, b, c, and d is R_a. For each a, b, c, and d, $(a, b) + (c, d) = (a + c, b + d)$.

L Study sentence Q in D5. To obtain statement Q from $(a, b) + (c, d) = (a + c, b + d)$, a was replaced by 6, b by 1, c by 4, and d by $\frac{1}{2}$. Is statement Q true? How do you know?

M What replacements were made for the variables in $(a, b) + (c, d) = (a + c, b + d)$ to obtain statement R? To obtain statement S? To obtain statement T? Is each of these statements true?

Use the definition of the sum of two rational numbers to decide whether each statement expressed below is true or false.

N $(1, 0) + (3, 5) = (4, 5)$.

O $(2, \frac{7}{2}) + (\frac{1}{2}, 6) = (8, \frac{8}{2})$.

P $(\frac{2}{9}, \frac{1}{9}) + (8, 8) = (\frac{3}{9}, 16)$.

Q $(6, 4) + (4, 6) = (10, 10)$.

R $(6, 0) + (0, 6) = (6, 6)$.

Now you will study some of the properties of addition of rational numbers.

A You know that $(a + c, b + d)$ is the sum of (a, b) and (c, d). The universe for a, b, c, and d is R_a. What property of addition of rational numbers of arithmetic tells you that, for each a, b, c, and d, $a + c$ and $b + d$ are members of R_a?

B Use your answer for exercise A and the definition of the sum of two rational numbers to explain why the sum of any two rational numbers is a rational number.

C If x and y are any two rational numbers, is $x + y$ always a rational number?

You have seen that the sum of any two elements of R is an element of R. Therefore, R is closed under addition. This property is the *closure property of addition of rational numbers.* The property is expressed below.

The universe for x *and* y *is R. For each* x *and* y, $x + y \in R$.

D Now look at D6. Does each sentence express a true statement? In statement A, $(\frac{3}{4}, 2)$ is the sum of what two rational numbers? Remember that the components of the ordered pairs are members of R_a.

A $(\frac{3}{4}, 0) + (0, 2) = (\frac{3}{4}, 2)$.
B $(\frac{3}{2}, \frac{3}{4}) + (1, 3) = (\frac{5}{2}, \frac{15}{4})$.
C $(\frac{3}{4}, 0) + (3, 5) = (\frac{15}{4}, 5)$.
D $(\frac{7}{4}, 1) + (0, 2) = (\frac{7}{4}, 3)$.

D6

sum of two rational numbers. The universe for a, b, c, and d is R_a. For each a, b, c, and d, $(a, b) + (c, d) = (a + c, b + d)$.

E Look again at D6. To obtain statement B from statement A, $(\frac{3}{4}, 0)$ was replaced by $(\frac{3}{2}, \frac{3}{4})$. Is the rational number $(\frac{3}{2}, \frac{3}{4})$ the same rational number as $(\frac{3}{4}, 0)$? $(0, 2)$ was replaced by $(1, 3)$. Is $(1, 3) = (0, 2)$? Explain your answers. Remember that when we ask "Is $(1, 3) = (0, 2)$?" we mean "Is the rational number indicated by $(1, 3)$ equal to the rational number indicated by $(0, 2)$?"

F To obtain statement C from statement A, what replacement was made for $(0, 2)$? Is $(3, 5) = (0, 2)$?

G To obtain statement D from statement A, what replacement was made for $(\frac{3}{4}, 0)$? Is $(\frac{7}{4}, 1) = (\frac{3}{4}, 0)$?

H You know that each sentence in D6 expresses a true statement. Now think about the sums expressed at the right of the symbols for equality. Are all these sums the same rational number? How do you know?

I If one or both of the two rational numbers that form a sum are replaced by equal rational numbers, does the sum remain the same?

Now you know that, if one or both of the two rational numbers that form a sum are replaced by equal rational numbers, the sum is the same rational number. This property is the

Universe for m and n = Z.

E $(m, 0) + (n, 0) = (m + n, 0)$.

F $(0, m) + (0, n) = (0, m + n)$.

G $(m, 0) + (0, n) = (m, n)$.

D7

well-defined property of addition of rational numbers. The property is expressed below.

If one or both of the numbers used in forming the sum of two rational numbers are replaced by equal rational numbers, the sum is still the same rational number.

Although you can use the idea of trips to suggest whether the sum of two rational numbers is positive or negative, it is sounder mathematically to use the definition of the sum of two rational numbers. The definition assures you of the following facts: the sum of two positive rational numbers is positive; the sum of two negative rational numbers is negative; and the sum of a positive rational number and a negative rational number is one of three numbers—a positive number, a negative number, or zero.

A Study sentence E in D7. Notice that the universe for m and n is Z. Remember that Z is the set of non-zero rational numbers of arithmetic. For each m and n, is $(m, 0)$ a positive rational number? Is $(n, 0)$ a positive rational number?

B For each m and n, is $(m, 0) + (n, 0) = (m + n, 0 + 0)$? Is $(m + n, 0 + 0)$ the same rational number as $(m + n, 0)$? Is $(m + n, 0)$ a positive rational number, or is it a negative rational number? Explain your answers.

C Is the sum of any two positive rational numbers a positive rational number?

D Read sentence F in D7. For each m and n, are both $(0, m)$ and $(0, n)$ negative, or are they positive? How do you know?

E For each m and n, is $(0, m) + (0, n) = (0 + 0, m + n)$? Is $(0 + 0, m + n) = (0, m + n)$? Is $(0, m + n)$ negative, or is it positive?

F Is the sum of any two negative rational numbers a negative rational number?

G Study sentence G. For each m and n, what kind of rational number is indicated by $(m, 0)$? By $(0, n)$?

H For each m and n, is $(m, 0) + (0, n) = (m + 0, 0 + n)$? Is $(m + 0, 0 + n) = (m, n)$? Suppose that $m < n$. Is (m, n) positive, or is it negative?

I Suppose that $m > n$. Is (m, n) positive, or is it negative? If $m = n$, what rational number is (m, n)?

J Is the sum of a positive rational number and a negative rational number always a positive rational number? Always a negative rational number? Explain your answers. When is the sum zero?

K For each c and d, is $(0, 0) + (c, d) = (c, d)$? The universe for c and d is R_a. What is the sum of zero and any given rational number?

In this lesson you learned to find the sum of two rational numbers. You also studied the closure property and the well-defined property of addition of rational numbers.

On your own

Each sentence in exercises 1 through 6 expresses a statement about the sum of two rational numbers. Use the definition of the sum of two rational numbers to tell whether the statement is true or whether it is false.

1 $(2, 3) + (4, 7) = (8, 21)$.

2 $(1, 3) + (\frac{1}{2}, 2) = (\frac{3}{2}, 5)$.

3 $(6, \frac{1}{2}) + (\frac{1}{4}, \frac{2}{3}) = (\frac{25}{4}, \frac{7}{6})$.

4 $(\frac{3}{8}, 1) + (2, 2) = (\frac{11}{8}, 4)$.

5 $(\frac{1}{6}, 0) + (0, \frac{4}{6}) = (0, \frac{5}{6})$.

6 $(27, 13) + (49, 126) = (76, 139)$.

The sentences in exercises 7 through 11 express true statements about the sum of two rational numbers. For each exercise, write two other sentences that express the same statement. Do this by first replacing one or both of the rational numbers by an equal rational number. Then find the sum of the replacements. For exercise 7 you could write:

$(1, 4) + (\frac{1}{3}, \frac{2}{3}) = (\frac{4}{3}, \frac{14}{3})$.

$(3, 6) + (\frac{2}{3}, 1) = (\frac{11}{3}, 7)$.

7 $(0, 3) + (0, \frac{1}{3}) = (0, \frac{10}{3})$.

8 $(\frac{3}{2}, 0) + (5, 0) = (\frac{13}{2}, 0)$.

9 $(7, 0) + (0, 21) = (7, 21)$.

10 $(0, \frac{1}{4}) + (\frac{2}{9}, 0) = (\frac{2}{9}, \frac{1}{4})$.

11 $(16, 0) + (14, 0) = (30, 0)$.

Find the sum of the numbers named in each of the exercises below. Give a standard name of the number that is the sum.

12 $^+15, ^+22$	20 $^+7, ^+14$	28 $^-576, ^+112$
13 $^-4, ^+12$	21 $^-37, ^+256$	29 $^-15, ^+15$
14 $^+16, ^+16$	22 $^-14, ^-75$	30 $^+72, ^+17$
15 $^-11, ^+11$	23 $^-64, ^+38$	31 $^-16, ^+39$
16 $^-6, ^-7$	24 $^+\frac{1}{3}, ^+\frac{4}{5}$	32 $^-183, ^+183$
17 $^-8, ^+2$	25 $^-9, ^-8$	33 $^-\frac{1}{3}, ^+\frac{1}{4}$
18 $^+5, ^-8$	26 $^-17, ^+26$	34 $^+\frac{12}{7}, ^-\frac{8}{21}$
19 $^+27, 0$	27 $^-\frac{4}{9}, ^+\frac{4}{9}$	35 $^-6, ^-\frac{4}{3}$

36 To obtain a sum of 0, what number do you add to $^+9$? To $^-9$? To $^+13$? To $^-13$?

KEEPING SKILFUL

Tabulate the solution set of each condition expressed below. U = Z.

1 $\frac{5}{6}/1 \sim c/8$.	6 $\frac{3}{4} \div (2\frac{1}{8} \times 3) = r$.
2 $25\frac{1}{2} + x = 30\frac{1}{4}$.	7 $h + .001 = .094$.
3 $w - 9.9 = 1.2$.	8 $1.25/5 \sim 6.25/d$.
4 $87\frac{1}{2}/100 \sim n/90$.	9 $7/5 \sim p/50.5$.
5 $1.52 \div .8 = m$.	10 $100.25 - a = 5.07$

11 $\frac{1}{8}/1 \sim x/2\frac{1}{2} \vee \frac{1}{5}/1 \sim x/5$.

12 $\frac{2}{5} + \frac{3}{4} > y \wedge \frac{1}{3} + \frac{1}{8} < y$.

13 $a - 6\frac{1}{2} = 10\frac{3}{4} \vee a - 6\frac{1}{2} = 12\frac{3}{4}$.

Preparation for multiplication

In lesson 130 we used the idea of trips to suggest what the sum of two rational numbers should be. We cannot use trips to suggest what the product of two rational numbers should be, because it does not make sense to think of a trip as being multiplied by another trip. Instead, we will rely on our knowledge of certain properties.

You will recall that, for R_a, we first defined the sum of two members and the product of two members. We then found that the operations of addition and multiplication had certain properties. We want certain properties to hold true for rational numbers. These properties will be used to suggest what the product of two members of R should be. Then, in later lessons, after we have developed a precise definition of the product of two rational numbers, we will decide whether or not these properties really do hold true.

A What is the sum of $\frac{1}{2}$ and $\frac{1}{4}$? What is the sum of $^+\frac{1}{2}$ and $^+\frac{1}{4}$?

You can see that, under addition, the positive rational numbers behave in much the same way as members of R_a. We want the positive rational numbers also to behave in much the same way as members of R_a under multiplication. We know, for example, that $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$. So we want the product of $^+\frac{1}{2}$ and $^+\frac{1}{4}$ to be $^+\frac{1}{8}$.

B What is the product of 2 and 3? What should be the product of $^+2$ and $^+3$?

C What should be the product of $^+6$ and $^+\frac{1}{5}$? Of $^+\frac{1}{3}$ and $^+\frac{4}{7}$? Should the product of any two

positive rational numbers be positive, or should it be negative?

D You know that, if x is any member of R_a, then $x(0) = 0$. We want rational numbers to have the zero property of multiplication. What should be the product of any given rational number and 0?

E What should be the product of $^{+}\frac{5}{6}$ and 0? Of $^{-}7$ and 0? Of $^{+}2$ and 0?

So far, we want the rational numbers to behave in such a way that the product of any two positive numbers is positive and the product of any rational number and 0 is 0. Next you will see what number we want for the product of a positive number and a negative number.

F Look at D1. What is the sum of $^{+}3$ and $^{-}3$? Is statement A true?

G What should be the product of $^{+}2$ and 0? Should statement B be true?

H Now compare statements B and C. We want multiplication of rational numbers to be well defined. This means that we should be able to replace one or both of the rational numbers in $^{+}2(0)$ by equal rational numbers without affecting the product. Now explain how we used statements A and B to obtain statement C.

I Should statement C be true?

J You know that, in R_a, multiplication distributes over addition. If x, y, and z are any three members of R_a, then $x(y + z) = xy + xz$. We also want the rational numbers to have the distributive property. Should statement D be true?

K What property makes it possible to obtain statement E from statements C and D?

L We have already decided that $^{+}6$ should be the product of $^{+}2$ and $^{+}3$. If statement E is to be true, should $^{-}6$ be the product of $^{+}2$ and $^{-}3$? Explain your answer.

We obtained the statements expressed in D1 by using some of the properties that we want to hold true for addition and multiplication of rational numbers. Statement E suggests that $^{-}6$ should be the product of $^{+}2$ and $^{-}3$.

M We also want multiplication of rational numbers to have the commutative property. If multiplication in R is commutative, what should be the product of $^{-}3$ and $^{+}2$?

N What should be the product of $^{+}\frac{7}{5}$ and $^{-}\frac{2}{3}$? Of $^{-}\frac{2}{3}$ and $^{+}\frac{7}{5}$? Should the product of any positive rational number and any negative rational number be positive, or should it be negative?

Next you will see what the product of any two negative rational numbers should be.

O Look at D2. What is the sum of $^{+}2$ and $^{-}2$? Is statement F true?

P Why should statement G be true?

Q Compare statements F and G with statement H. Statement H should be true because we want a certain property to hold true for multiplication of rational numbers. Name the property. How was statement H obtained?

R What property that we want to hold true for rational numbers tells you that statement I should be true?

A $^{+}3 + {}^{-}3 = 0.$
B $^{+}2(0) = 0.$
C $^{+}2(^{+}3 + {}^{-}3) = 0.$
D $^{+}2(^{+}3 + {}^{-}3) = (^{+}2 \cdot {}^{+}3) + (^{+}2 \cdot {}^{-}3).$
E $(^{+}2 \cdot {}^{+}3) + (^{+}2 \cdot {}^{-}3) = 0.$

D1

F $^{+}2 + {}^{-}2 = 0.$
G $^{-}3(0) = 0.$
H $^{-}3(^{+}2 + {}^{-}2) = 0.$
I $^{-}3(^{+}2 + {}^{-}2) = (^{-}3 \cdot {}^{+}2) + (^{-}3 \cdot {}^{-}2).$
J $(^{-}3 \cdot {}^{+}2) + (^{-}3 \cdot {}^{-}2) = 0.$

D2

S What property makes it possible to obtain statement J from statements H and I?

T We have already decided that $^{-}6$ should be the product of $^{-}3$ and $^{+}2$. If statement J is to be true, should $^{+}6$ be the product of $^{-}3$ and $^{-}2$? Explain your answer.

U Statement J suggests that $^{+}6$ should be the product of $^{-}3$ and $^{-}2$. Why should $^{+}6$ also be the product of $^{-}2$ and $^{-}3$?

V What should be the product of $^{-}15$ and $^{-}4$? Of $^{-}\frac{7}{8}$ and $^{-}\frac{1}{3}$? Should the product of any two negative rational numbers be positive, or should it be negative?

Notice what we have done in this lesson. First we decided that we wanted the positive rational numbers to behave much as the members of R_a do. Therefore, the product of any two positive numbers should be positive. Then we decided that we wanted certain properties to hold true for the rational numbers. These properties suggested that the product of any rational number and 0 should be 0; the product of any positive number and any negative number should be negative; and the product of any two negative numbers should be positive.

In the next lesson we will develop a precise definition of the product of two rational numbers. Then, in later lessons, you will see whether or not we have defined both the sum and the product of two rational numbers in such a way that the usual properties actually do hold true for the operations of rational numbers.

In this lesson you learned that we want the product of two positive rational numbers to be positive; the product of two negative rational numbers to be positive; the product of any rational number and 0 to be 0; and the product of a positive number and a negative number to be negative.

A $^{+}6 + {}^{-}6 = 0.$

B $^{+}5(0) = 0.$

C $^{+}5(^{+}6 + {}^{-}6) = 0.$

D $^{+}5(^{+}6 + {}^{-}6) = (^{+}5 \circ {}^{+}6) + (^{+}5 \cdot {}^{-}6).$

E $(^{+}5 \cdot {}^{+}6) + (^{+}5 \cdot {}^{-}6) = 0.$

D3

On your own

1 What number should be the product of $^{+}5$ and $^{+}6$? Of $^{+}\frac{1}{2}$ and $^{+}4$? Of $^{+}\frac{10}{7}$ and $^{+}\frac{2}{5}$?

2 The statements expressed in D3 suggest what number should be the product of $^{+}5$ and $^{-}6$. Explain why statement A is true.

3 What property must hold true if statement B is true?

4 Explain how you can use statement A and statement B to obtain statement C.

5 What property must hold true for rational numbers for statement D to be true?

6 How can you use statements C and D to obtain statement E?

7 Think about statement E. What should be a standard name of $^{+}5 \cdot {}^{-}6$?

8 Explain why $^{-}6 \cdot {}^{+}5$ should be $^{-}30$.

9 What number should be the product of $^{+}7$ and $^{+}2$? Of $^{+}7$ and 0? Now develop five statements like the ones expressed in D3 to explain why the product of $^{+}7$ and $^{-}2$ should be $^{-}14$.

10 If $^{+}7 \cdot {}^{-}2$ should be $^{-}14$, explain why $^{-}2 \cdot {}^{+}7$ should also be $^{-}14$.

11 What number should be the product of $^{-}2$ and 0?

12 Develop five statements like the ones expressed in D3 to explain what number the product of $^{-}2$ and $^{-}7$ should be. Use your answers for exercises 10 and 11 and the true statement $^{+}7 + {}^{-}7 = 0$ to help you develop the statements.

13 What should be a standard name of the product $^{-}2 \cdot {}^{-}7$?

Product of two rational numbers

In lesson 132 you used certain properties of operations to suggest what number the product of two given rational numbers should be. In this lesson you will develop a precise definition of the product of two rational numbers.

A Look at sentences A, C, E, and G in D1. From your previous work, you know that you want the product of two rational numbers to be defined in such a way that statements A, C, E, and G are all true. How do you know that the same statement is expressed by sentences A and B? That the same statement is expressed by sentences C and D? By sentences E and F? By sentences G and H?

B Sentence B in D1 is repeated in D2. Keep this sentence in mind as you study diagram I in D2. What is the product of the first components of (3, 1) and (5, 2)? What is the product of the second components? Is the first component of (17, 11) the sum of these two products?

C Study diagram J. What is the product of the first component of (3, 1) and the second component of (5, 2)? What is the product of the second component of (3, 1) and the first component of (5, 2)? Is the second component of (17, 11) the sum of these two products?

D Now look at D3. Notice that sentence D in D1 is repeated. Keep this sentence in mind as you study diagram K. How was 12 obtained from the components of (4, 2) and (3, 3)? How was 6 obtained? Is the first component of (18, 18) the sum of 12 and 6?

A $^{+}2 \times {}^{+}3 = {}^{+}6$.
B $(3, 1) \times (5, 2) = (17, 11)$.
C $^{+}2 \times 0 = 0$.
D $(4, 2) \times (3, 3) = (18, 18)$.
E $^{+}2 \times {}^{-}3 = {}^{-}6$.
F $(5, 3) \times (1, 4) = (17, 23)$.
G $^{-}3 \times {}^{-}2 = {}^{+}6$.
H $(2, 5) \times (2, 4) = (24, 18)$.

D1

B $(3, 1) \times (5, 2) = (17, 11)$.

$3 \times 5 = 15.$ $3 \times 2 = 6.$

I $(3, 1) \times (5, 2)$ **J** $(3, 1) \times (5, 2)$

$1 \times 2 = 2.$ $1 \times 5 = 5.$

$15 + 2 = 17.$ $6 + 5 = 11.$

D2

D $(4, 2) \times (3, 3) = (18, 18)$.

12 12

K $(4, 2) \times (3, 3)$ **L** $(4, 2) \times (3, 3)$

6 6

D3

E Study diagram L in D3. How was 12 obtained from the components of (4, 2) and (3, 3)? How was 6 obtained? Is the second component of (18, 18) equal to $12 + 6$?

F Sentence F in D1 is repeated in D4. Study diagram M in D4. Explain how to obtain the

first component of (17, 23) from the components of (5, 3) and (1, 4).

G Study diagram N in D4. Explain how to obtain the second component of (17, 23) from the components of (5, 3) and (1, 4).

H Sentence H in D1 is repeated in D5. Make a diagram that shows how to obtain the first component of (24, 18) from the components of (2, 5) and (2, 4). Then make a diagram that shows how to obtain the second component of (24, 18).

You have already decided what the product of two given rational numbers should be. From exercises A through H above, you see that you can obtain this product by using the components of the ordered pairs that indicate the two rational numbers.

Now you will use variables to develop a definition of the product of two rational numbers.

I Look at D6. Think of (a, b) and (c, d) as any two rational numbers. What is the universe for each of the variables in condition O?

J Think about condition O. What is the first component of $(ac + bd, ad + bc)$? What is the second component?

K Now study diagrams P and Q in D6. How can you obtain the first component of $(ac + bd, ad + bc)$ from the components of (a, b) and (c, d)? How can you obtain the second component of $(ac + bd, ad + bc)$ from the components of (a, b) and (c, d)?

Now we can define the *product* of the rational numbers (a, b) and (c, d). The universe for a, b, c, and d is R_a. For each a, b, c, and d, $(a, b) \times (c, d) = (ac + bd, ad + bc)$.

L Look at D7. Sentence O in D6 is repeated.

? To obtain statement R from condition O, we replaced a by 3, b by 1, c by 0, and d by $\frac{2}{3}$. Is $3 \cdot 0 + 1 \cdot \frac{2}{3}$ the same number as $0 + \frac{2}{3}$? Is $3 \cdot \frac{2}{3} + 1 \cdot 0$ the same number as $2 + 0$? How do you know?

F $(5, 3) \times (1, 4) = (17, 23)$.

M $(5, 3) \times (1, 4)$ N $(5, 3) \times (1, 4)$

D4

H $(2, 5) \times (2, 4) = (24, 18)$.

D5

Universe for a, b, c, and $d = R_a$.

O $(a, b) \times (c, d) = (ac + bd, ad + bc)$.

P $(a, b) \times (c, d)$ (ac, bd) Q $(a, b) \times (c, d)$ (ad, bc)

D6

Universe for a, b, c, and $d = R_a$.

O $(a, b) \times (c, d) = (ac + bd, ad + bc)$.

R $(3, 1) \times (0, \frac{2}{3}) = (0 + \frac{2}{3}, 2 + 0)$.

S $(\frac{1}{2}, 2) \times (2, \frac{2}{3}) = (1 + \frac{4}{3}, \frac{1}{3} + 4)$.

T $(\frac{1}{2}, 2) \times (2, \frac{2}{3}) = (\frac{7}{3}, \frac{13}{3})$.

U $(\frac{3}{5}, 1) \times (\frac{1}{2}, \frac{5}{2}) = (\frac{14}{5}, 2)$.

D7

product of two rational numbers. The universe for a, b, c, and d is R_a. For each a, b, c, and d, $(a, b) \times (c, d) = (ac + bd, ad + bc)$.

M How do you know that statement R is true?

N What replacements were made for the variables in condition O to obtain statement S? Is statement S true?

O How do you know that sentence T expresses the same statement as sentence S?

P What replacements were made for the variables in condition O to obtain statement U? Is statement U true?

Use the definition of the product of two rational numbers to decide whether each of the following sentences expresses a true statement or a false statement.

Q $(\frac{1}{3}, 0) \times (\frac{5}{6}, 0) = (\frac{1}{3}, \frac{5}{6})$.

R $(0, 5) \times (\frac{2}{9}, 0) = (0, \frac{10}{9})$.

S $(0, 0) \times (\frac{4}{7}, \frac{3}{2}) = (0, 0)$.

T $(1, \frac{9}{4}) \times (\frac{1}{8}, \frac{3}{4}) = (\frac{29}{16}, \frac{33}{32})$.

Now that you have studied the definition of the product of two rational numbers, you can decide whether or not multiplication of rational numbers has these two important properties: the closure property and the well-defined property.

A Use the closure properties of addition and multiplication of rational numbers of arithmetic to explain why, for each a, b, c, and d, $(ac + bd, ad + bc)$ is an ordered pair in which the components are members of R_a. The universe for each variable is R_a.

B Use your answer for exercise A and the definition of the product of two rational numbers to explain why the product of any two rational numbers is a rational number.

You can see that the product of any two rational numbers is an element of R. Therefore, R is closed under multiplication. This property is the *closure property of multiplication of rational numbers*. The property is expressed as follows.

Universe for a, b, c, and $d = R_a$.

O $(a, b) \times (c, d) = (ac + bd, ad + bc)$.

R $(3, 1) \times (0, \frac{2}{3}) = (0 + \frac{2}{3}, 2 + 0)$.

S $(\frac{1}{2}, 2) \times (2, \frac{2}{3}) = (1 + \frac{4}{3}, \frac{1}{3} + 4)$.

T $(\frac{1}{2}, 2) \times (2, \frac{2}{3}) = (\frac{7}{3}, \frac{13}{3})$.

U $(\frac{3}{5}, 1) \times (\frac{1}{2}, \frac{5}{2}) = (\frac{14}{5}, 2)$.

D7

A $(4, 2) \cdot (1, 6) = (16, 26)$.

B $(\frac{5}{2}, \frac{1}{2}) \cdot (0, 5) = (\frac{5}{2}, \frac{25}{2})$.

C $(4, 2) \cdot (\frac{4}{3}, \frac{19}{3}) = (\frac{54}{3}, \frac{84}{3})$.

D $(20, 18) \cdot (1, 6) = (128, 138)$.

D8

The universe for x *and* y *is R. For each* x *and* y, $xy \in R$.

C Is multiplication of rational numbers a binary operation?

D Now look at D8. Does each sentence express a true statement? How do you know?

E In statement A, (16, 26) is the product of (4, 2) and what other rational number?

F Statement B was obtained from statement A. (4, 2) was replaced by $(\frac{5}{2}, \frac{1}{2})$. What replacement was made for (1, 6)? Is $(\frac{5}{2}, \frac{1}{2}) = (4, 2)$? Is $(0, 5) = (1, 6)$?

G To obtain statement C from statement A, what replacement was made for (1, 6)? Is the replacement equal to (1, 6)?

H To obtain statement D from statement A, what replacement was made for (4, 2)? Is the replacement equal to (4, 2)?

I Is each of the products $(\frac{5}{2}, \frac{25}{2})$, $(\frac{54}{3}, \frac{84}{3})$, and (128, 138) equal to the product (16, 26)? How do you know?

J If one or both of the two rational numbers that are used to obtain a product are replaced

by equal rational numbers, does the product remain the same?

If two rational numbers that form a product are replaced by equal rational numbers, the product is still the same rational number. This property is the *well-defined property of multiplication of rational numbers.* The property is expressed below.

If one or both of the numbers used in forming the product of two rational numbers are replaced by equal rational numbers, the product is still the same rational number.

Next you will use the definition of the product of two rational numbers to decide whether the product of two given rational numbers is a positive number, a negative number, or zero.

A Look at D9. What is the universe for m and n? Now study sentence E. How do you know that, for each m and n, both $(m, 0)$ and $(n, 0)$ are positive rational numbers?

B Explain why $(mn, 0)$ is the product of $(m, 0)$ and $(n, 0)$.

C For each m and n, is $(mn, 0)$ positive, or is it negative? What can you say about the product of any two positive numbers?

D Look at sentence F in D9. You know that $(m, 0)$ is a positive rational number. For each n, is $(0, n)$ positive, or is it negative? Explain why $(0, mn)$ is the product of $(m, 0)$ and $(0, n)$. For each m and n, is $(0, mn)$ positive, or is it negative?

E What can you say about the product of any positive number and any negative number?

F Look at sentence G. How can you obtain the product $(0, mn)$? What can you say about the product of any negative number and any positive number?

G Look at sentence H. How can you obtain the product $(mn, 0)$? What can you say about the product of any two negative numbers?

Universe for m and n = Z.

E $(m, 0) \cdot (n, 0) = (mn, 0)$.

F $(m, 0) \cdot (0, n) = (0, mn)$.

G $(0, m) \cdot (n, 0) = (0, mn)$.

H $(0, m) \cdot (0, n) = (mn, 0)$.

D9

I $(\frac{2}{7}, 0) \cdot (14, 0) = (4, 0)$.

J $(0, \frac{3}{2}) \cdot (\frac{5}{8}, 0) = (0, \frac{15}{16})$.

K $(0, 10) \cdot (0, 7) = (70, 0)$.

D10

H Now think of (a, b) as any rational number. The universe for a and b is R_a. What is the product of (a, b) and $(0, 0)$? What is the product of $(0, 0)$ and (a, b)?

I What is the product of any rational number and 0? What is the product of 0 and any rational number?

Tell whether each product named below is a positive number, a negative number, or 0.

J $+3 \times {}^{-}8$ **L** $\frac{-1}{3} \times \frac{-7}{5}$ **N** $+52 \times \frac{+6}{7}$

K ${}^{-}26 \times 0$ **M** $\frac{-11}{13} \times +3$ **O** $\frac{-2}{3} \times {}^{-}98$

In the following exercises you will learn a short way to find the product of two rational numbers. You will use basic pairs and the definition of the product of two rational numbers.

A Look at D10. Use the definition of product to explain why statement I is true. Does $(\frac{2}{7}, 0)$ indicate $\frac{+2}{7}$? What is a standard name of $(14, 0)$? What is the product of $\frac{+2}{7}$ and $+14$?

B Look again at sentence I. This time think only of the rational numbers of arithmetic that are components of the basic pairs. How can you obtain 4 from $\frac{2}{7}$ and 14?

C You can find the product of $\frac{+2}{7}$ and $+14$ by first finding the product of $\frac{2}{7}$ and 14 and then

I $(\frac{2}{7}, 0) \cdot (14, 0) = (4, 0)$.

J $(0, \frac{3}{2}) \cdot (\frac{5}{8}, 0) = (0, \frac{15}{16})$.

K $(0, 10) \cdot (0, 7) = (70, 0)$.

D10

deciding whether the product is positive or negative. Is the product of any two positive rational numbers positive, or is it negative?

D Does sentence J in D10 express a true statement? What is a standard name of $(0, \frac{3}{2})$? Of $(\frac{5}{8}, 0)$?

E Now think about the non-zero components of $(0, \frac{3}{2})$ and $(\frac{5}{8}, 0)$. What is the product of $\frac{3}{2}$ and $\frac{5}{8}$? Is the product of $^{-}\frac{3}{2}$ and $^{+}\frac{5}{8}$ positive, or is it negative? Give a standard name of the product of $^{-}\frac{3}{2}$ and $^{+}\frac{5}{8}$.

To obtain the product of $^{-}\frac{3}{2}$ and $^{+}\frac{5}{8}$, you can first find the product of $\frac{3}{2}$ and $\frac{5}{8}$ and then decide if the product of the rational numbers is positive or negative.

F Does sentence K express a true statement? What is a standard name of (0, 10)? Of (0, 7)?

G Think about the non-zero components of (0, 10) and (0, 7). What is the product of 10 and 7? Is the product of any two negative rational numbers positive, or is it negative? What is the product of $^{-}10$ and $^{-}7$?

A short way to find the product of $^{-}10$ and $^{-}7$ is first to find the product of 10 and 7 and then to decide whether the product of $^{-}10$ and $^{-}7$ is positive or negative.

Use the short way that you have just learned to find the product of each pair of numbers named below.

H $^{+}\frac{1}{5}, ^{+}\frac{7}{9}$	J $^{-}9, ^{-}6$	L $^{-}\frac{3}{4}, ^{+}\frac{1}{8}$	N $^{+}12, ^{-}\frac{4}{3}$
I $^{+}3, ^{-}\frac{3}{2}$	K $0, ^{-}\frac{4}{5}$	M $^{+}45, 0$	O $^{-}6, ^{-}9$

In this lesson you learned how to find the product of two rational numbers. You also studied the closure property and the well-defined property of multiplication of rational numbers.

On your own

Use the definition of the product of two rational numbers to decide whether each of the sentences in exercises 1 through 6 expresses a true statement or a false statement.

1 $(1, 3) \times (3, 1) = (4, 4)$.
2 $(\frac{2}{3}, \frac{1}{3}) \times (\frac{2}{3}, 0) = (\frac{4}{9}, \frac{2}{9})$.
3 $(5, 5) \cdot (2, 2) = (10, 10)$.
4 $(9, 0) \times (\frac{1}{9}, 0) = (1, 0)$.
5 $(\frac{2}{7}, 0) \cdot (\frac{5}{9}, 0) = (0, \frac{10}{63})$.
6 $(0, \frac{4}{5}) \times (0, \frac{9}{8}) = (\frac{9}{10}, 0)$.

For each of exercises 7 through 11, tell whether the statement expressed is true or false. For each true statement, explain how you can use the well-defined property of multiplication of rational numbers to make your decision.

7 $(2, 3) \cdot (\frac{1}{2}, 4) = (\frac{1}{3}, \frac{4}{3}) \cdot (\frac{1}{2}, 4)$.
8 $(8, \frac{1}{8}) \times (0, 3) = (4, \frac{1}{4}) \times (0, 3)$.
9 $(\frac{13}{2}, 0) \times (\frac{2}{5}, 0) = (\frac{15}{2}, 1) \times (\frac{68}{5}, 14)$.
10 $(3, 5) \cdot (\frac{3}{7}, 1) = (6, 10) \cdot (\frac{6}{7}, 2)$.
11 $(\frac{11}{9}, 3) \times (0, \frac{10}{7}) = (0, \frac{16}{9}) \times (4, \frac{38}{7})$.

In each exercise below, find the product of the pair of numbers named.

12 $^{-}3, ^{+}9$	15 $^{+}13, ^{-}4$	18 $^{-}4, ^{+}25$
13 $^{+}12, ^{+}4$	16 $^{-}5, ^{-}11$	19 $^{-}\frac{5}{3}, ^{+}\frac{7}{8}$
14 $^{-}6, ^{-}6$	17 $^{+}12, ^{+}12$	20 $^{-}\frac{3}{5}, ^{-}\frac{1}{4}$

KEEPING SKILFUL

For each of exercises 1 through 9, find the sum of the numbers named in the exercise.

1 $^{+}\frac{6}{5}, ^{+}\frac{13}{10}$	4 $^{+}\frac{1}{6}, ^{-}\frac{4}{15}$	7 $^{-}8, ^{-}\frac{1}{7}$
2 $^{-}\frac{27}{16}, ^{-}\frac{7}{2}$	5 $^{-}\frac{3}{7}, \frac{0}{12}$	8 $^{+}\frac{15}{17}, ^{-}\frac{8}{3}$
3 $^{-}\frac{16}{5}, ^{-}\frac{48}{5}$	6 $^{-}25, ^{+}\frac{3}{14}$	9 $^{+}\frac{18}{35}, ^{-}\frac{21}{5}$

For each of exercises 10 through 18, find the product of the numbers named in the exercise.

10 $^{+}\frac{13}{2}, ^{+}\frac{18}{5}$	13 $^{+}40, ^{-}\frac{5}{4}$	16 $^{-}\frac{12}{13}, ^{-}\frac{5}{3}$
11 $^{+}\frac{5}{4}, ^{-}\frac{21}{4}$	14 $^{-}\frac{7}{3}, ^{+}\frac{20}{3}$	17 $^{+}\frac{3}{4}, 0$
12 $^{-}6, ^{-}\frac{6}{5}$	15 $^{-}2, ^{-}\frac{36}{7}$	18 $^{+}\frac{27}{28}, ^{-}\frac{1}{2}$

The elements of R_a related to certain elements of R

You know how the set of natural numbers is related to a subset of the set of rational numbers of arithmetic. Now you will learn how the set of rational numbers of arithmetic is related to a subset of the set of rational numbers.

A Study the mappings represented in D1. Picture A represents a number line for the rational numbers of arithmetic. Picture B represents a number line for the rational numbers. The number 0 in R_a is mapped onto the number 0 in R. Onto what member of R is $\frac{3}{2}$ mapped? Onto what member of R_a is $^+3$ mapped?

B How many elements of R_a that are named in D1 are not mapped onto rational numbers? Which rational numbers named in D1 are not mapped onto elements of R_a?

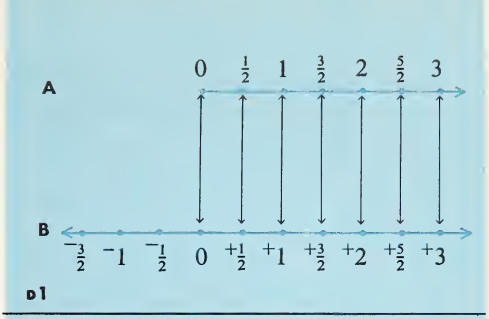
C If you were to continue the mapping represented in D1, each element of R_a would be mapped onto a rational number. Onto what rational number would $\frac{5}{8}$ be mapped? Onto what rational number would $\frac{15}{3}$ be mapped?

D In lesson 128 you learned that the set of rational numbers is the union of R_p, R_n, and {0}. Is $\{0\} \cup R_p$ a subset of R?

E Is every positive rational number a member of $\{0\} \cup R_p$? How many negative rational numbers are members of $\{0\} \cup R_p$?

Any rational number that is not a negative number is a *non-negative rational number*.

F How do you know that 0 is a non-negative rational number? How do you know that $\{0\} \cup R_p$ is a proper subset of R?



D1

$\overline{R}_n = \{0\} \cup R_p.$

D2

G The universe is R. Is $\{0\} \cup R_p$ the complement of R_n? How do you know?

> REMINDER
> The set that contains all the members of the universe, and only those members, that do not belong to a given subset is the complement of the given subset.
> *See lesson 112, page 78.*

When the universe is R, the complement of R_n is the set of non-negative rational numbers. The set of non-negative rational numbers contains zero and all the positive rational numbers. We will use the symbol shown at the right as a name of this set. The sentence in D2 expresses a true statement about $\overline{R}_n$.

$\overline{R}_n$

H Think about the mapping that is represented in D1. How do you know that there is a one-to-one correspondence between R_a and $\overline{R}_n$? How do you know that R_a and $\overline{R}_n$ are equivalent sets?

I What member of $\overline{R}_n$ is mapped onto $\frac{1}{2}$? What member of R_a is mapped onto $^+\frac{1}{2}$?

Since $\frac{1}{2}$ and $^+\frac{1}{2}$ are mapped onto each other, $\frac{1}{2}$ and $^+\frac{1}{2}$ are *mates* of each other. In the one-

to-one correspondence that we have been discussing, any two numbers that map onto each other are mates. Each number is the mate of the other. One of the mates is a rational number of arithmetic, and the other is a non-negative rational number.

J How do you know that $\frac{5}{6}$ and $^{+}\frac{10}{12}$ are mates? How do you know that $\frac{5}{6}$ and $^{+}\frac{2}{3}$ are not mates? That $\frac{5}{6}$ and $^{-}\frac{5}{6}$ are not mates?

K How many members of $\overline{R}_n$ are mates of $\frac{7}{8}$? How many members of R_a are mates of $^{+}\frac{3}{5}$?

L What is the basic pair for $^{+}\frac{8}{9}$? Is $\frac{8}{9}$ the mate of the rational number $(\frac{8}{9}, 0)$?

M You know that you can use either standard names or names of ordered pairs to express rational numbers. When the universe is R_a, we will use the symbol ^{+}a to name $(a, 0)$. What is the mate of ^{+}a?

N Is $^{+}(a + b)$ the same as $(a + b, 0)$ when the universe for a and b is R_a? Is $^{+}(a + b)$ the mate of $a + b$? Is $^{+}(a \cdot b)$ the mate of $a \cdot b$?

Remember that mates are members of two different sets. For example, $\frac{3}{4}$ and $^{+}\frac{3}{4}$ are mates. $\frac{3}{4} \in R_a$, and $^{+}\frac{3}{4} \in \overline{R}_n$.

O What is the sum of $\frac{3}{4}$ and $\frac{1}{2}$? What is the sum of $^{+}\frac{3}{4}$ and $^{+}\frac{1}{2}$?

P Are $\frac{5}{4}$ and $^{+}\frac{5}{4}$ mates?

Q What is the mate of $^{+}\frac{2}{5}$? What is the mate of $^{+}\frac{1}{5}$? What is the mate of $^{+}\frac{2}{5} + {}^{+}\frac{1}{5}$?

R Think of a and ^{+}a and b and ^{+}b as any two pairs of mates. The universe for a and b is R_a. You know that $^{+}(a + b)$ and $a + b$ are mates. How do you know that, for each a and b, $a + b$ is the mate of $^{+}a + {}^{+}b$?

If you add two rational numbers of arithmetic and then add the non-negative rational numbers that are their mates, the sum of the two elements of R_a is the mate of the sum of the two elements of $\overline{R}_n$.

S What is the product of $\frac{2}{9}$ and $\frac{4}{3}$? What is the product of $^{+}\frac{2}{9}$ and $^{+}\frac{4}{3}$?

T Is $\frac{8}{27}$ the mate of $^{+}\frac{8}{27}$?

U What is the mate of $^{+}\frac{1}{6}$? What is the mate of $^{+}\frac{1}{2}$? What is the mate of $^{+}\frac{1}{6} \cdot {}^{+}\frac{1}{2}$?

V Think of a and ^{+}a and b and ^{+}b as any two pairs of mates. The universe for a and b is R_a. You know that $^{+}(a \cdot b)$ and $a \cdot b$ are mates. How do you know that, for each a and b, $a \cdot b$ is the mate of $^{+}a \cdot {}^{+}b$?

If you multiply two rational numbers of arithmetic and then multiply the non-negative rational numbers that are their mates, the product of the two elements of R_a is the mate of the product of the two elements of $\overline{R}_n$.

Because the rational number of arithmetic a and the non-negative rational number ^{+}a behave in much the same way in addition and multiplication, it is often convenient to use the same numeral to name a member of $\overline{R}_n$ and its mate in R_a. For example, you can use the symbol 3 to name $^{+}3$ or the symbol $\frac{1}{2}$ to name $^{+}\frac{1}{2}$.

In this lesson you learned that R_a and $\overline{R}_n$ can be put in one-to-one correspondence and that element a of R_a is the mate of element ^{+}a of $\overline{R}_n$. You also learned that, if a and b are rational numbers of arithmetic, then $a + b$ and $^{+}a + {}^{+}b$ are mates and $a \cdot b$ and $^{+}a \cdot {}^{+}b$ are mates. The universe for the variables is R_a.

On your own

Name the rational number of arithmetic that is the mate of the rational number named in each of exercises 1 through 5.

1 $^{+}\frac{7}{8}$ **2** $^{+}\frac{11}{7}$ **3** $^{+}\frac{5}{2}$ **4** $^{+}\frac{3}{19}$ **5** $^{+}\frac{9}{10}$

For each of exercises 6 through 10, name the element of $\overline{R}_n$ that is the mate of the element of R_a named.

6 0 **7** $\frac{1}{4}$ **8** $\frac{13}{6}$ **9** $\frac{4}{15}$ **10** $\frac{7}{100}$

Two rational numbers of arithmetic are named in each of exercises 11 through 20. For each exercise, first write a sentence that expresses a true statement about the sum of the

two numbers. Then replace the members of R_a by their rational-number mates and write a sentence that expresses a true statement about the sum of the mates. For exercise 11, you could write the two sentences below.

$\frac{5}{9} + \frac{7}{3} = \frac{26}{9}$. $^{+}\frac{5}{9} + {}^{+}\frac{7}{3} = {}^{+}\frac{26}{9}$.

11 $\frac{5}{9}, \frac{7}{3}$ **13** $\frac{3}{2}, 2$ **15** $\frac{1}{3}, \frac{4}{5}$ **17** $6, \frac{4}{5}$ **19** $\frac{5}{3}, \frac{10}{12}$

12 $\frac{11}{10}, \frac{3}{10}$ **14** $\frac{1}{4}, \frac{5}{8}$ **16** $9, 4$ **18** $\frac{2}{9}, 7$ **20** $\frac{6}{7}, \frac{3}{10}$

Two non-negative rational numbers are named in each of exercises 21 through 26. For each exercise, first write a sentence that expresses a true statement about the product of the two numbers. Then replace the members of $\overline{R}_n$ by their mates in R_a and write a sentence that expresses a true statement about the product of the mates.

21 $0, {}^{+}\frac{1}{2}$ **23** $^{+}\frac{7}{10}, {}^{+}1$ **25** $^{+}\frac{3}{5}, {}^{+}\frac{4}{15}$

22 $^{+}5, {}^{+}\frac{1}{4}$ **24** $^{+}\frac{5}{6}, {}^{+}\frac{5}{12}$ **26** $^{+}\frac{7}{1}, {}^{+}\frac{8}{1}$

Tell which sentences in exercises 27 through 32 express true statements and which express false statements.

27 $R = R_p \cup R_n \cup \{0\}$.

28 $I_n \cup \{0\} \cup R_p = R$.

29 $\overline{R}_n = \{0\} \cup R_n$.

30 $\overline{R}_n \cap I_p = I_p$.

31 $I_n \cap R_n = I_n$.

32 $I_p \cup I_n = I$.

KEEPING SKILFUL

For each condition expressed below, tabulate the solution set if possible. Otherwise, give a standard description of the solution set. $U = R_a$.

1 $15.5 - s > 10.5 \wedge .4 + 1.4 < s$.

2 $\sim (n - \frac{1}{2} > 4\frac{1}{2})$.

3 $.5 + .3 = m \vee 16.9 - 16.2 = m$.

4 $3\frac{4}{5} < 2 + r \wedge r + \frac{3}{10} < 4$.

5 $\sim (p + .53 \neq 7.53)$.

6 $8\frac{2}{3} + q > 9\frac{1}{2} \vee q + \frac{1}{2} = 3$.

7 $z - 4 \geqq 10\frac{1}{2} \wedge z + 3 > 16\frac{1}{2}$.

8 $m + .5 < 3.6 \vee m - 2.5 = .2$.

9 $16.8 - w > 13.4 \wedge w + .2 \geqq 1$.

CHECKING UP

If you have trouble with these two tests, you can find help in lessons 130 through 134.

Test 124

Name the element of $\overline{R}_n$ that is the mate of each rational number of arithmetic expressed in exercises 1 through 5.

1 $\frac{5}{6}$ **2** $\frac{8}{9}$ **3** $\frac{47}{10}$ **4** 0 **5** 67

Name the element of R_a that is the mate of each rational number expressed in exercises 6 through 10.

6 $^{+}\frac{4}{9}$ **7** 0 **8** $^{+}81$ **9** $^{+}\frac{17}{8}$ **10** $^{+}\frac{1}{100}$

For each of exercises 11 through 19, decide which one of the symbols, R_n, R_p, or $\{0\}$, correctly completes the sentence. The universe for x and y is Z.

11 For each x and y, $(x, 0) + (y, 0) \in$ ~~~~.

12 For each x and y, $(0, x) + (0, y) \in$ ~~~~.

13 For each x and y, $(x, 0) \times (0, y) \in$ ~~~~.

14 For each x and y, $(0, x) \times (y, 0) \in$ ~~~~.

15 For each x and y, $(x, 0) \times (y, 0) \in$ ~~~~.

16 For each x and y, $(0, x) \times (0, y) \in$ ~~~~.

17 Suppose that x is less than y. For each x and y, $(0, x) + (y, 0) \in$ ~~~~.

18 Suppose that x is equal to y. For each x and y, $(0, x) + (y, 0) \in$ ~~~~.

19 Suppose that x is greater than y. For each x and y, $(0, x) + (y, 0) \in$ ~~~~.

Test 125

For each of exercises 20 through 28, find the sum of the numbers named.

20 $^{+}13, {}^{+}42$ **23** $^{+}7, {}^{-}8$ **26** $^{+}\frac{8}{15}, {}^{+}\frac{4}{15}$

21 $^{-}17, {}^{-}5$ **24** $^{-}26, {}^{+}52$ **27** $^{-}\frac{18}{7}, {}^{+}\frac{11}{14}$

22 $^{-}\frac{3}{2}, {}^{-}\frac{2}{5}$ **25** $^{+}\frac{3}{4}, {}^{-}\frac{1}{8}$ **28** $^{-}\frac{1}{3}, {}^{-}\frac{5}{6}$

For each of exercises 29 through 37, find the product of the numbers named.

29 $^{+}13, {}^{+}42$ **32** $^{-}\frac{5}{6}, {}^{-}\frac{1}{2}$ **35** $^{+}\frac{1}{3}, {}^{-}\frac{2}{5}$

30 $^{-}7, {}^{-}2$ **33** $^{-}\frac{2}{9}, {}^{+}3$ **36** $^{+}\frac{1}{2}, {}^{+}17$

31 $^{-}23, {}^{+}\frac{3}{4}$ **34** $^{+}\frac{9}{10}, {}^{+}33$ **37** $^{+}\frac{1}{6}, {}^{-}\frac{5}{3}$

The commutative properties

You know that the order in which you add two members of N or two members of R_a does not affect their sum. You also know that the order in which you multiply two members of N or two members of R_a does not affect their product. In this lesson you will decide if addition and multiplication of rational numbers have the commutative properties.

A Look at D1. Sentence A expresses a statement about the sum of two members of R_a. Is $\frac{5}{6}$ the mate of the rational number $^+\frac{5}{6}$? What rational number is the mate of $\frac{1}{4}$? Of $\frac{13}{12}$?

Since R_a and $\overline{R}_n$ behave in the same way with respect to the operations of addition and multiplication, you may use the same numeral to name a member of $\overline{R}_n$ and its mate in R_a. This means that you may omit the symbol for "positive" in standard names of positive rational numbers. Sentence A in D1 can express the same statement about rational numbers as does the sentence $^+\frac{5}{6} + {}^+\frac{1}{4} = {}^+\frac{13}{12}$.

B Now think of each sentence in D1 as expressing a statement about the sum of two rational numbers. Are all of the statements true?

From the transitive property of equality, you know that, if a first number is equal to a second number and the second number is equal to a third number, then the first number is equal to the third number. Using the transitive property of equality and statements A and B expressed in D1, you can obtain the true statement $\frac{5}{6} + \frac{1}{4} = \frac{1}{4} + \frac{5}{6}$.

D1

A $\frac{5}{6} + \frac{1}{4} = \frac{13}{12}$.

B $\frac{1}{4} + \frac{5}{6} = \frac{13}{12}$.

C $^-\frac{3}{8} + {}^-\frac{1}{2} = {}^-\frac{7}{8}$.

D $^-\frac{1}{2} + {}^-\frac{3}{8} = {}^-\frac{7}{8}$.

E $\frac{2}{5} + {}^-\frac{1}{3} = \frac{1}{15}$.

F $^-\frac{1}{3} + \frac{2}{5} = \frac{1}{15}$.

D2

Universe for a, b, c, and $d = R_a$.

G $(a, b) + (c, d) = (a + c, b + d)$.

H $(c, d) + (a, b) = (c + a, d + b)$.

I $(a + c, b + d) = (c + a, d + b)$.

J $(a, b) + (c, d) = (c + a, d + b)$.

K $(a, b) + (c, d) = (c, d) + (a, b)$.

C What does $\frac{5}{6} + \frac{1}{4} = \frac{1}{4} + \frac{5}{6}$ tell you about the order in which you add $\frac{5}{6}$ and $\frac{1}{4}$?

D From which two statements expressed in D1 can you obtain $^-\frac{3}{8} + {}^-\frac{1}{2} = {}^-\frac{1}{2} + {}^-\frac{3}{8}$? Is $^-\frac{3}{8} + {}^-\frac{1}{2} = {}^-\frac{1}{2} + {}^-\frac{3}{8}$ a true statement? What do you know about the order in which you add $^-\frac{3}{8}$ and $^-\frac{1}{2}$?

E Using the transitive property of equality, what other true statement can you obtain from statements E and F? What do you know about the order in which you add $\frac{2}{5}$ and $^-\frac{1}{3}$?

You can use variables to help you decide if the order in which you add any two rational numbers affects the sum.

F Look at D2. Think of (a, b) and (c, d) as any two rational numbers. What is the universe for each of the variables?

If you can show that, for each a, b, c, and d, the sum of (a, b) and (c, d) is the same as the sum of (c, d) and (a, b), then you know that the order in which you add any two rational numbers does not affect the sum.

G Look at sentences G and H in D2. For each a, b, c, and d, is $(a + c, b + d)$ the sum of (a, b) and (c, d)? How do you know? What is the sum of (c, d) and (a, b)?

H Look at sentence I in D2. Remember that the universe is R_a. For each a, b, c, and d, is $a+c=c+a$? Is $b+d=d+b$? Upon what property of addition of rational numbers of arithmetic do your answers depend? Is $(a+c, b+d)=(c+a, d+b)$? Remember that two ordered pairs are equal when their first components are equal and their second components are equal.

I Look again at D2. You can obtain condition J from conditions G and I by using the transitive property of equality. Since true statements are obtained from conditions G and I for each a, b, c, and d, a true statement is also obtained from condition J. From which conditions expressed in D2 can you obtain condition K?

J How do you know that, for each a, b, c, and d, $(a, b)+(c, d)=(c, d)+(a, b)$?

You have just observed that the order in which you add any two rational numbers does not affect the sum. This important property is the *commutative property of addition of rational numbers*. The property is expressed below.

The universe for x *and* y *is R. For each* x *and* y, $x+y=y+x$.

Notice that you have developed, or derived, the commutative property of addition of rational numbers from a definition and properties that you studied previously. By using the definition and these properties, you were able to prove that $x+y=y+x$ is actually a property of rational numbers. Remember that, when a condition is a property, you obtain a true statement from the condition for each replacement of the variables.

Decide whether each sentence below expresses a true statement or a false statement.

K $\frac{3}{4}+\frac{-4}{5}=\frac{-4}{5}+\frac{3}{4}$.

L $\frac{-1}{6}+\frac{3}{8}=\frac{-3}{8}+\frac{1}{6}$.

M $\frac{-5}{9}+\frac{-7}{10}=\frac{-7}{10}+\frac{-5}{9}$.

N $\frac{2}{5}+\frac{-7}{8}=\frac{-7}{8}+\frac{2}{5}$.

O $\frac{-2}{9}\times\frac{-3}{5}=\frac{6}{45}$.

P $\frac{2}{3}\times\frac{-4}{7}=\frac{-8}{21}$.

Q $\frac{-4}{7}\times\frac{2}{3}=\frac{-8}{21}$.

D3

Universe for a, b, c, and $d=R_a$.

R $(a, b)\cdot(c, d)=(ac+bd, ad+bc)$.

S $(c, d)\cdot(a, b)=(ca+db, cb+da)$.

D4

In lesson 132 you used the idea of a commutative property of multiplication to help you find a useful definition of the product of two rational numbers. Now you can decide whether or not you have defined the product in such a way that multiplication of rational numbers has the commutative property. In other words, you will decide if the order in which you multiply two rational numbers affects the product.

A Each statement expressed in D3 concerns the product of two rational numbers. Are all of the statements true?

B Using the transitive property of equality, can you obtain $\frac{1}{3}\times\frac{4}{5}=\frac{4}{5}\times\frac{1}{3}$ from statements L and M? Is $\frac{1}{3}\times\frac{4}{5}=\frac{4}{5}\times\frac{1}{3}$?

C Using the transitive property, what other true statement can you obtain from statements N and O? What other true statement can you obtain from statements P and Q?

D What do you know about the order in which you multiply $\frac{1}{3}$ and $\frac{4}{5}$? $\frac{-3}{5}$ and $\frac{-2}{9}$? $\frac{-4}{7}$ and $\frac{2}{3}$?

You can use variables to help you decide if the order in which you multiply any two rational numbers affects the product.

E Look at D4. Think of (a, b) and (c, d) as any two rational numbers. What is the universe for each of the variables?

Universe for a, b, c, and $d = R_a$.

R $(a, b) \cdot (c, d) = (ac + bd, ad + bc)$.

S $(c, d) \cdot (a, b) = (ca + db, cb + da)$.

D4

F What is the product of (a, b) and (c, d)? Of (c, d) and (a, b)? How do you know?

G Now try to prove that, for each a, b, c, and d, $(a, b) \cdot (c, d) = (c, d) \cdot (a, b)$. The universe for the variables is R_a. In your proof, use the transitive property of equality and the commutative properties of addition and multiplication of rational numbers of arithmetic. You can refer to D2 and to the questions that concern D2 to help you in your proof.

You have learned that the order in which you multiply two rational numbers does not affect the product. This property is the *commutative property of multiplication of rational numbers*. The property is expressed below.

The universe for x *and* y *is R. For each* x *and* y, $xy = yx$.

Decide whether each of the following sentences expresses a true statement or a false statement.

H $\frac{3}{7} \cdot \frac{-2}{5} = \frac{-2}{5} \cdot \frac{-3}{7}$.

I $\frac{-1}{4} \times \frac{3}{8} = \frac{3}{8} \times \frac{-1}{4}$.

J $\frac{-3}{10} \times \frac{-2}{3} = \frac{-2}{3} \times \frac{-3}{10}$.

K $\frac{5}{6} \cdot \frac{1}{3} = \frac{-1}{3} \cdot \frac{+5}{6}$.

In this lesson you learned that both addition and multiplication of rational numbers have the commutative property.

On your own

Decide whether each of the following sentences expresses a true statement or a false statement.

1 $\frac{-1}{2} \times \frac{3}{4} = \frac{3}{4} \times \frac{-1}{2}$.

2 $\frac{-4}{5} + \frac{-2}{3} = \frac{-3}{2} + \frac{-5}{4}$.

3 $\frac{5}{8} + \frac{7}{8} = \frac{7}{1} + \frac{5}{1}$.

4 $\frac{-1}{2} \times \frac{+1}{4} = \frac{1}{4} \times \frac{1}{2}$.

5 $\frac{-5}{6} + \frac{3}{4} = \frac{3}{4} + \frac{-5}{6}$.

6 $0 + \frac{-5}{9} = \frac{5}{9} + 0$.

7 $\frac{-4}{7} \times \frac{-3}{7} = \frac{-3}{7} \times \frac{-4}{7}$.

8 $\frac{-3}{5} \times \frac{-1}{1} = \frac{1}{1} \times \frac{+3}{5}$.

Two rational numbers are named in each of exercises 9 through 17. For each exercise, write two sentences that express true statements about the sum of the two numbers. Use the commutative property of addition of rational numbers to obtain your second statement. For exercise 11, you could write the following sentences: $\frac{7}{10} + \frac{-3}{4} = \frac{-1}{20}$. $\frac{-3}{4} + \frac{7}{10} = \frac{-1}{20}$.

9 $\frac{7}{10}, \frac{-3}{4}$

10 $\frac{5}{8}, \frac{5}{6}$

11 $\frac{-1}{2}, \frac{-3}{7}$

12 $\frac{2}{3}, \frac{-7}{12}$

13 $\frac{-3}{5}, \frac{2}{11}$

14 $\frac{3}{16}, \frac{5}{12}$

15 $\frac{-3}{10}, \frac{-7}{8}$

16 $\frac{1}{4}, \frac{-5}{9}$

17 $\frac{-7}{15}, \frac{-3}{15}$

Two rational numbers are named in each of exercises 18 through 29. For each exercise, write two sentences that express true statements about the product. Use the commutative property of multiplication of rational numbers to obtain your second statement.

18 $\frac{-3}{8}, \frac{3}{4}$

19 $\frac{-7}{9}, \frac{-2}{15}$

20 $\frac{+1}{6}, \frac{-5}{12}$

21 $\frac{3}{10}, \frac{5}{9}$

22 $\frac{2}{3}, \frac{-1}{2}$

23 $\frac{-7}{20}, \frac{-4}{7}$

24 $\frac{-2}{13}, \frac{3}{4}$

25 $\frac{3}{16}, \frac{4}{5}$

26 $\frac{1}{10}, \frac{1}{7}$

27 $-5, \frac{+7}{8}$

28 $-12, -6$

29 $\frac{4}{3}, \frac{1}{16}$

KEEPING SKILFUL

For each exercise below, find a mixed numeral that expresses the rational number of arithmetic named.

1 $\frac{19}{8}$

2 $\frac{150}{12}$

3 $\frac{45}{6}$

4 $\frac{103}{11}$

5 $\frac{89}{9}$

6 $\frac{125}{3}$

7 $\frac{235}{7}$

8 $\frac{139}{11}$

In each exercise below, use a decimal to express the member of R_a named.

9 $\frac{7}{10}$

10 $\frac{3}{8}$

11 $\frac{5}{12}$

12 $\frac{7}{25}$

13 $\frac{37}{4}$

14 $\frac{78}{5}$

15 $\frac{63}{2}$

16 $\frac{70}{8}$

For each exercise below, tabulate the solution set of the condition. U = Z.

17 $5/2 \sim x/\frac{1}{2}$.

18 $87\frac{1}{2}/100 \sim 56/x$.

19 $18/x \sim .4/1$.

20 $10\frac{1}{3} - x = 4\frac{5}{6}$.

21 $x + 19\frac{3}{5} = 24\frac{2}{5}$.

22 $x/7 \sim 23/2$.

23 $x/104 \sim 12\frac{1}{2}/100$.

24 $27/x \sim .75/1$.

25 $x - 15\frac{1}{4} = 18\frac{3}{4}$.

26 $x = 6\frac{1}{8} \div 2\frac{7}{10}$.

The associative properties

You know that the way in which you group any three members of N or of R_a does not affect their sum or their product. Now you will decide if addition and multiplication of rational numbers have the associative properties.

Look at symbol A in D1. To find the sum named by symbol A, you must work with one pair of numbers at a time.

A Study sentence B in D1. In the numeral $(\frac{2}{5} + \bar{}\frac{1}{3}) + \frac{1}{2}$, the sum of which pair of numbers is expressed within parentheses? Is $\frac{1}{15}$ the sum of $\frac{2}{5}$ and $\bar{}\frac{1}{3}$?

D1

A $\frac{2}{5} + \bar{}\frac{1}{3} + \frac{1}{2}$

B $(\frac{2}{5} + \bar{}\frac{1}{3}) + \frac{1}{2} = \frac{2}{5} + (\bar{}\frac{1}{3} + \frac{1}{2})$.

D2

C $(\bar{}\frac{9}{4} + \frac{2}{3}) + \bar{}\frac{5}{8} = \bar{}\frac{9}{4} + (\frac{2}{3} + \bar{}\frac{5}{8})$.

D $(\bar{}\frac{5}{2} + \bar{}\frac{5}{6}) + \bar{}\frac{7}{12} = \bar{}\frac{5}{2} + (\bar{}\frac{5}{6} + \bar{}\frac{7}{12})$.

B What is the sum of $\frac{1}{15}$ and $\frac{1}{2}$? What is a standard name of $(\frac{2}{5} + \bar{}\frac{1}{3}) + \frac{1}{2}$?

C Look again at sentence B. In the numeral $\frac{2}{5} + (\bar{}\frac{1}{3} + \frac{1}{2})$, the sum of which pair of numbers is expressed within parentheses? What is the sum of $\bar{}\frac{1}{3}$ and $\frac{1}{2}$?

D What is the sum of $\frac{2}{5}$ and the sum that you found for exercise C? What is a standard name of $\frac{2}{5} + (\bar{}\frac{1}{3} + \frac{1}{2})$?

E Is statement B true? Does the way in which you group $\frac{2}{5}$, $\bar{}\frac{1}{3}$, and $\frac{1}{2}$ affect the sum?

F Now read sentence C in D2. What is the sum of $\bar{}\frac{9}{4}$ and $\frac{2}{3}$? The sum of $\bar{}\frac{19}{12}$ and $\bar{}\frac{5}{8}$?

G What is the sum of $\frac{2}{3}$ and $\bar{}\frac{5}{8}$? What is the sum of $\bar{}\frac{9}{4}$ and $\frac{1}{24}$?

H Is statement C true? Does the way in which you group $\bar{}\frac{9}{4}$, $\frac{2}{3}$, and $\bar{}\frac{5}{8}$ affect the sum?

I Read sentence D in D2. What is the sum of $\bar{}\frac{5}{2}$ and $\bar{}\frac{5}{6}$? What is the sum of the number that you just found and $\bar{}\frac{7}{12}$?

J What is the sum of $\bar{}\frac{5}{6}$ and $\bar{}\frac{7}{12}$? What is the sum of $\bar{}\frac{5}{2}$ and the number that you just found?

K Is statement D true? Does the way in which you group $\bar{}\frac{5}{2}$, $\bar{}\frac{5}{6}$, and $\bar{}\frac{7}{12}$ affect the sum?

You can use variables to help you prove that the way in which you group any three rational numbers does not affect the sum.

A Look at D3. Think of (a, b), (c, d), and (e, f) as any three rational numbers. What is the universe for each of the variables?

D3

Universe for a, b, c, d, e, and $f = R_a$.

E $[(a, b) + (c, d)] + (e, f) = \big((a + c) + e, (b + d) + f\big)$.

F $(a, b) + [(c, d) + (e, f)] = \big(a + (c + e), b + (d + f)\big)$.

G $\big((a + c) + e, (b + d) + f\big) = \big(a + (c + e), b + (d + f)\big)$.

H $[(a, b) + (c, d)] + (e, f) = (a, b) + [(c, d) + (e, f)]$.

You know that, to find the sum of three numbers, you must work with one pair of numbers at a time. The symbol below suggests one way to group (a, b), (c, d), and (e, f) to obtain their sum.

$[(a, b) + (c, d)] + (e, f)$

Notice that the sum of (a, b) and (c, d) is expressed within brackets. This means that you will first find the rational number that is the sum of (a, b) and (c, d). Then you will find the sum of this number and (e, f).

B Is $(a + c, b + d)$ the sum of (a, b) and (c, d)? Remember that, when we say that $(a + c, b + d)$ is the sum of (a, b) and (c, d), we mean that, for each replacement of the variables, $(a, b) + (c, d) = (a + c, b + d)$.

C Is $\big((a + c) + e, (b + d) + f\big)$ the sum of $(a + c, b + d)$ and (e, f)? Use the definition of the sum of two rational numbers to explain your answer.

Sentence E in D3 expresses the sum you obtain when you group (a, b), (c, d), and (e, f) in the way just explained.

You can find the sum of (a, b), (c, d), and (e, f) by grouping the numbers in another way. In the symbol below, the brackets help you think about how to group the three numbers.

$(a, b) + [(c, d) + (e, f)]$

D Is $(c + e, d + f)$ the sum of (c, d) and (e, f)?

E Is $\big(a + (c + e), b + (d + f)\big)$ the sum of (a, b) and $(c + e, d + f)$? Use the definition of the sum of two rational numbers to explain your answer.

Sentence F in D3 expresses the sum you obtain when you group (a, b), (c, d), and (e, f) in the way just explained.

F You have used two ways of grouping three rational numbers to find their sum. Now you will decide if the sum you obtain in each case is the same rational number. Look at sentence G. Think about the first components of the ordered pairs. For each a, c, and e, is $(a + c) + e = a + (c + e)$? Upon what property of rational numbers of arithmetic does your answer depend?

G Think about the second components of the two ordered pairs. For each b, d, and f, is $(b + d) + f = b + (d + f)$? How do you know?

H How do you know that, for each a, b, c, d, e, and f, condition G is true? For each grouping of (a, b), (c, d), and (e, f), do you obtain the same rational number for the sum?

I Use conditions E, F, and G and the transitive property of equality to show that sentence H in D3 expresses a true statement for each a, b, c, d, e, and f.

In exercises A through I, you have proved that the way in which you group any three rational numbers does not affect the sum. The property that you have derived is the *associative property of addition of rational numbers.* The property is expressed as follows.

Universe for a, b, c, d, e, and $f = R_a$.

E $[(a, b) + (c, d)] + (e, f) = \big((a + c) + e, (b + d) + f\big)$.

F $(a, b) + [(c, d) + (e, f)] = \big(a + (c + e), b + (d + f)\big)$.

G $\big((a + c) + e, (b + d) + f\big) = \big(a + (c + e), b + (d + f)\big)$.

H $[(a, b) + (c, d)] + (e, f) = (a, b) + [(c, d) + (e, f)]$.

D3

I $\frac{-7}{5} \cdot \frac{-3}{5} \cdot \frac{-1}{4}$

J $(\frac{-7}{5} \cdot \frac{-3}{5}) \frac{-1}{4} = \frac{-7}{5}(\frac{-3}{5} \cdot \frac{-1}{4})$.

D4

K $(\frac{-3}{5} \cdot \frac{3}{8}) \frac{-7}{2} = \frac{-3}{5}(\frac{3}{8} \cdot \frac{-7}{2})$.

L $(\frac{4}{3} \cdot \frac{-2}{9}) -5 = \frac{4}{3}(\frac{-2}{9} \cdot -5)$.

D5

The universe for x, y, *and* z *is R. For each* x, y, *and* z, $(x + y) + z = x + (y + z)$.

Tell whether each of the following sentences expresses a true statement or a false statement.

J $(\frac{13}{10} + \frac{-1}{4}) + \frac{-7}{10} = \frac{13}{10} + (\frac{-1}{4} + \frac{-7}{10})$.

K $(\frac{-5}{9} + \frac{-7}{3}) + \frac{-2}{5} = \frac{5}{9} + (\frac{7}{3} + \frac{2}{5})$.

L $\frac{9}{10} + (\frac{-3}{8} + \frac{-1}{3}) = (\frac{9}{10} + \frac{-3}{8}) + \frac{-1}{3}$.

Next you will decide if multiplication of rational numbers also has the associative property.

Look at symbol I in D4. To find the product named by symbol I, you must work with one pair of numbers at a time.

A Study sentence J in D4. In the numeral $(\frac{-7}{5} \cdot \frac{-3}{5}) \frac{-1}{4}$, the product of which pair of numbers is expressed within parentheses? Is $\frac{21}{25}$ the product of $\frac{-7}{5}$ and $\frac{-3}{5}$?

B What is the product of $\frac{21}{25}$ and $\frac{-1}{4}$? What is a standard name of $(\frac{-7}{5} \cdot \frac{-3}{5}) \frac{-1}{4}$?

C Look again at sentence J. In the numeral $\frac{-7}{5}(\frac{-3}{5} \cdot \frac{-1}{4})$, the product of which pair of numbers is expressed within parentheses? What is the product of the two numbers?

D What is the product of $\frac{-7}{5}$ and the number you found for exercise C? Is $\frac{-7}{5}(\frac{-3}{5} \cdot \frac{-1}{4}) = \frac{-21}{100}$?

E Is statement J true? Does the way in which you group $\frac{-7}{5}$, $\frac{-3}{5}$, and $\frac{-1}{4}$ affect the product?

F Now read sentence K in D5. What is the product of $\frac{-3}{5}$ and $\frac{3}{8}$? What is the product of the number that you just found and $\frac{-7}{2}$?

G What is the product of $\frac{3}{8}$ and $\frac{-7}{2}$? What is the product of $\frac{-3}{5}$ and the number that you just found?

H Is statement K true? Does the way in which you group $\frac{-3}{5}$, $\frac{3}{8}$, and $\frac{-7}{2}$ affect the product?

I Read sentence L in D5. How do you know that statement L is true? Does the way in which you group $\frac{4}{3}$, $\frac{-2}{9}$, and -5 affect the product?

As you continue to study statements like those expressed in D5, you will see that the way in which you group any three rational numbers does not affect the product. We will accept without proof the *associative property of multiplication of rational numbers*. The property is expressed below.

The universe for x, y, *and* z *is R. For each* x, y, *and* z, $(xy)z = x(yz)$.

Tell whether each of the following sentences expresses a true statement or a false statement.

J $\frac{-1}{10}(\frac{2}{3} \cdot \frac{1}{8}) = (\frac{-1}{10} \cdot \frac{2}{3}) \frac{1}{8}$.

K $(\frac{4}{5} \cdot \frac{3}{8}) \frac{2}{3} = \frac{4}{5}(\frac{3}{8} \cdot \frac{2}{3})$.

L $\frac{-3}{4}(\frac{-4}{3} \cdot \frac{1}{7}) = (\frac{3}{4} \cdot \frac{4}{3}) \frac{-1}{7}$.

M $(4 \cdot \frac{1}{3}) \frac{1}{6} = 4(\frac{1}{3} \cdot \frac{-1}{6})$.

N $(\frac{9}{8} \cdot \frac{1}{4}) \frac{4}{6} = \frac{9}{8}(\frac{1}{4} \cdot \frac{4}{6})$.

O $\frac{1}{9}(\frac{3}{8} \cdot 2) = (\frac{-1}{9} \cdot \frac{3}{8}) 2$.

In this lesson you learned that both addition and multiplication of rational numbers have the associative property.

On your own

First decide whether each statement expressed in exercises 1 through 8 on page 138 is true or false. Then, for each true statement, tell which of the following properties you used to make your decision.

Commutative property of addition
Commutative property of multiplication
Associative property of addition
Associative property of multiplication

1 $\overline{6}/7(5/8 \cdot \overline{2}/3) = (\overline{6}/7 \cdot 5/8)\overline{2}/3$.

2 $(3/4 + \overline{8}/9) + \overline{4}/5 = 3/4 + (\overline{8}/9 + \overline{4}/5)$.

3 $7/8 + 1/10 = 1/10 + 7/8$.

4 $5/3 \cdot \overline{5}/9 = \overline{5}/9 \cdot 5/3$.

5 $\overline{3}/10(\overline{1}/2 \cdot \overline{3}/4) = (3/10 + 1/2)3/4$.

6 $4/5 + (\overline{5}/6 + \overline{1}/4) = 4/5 + (1/4 + \overline{5}/6)$.

7 $(7/12 + \overline{4}/9) + \overline{3}/5 = \overline{3}/5(7/12 + \overline{4}/9)$.

8 $\overline{3}/8 + (3/10 \cdot \overline{4}) = (\overline{4} \cdot 3/10) + \overline{3}/8$.

For each of exercises 9 through 14, first show two ways in which the rational numbers expressed in the exercise can be grouped without affecting the sum. Then give a standard name of the sum.

9 $\overline{6}/7 + \overline{3}/2 + \overline{3}/2$

10 $4/5 + \overline{7}/16 + 2$

11 $\overline{1}/9 + 6 + 11/18$

12 $7/10 + 0 + \overline{7}/10$

13 $\overline{5}/3 + \overline{7}/6 + 1/4$

14 $\overline{6}/7 + \overline{21}/12 + \overline{7}$

For each of exercises 15 through 23, first show two ways in which the rational numbers expressed in the exercise can be grouped without affecting the product. Then give a standard name of the product.

15 $\overline{4} \cdot \overline{9} \cdot \overline{1}$

16 $\overline{9}/10 \cdot \overline{2}/3 \cdot 4/5$

17 $5/6 \cdot 7/8 \cdot 3/2$

18 $\overline{8}/9 \cdot 0 \cdot 35/44$

19 $1/6 \cdot \overline{3}/8 \cdot \overline{12}/7$

20 $3/10 \cdot \overline{1} \cdot 1/3$

21 $\overline{6} \cdot 19 \cdot 1/4$

22 $7 \cdot \overline{2}/3 \cdot 2/3$

23 $\overline{1}/9 \cdot \overline{8} \cdot 7$

KEEPING SKILFUL

For each of exercises 1 through 12, find the sum of the numbers named in the exercise.

1 $+7/6, +13/12$

2 $\overline{10}, +4/5$

3 $+16/17, \overline{5}/4, +3/2$

4 $\overline{20}, \overline{1}/9$

5 $\overline{7}/4, \overline{17}/4, +1/8$

6 $+2/3, \overline{9}/21$

7 $\overline{4}/21, 0/10$

8 $\overline{11}, \overline{8}/7$

9 $\overline{11}/15, +13/30$

10 $+32, +8/15$

11 $+19/36, \overline{19}/3$

12 $+24/25, +2/5$

For each of exercises 13 through 21, find the product of the numbers named in the exercise.

13 $+32/7, +49/10$

14 $\overline{20}/21, \overline{3}/7, \overline{2}/3$

15 $+5/6, \overline{12}/13$

16 $\overline{18}, \overline{65}/9$

17 $\overline{12}/13, \overline{13}/3$

18 $\overline{4}, \overline{3}/10$

19 $\overline{7}/3, +40/21, +2/7$

20 $+78/35, +10$

21 $+15, \overline{5}/6$

The distributive property and the system of rational numbers

In *Book 1* you learned that multiplication distributes over addition in both the set of natural numbers and the set of rational numbers of arithmetic. If x, y, and z are any three members of N or any three members of R_a, then $x(y + z) = xy + xz$.

In lesson 132 we used the idea of a distributive property to suggest how we should define the product of two rational numbers. In this lesson you will decide whether or not we did, in fact, define the product in such a way that multiplication distributes over addition in the set of rational numbers.

A Look at sentence A in D1. What product is expressed at the left of the symbol for equality? What sum is expressed at the right of the symbol for equality?

To decide if statement A is true, you will first work with the product $3/8(3/2 + \overline{2})$.

B Study sentence B in D2. Is $\overline{1}/2$ the sum of $3/2$ and $\overline{2}$? Is statement B true?

C What is the product of $3/8$ and $\overline{1}/2$? Is statement C true?

D Look at sentence D in D2. Use the transitive property of equality to explain why statement D is true.

So far, you know that the product $3/8(3/2 + \overline{2})$ is equal to $\overline{3}/16$. Next you will work with the sum $3/8(3/2) + 3/8(\overline{2})$.

E Study sentence E in D2. Is $9/16$ the product of $3/8$ and $3/2$? Is $\overline{6}/8$ the product of $3/8$ and $\overline{2}$? Is statement E true?

A $\frac{3}{8}(\frac{3}{2} + {}^{-}2) = \frac{3}{8}(\frac{3}{2}) + \frac{3}{8}({}^{-}2)$.

D1

B $\frac{3}{8}(\frac{3}{2} + {}^{-}2) = \frac{3}{8}(\frac{-1}{2})$.
C $\frac{3}{8}(\frac{-1}{2}) = \frac{-3}{16}$.
D $\frac{3}{8}(\frac{3}{2} + {}^{-}2) = \frac{-3}{16}$.
E $\frac{3}{8}(\frac{3}{2}) + \frac{3}{8}({}^{-}2) = \frac{9}{16} + \frac{-6}{8}$.
F $\frac{9}{16} + \frac{-6}{8} = \frac{-3}{16}$.
G $\frac{3}{8}(\frac{3}{2}) + \frac{3}{8}({}^{-}2) = \frac{-3}{16}$.

D2

H $\frac{-5}{6}(\frac{-1}{2} + \frac{-3}{4}) = \frac{-5}{6}(\frac{-1}{2}) + \frac{-5}{6}(\frac{-3}{4})$.
I $\frac{-2}{3}(+1 + \frac{+3}{2}) = \frac{-2}{3}(+1) + \frac{-2}{3}(\frac{+3}{2})$.
J $6(\frac{-1}{4} + {}^{-}3) = 6(\frac{-1}{4}) + 6({}^{-}3)$.
K $\frac{+6}{5}(\frac{+1}{6} + \frac{+2}{3}) = \frac{+6}{5}(\frac{+1}{6}) + \frac{+6}{5}(\frac{+2}{3})$.

D3

Universe for x, y, and z = R.
$x(y + z) = xy + xz$.
L $\frac{+12}{5}(\frac{-1}{7} + \frac{+5}{8}) = \frac{+12}{5}(\frac{-1}{7}) + \frac{+12}{5}(\frac{+5}{8})$.
M $\frac{-4}{9}(\frac{+3}{2} + \frac{+1}{3}) = \frac{-4}{9}(\frac{+3}{2}) + \frac{-4}{9}(\frac{+1}{3})$.
N $\frac{-3}{4}(\frac{-11}{12} + \frac{-7}{10}) = \frac{-3}{4}(\frac{-11}{12}) + \frac{-3}{4}(\frac{-7}{10})$.

D4

Universe for x, y, and z = R.
$x + (yz) = (x + y)(x + z)$.
O $\frac{+4}{5} + (\frac{-2}{3} \cdot \frac{+3}{5}) = (\frac{+4}{5} + \frac{-2}{3})(\frac{+4}{5} + \frac{+3}{5})$.

D5

F What is the sum of $\frac{9}{16}$ and $\frac{-6}{8}$? Is statement F true?

G Use the transitive property of equality to explain why statement G is true.

H Look again at sentence A in D1. You know that both the product $\frac{3}{8}(\frac{3}{2} + {}^{-}2)$ and the sum $\frac{3}{8}(\frac{3}{2}) + \frac{3}{8}({}^{-}2)$ are equal to $\frac{-3}{16}$. Is statement A true?

I Now, for each sentence in D3, first find the product expressed at the left of the symbol for equality. Next, find the sum expressed at the right of the symbol for equality. Then decide if the sentence expresses a true statement.

Notice that, for each sentence in D3, the product expressed at the left of the symbol for equality is the same number as the sum expressed at the right of the symbol. Each statement expressed in D3 is an example of the *distributive property of multiplication over addition of rational numbers*. We will accept this property without proof. The property is expressed below.

The universe for x, y, *and* z *is R. For each* x, y, *and* z, $x(y + z) = xy + xz$.

J Look at D4. What replacements were made for the variables in $x(y + z) = xy + xz$ to obtain statements L, M, and N? How do you know, without computing, that each statement is true?

You have learned that multiplication distributes over addition in the set of rational numbers. Next you will decide if addition distributes over multiplication.

A Look at D5. What condition is expressed? What is the universe for the variables?

B What replacements were made for the variables in $x + (yz) = (x + y)(x + z)$ to obtain statement O?

If statement O is true, then the number you obtain when you add the product $\frac{-2}{3} \cdot \frac{+3}{5}$ to $\frac{+4}{5}$ should be equal to the number you obtain when you add each of the numbers $\frac{-2}{3}$ and $\frac{+3}{5}$ to $\frac{+4}{5}$ and then multiply the sums.

C Is statement O true?

Remember that, if a condition is a property, you must obtain a true statement from the condition for each replacement of the variables.

D How many false statements do you need to obtain from the condition expressed in D5 to decide that the condition is not a property?

E Does addition distribute over multiplication of rational numbers?

As you know, a number system consists of a set of numbers and two operations that are closed, commutative, and associative. One of the operations distributes over the other. In *Book 1* you studied the system of natural numbers and the system of rational numbers of arithmetic. Now you can decide if the set of rational numbers, together with addition and multiplication, also forms a number system.

F Study D6. Is set R closed under each of the operations named in D6?

G Which condition expressed in D6 is the commutative property of addition? Which is the commutative property of multiplication?

H Which condition expressed in D6 is the associative property of addition? Which is the associative property of multiplication? Which property relates multiplication and addition?

I Does R, together with addition and multiplication, form a number system?

J Is R_p a subset of R? Does R_p, together with addition and multiplication, form a number system? Explain your answer.

K Is R_n a subset of R? Does R_n, together with addition and multiplication, form a number system? Explain your answer.

In this lesson you learned that multiplication distributes over addition of rational numbers. You also learned that the set of rational numbers, together with the operations of addition and multiplication and some of their properties, forms a number system.

On your own

For each exercise below, tell whether the sentence expresses a true statement or a false statement. For each true statement, name the property that you used to make your decision.

1 $^{-}\frac{9}{10}(6 + {}^{-}\frac{3}{4}) = {}^{-}\frac{9}{10}(6) + {}^{-}\frac{9}{10}({}^{-}\frac{3}{4})$.

2 $\frac{3}{2} + ({}^{-}\frac{57}{8} \cdot \frac{1}{3}) = (\frac{3}{2} + {}^{-}\frac{57}{8})(\frac{3}{2} + \frac{1}{3})$.

3 $({}^{-}3 \cdot {}^{-}\frac{4}{5})\frac{3}{8} = {}^{-}3({}^{-}\frac{4}{5} \cdot \frac{3}{8})$.

4 $\frac{1}{2}({}^{-}\frac{13}{4} + 2) = {}^{-}\frac{1}{2}({}^{-}\frac{13}{4}) + {}^{-}\frac{1}{2}(2)$.

5 $(\frac{5}{6} + {}^{-}\frac{1}{3}) + \frac{1}{3} = \frac{5}{6} + ({}^{-}\frac{1}{3} + \frac{1}{3})$.

6 $^{-}\frac{5}{3} + \frac{1}{3} = \frac{1}{3} + {}^{-}\frac{5}{3}$.

7 $\frac{31}{2} + ({}^{-}10 + \frac{28}{5}) = (\frac{31}{2} + {}^{-}10) + \frac{28}{5}$.

Universe for x, y, and z = R.

$x + (yz) = (x + y)(x + z)$.

O $^{+}\frac{4}{5} + ({}^{-}\frac{2}{3} \cdot {}^{+}\frac{3}{5}) = ({}^{+}\frac{4}{5} + {}^{-}\frac{2}{3})({}^{+}\frac{4}{5} + {}^{+}\frac{3}{5})$.

D5

Universe for x, y, and z = R.		
Operations ⟶	Addition	Multiplication
Closure properties ⟶	$x + y \in$ R.	$xy \in$ R.
Commutative properties ⟶	$x + y = y + x$.	$xy = yx$.
Associative properties ⟶	$(x + y) + z = x + (y + z)$.	$(xy)z = x(yz)$.
Distributive property ⟶	$x(y + z) = xy + xz$.	

D6

8 $(\frac{7}{10} + \bar{}\frac{3}{8})(\frac{7}{10} + \bar{}\frac{1}{8}) = \frac{7}{10} + (\bar{}\frac{3}{8} + \bar{}\frac{1}{8})$.

9 $\frac{7}{5}(\bar{}\frac{7}{2}) = \bar{}\frac{7}{2}(\frac{7}{5})$.

10 $\bar{}\frac{2}{3}(6) + \bar{}\frac{2}{3}(\frac{3}{2}) = \bar{}\frac{2}{3}(6 + \frac{3}{2})$.

For each condition expressed below, tell what replacement you could make for the variable to obtain a true statement. Instead of computing to find the replacement, use your knowledge of the distributive property. U = R.

11 $\frac{13}{2}(\frac{1}{4} + z) = \frac{13}{2}(\frac{1}{4}) + \frac{13}{2}(\bar{}\frac{3}{2})$.

12 $(\bar{}\frac{4}{9} + \frac{3}{8})x = \bar{}\frac{4}{9}(\bar{}3) + \frac{3}{8}(\bar{}3)$.

13 $\frac{1}{5}(y) + \frac{1}{5}(\bar{}\frac{5}{2}) = \frac{1}{5}(\frac{7}{10} + \bar{}\frac{5}{2})$.

14 $x(\bar{}5 + \bar{}\frac{12}{5}) = \bar{}\frac{3}{4}(\bar{}5) + \bar{}\frac{3}{4}(\bar{}\frac{12}{5})$.

15 $\frac{13}{10}(\frac{2}{3}) + \frac{13}{10}(\bar{}\frac{8}{10}) = \frac{13}{10}(\frac{2}{3} + z)$.

16 $\bar{}\frac{2}{3}(y + \bar{}\frac{5}{6}) = \bar{}\frac{2}{3}(\frac{3}{4}) + \bar{}\frac{2}{3}(\bar{}\frac{5}{6})$.

17 $\frac{39}{8}(6 + \bar{}\frac{7}{2}) = x(6) + x(\bar{}\frac{7}{2})$.

18 $\frac{8}{11}(\bar{}\frac{11}{20} + \frac{2}{5}) = \frac{8}{11}(y) + \frac{8}{11}(\frac{2}{5})$.

19 Set I = {. . . , $\bar{}3$, $\bar{}2$, $\bar{}1$, 0, +1, +2, +3, . . .}. Does set I, together with addition and multiplication, form a number system?

SPECIAL CHALLENGE

In the Special Challenge on page 129 you studied a finite geometry (the geometry of nine points). In the Special Challenge on pages 169 and 170, you found solution sets of conditions involving addivision and muldivision. Now you will discover how a finite geometry is related to a finite number system.

A F = {0, 1, 2}. Tabulate F × F.

Each member of F × F can be associated with a point in the geometry of nine points, and each point in the geometry of nine points can be associated with a member of F × F.

B Look at D1. Which member of F × F is associated with point S? Which point is associated with (0, 2)? Is (0, 2) ∈ F × F?

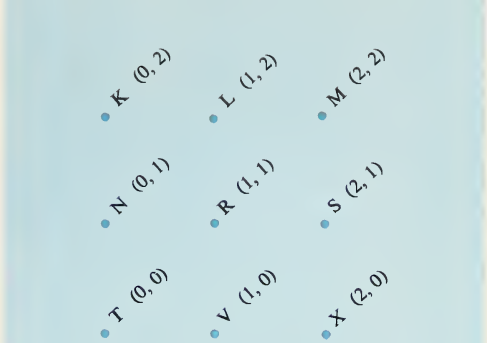
C Look at D2. The solution set of condition A is {(0, 0), (1, 2), (2, 1)}. The members of this set are associated with the points in ℓ_{TLS}. Tabulate the solution set of each of the other conditions expressed in D2.

D For each condition expressed in D2, name the "line" whose points are associated with the members of the solution set of that condition.

E Are the 12 "lines" that you named in exercise D the same "lines" that were represented in the Special Challenge on page 71?

F Think of each condition expressed in D2 as describing a "line" in the geometry of nine points. Which condition describes ℓ_{VRL}? Which condition describes ℓ_{KVS}?

G Tabulate the solution set of this condition: $x \oplus 2 = 0 \wedge (2 \otimes x) \oplus y \oplus 1 = 0$. What does this solution set tell you about ℓ_{VRL} and ℓ_{KVS}? Explain your answer.



D1

Universe for (x, y) = F × F.

A $x \oplus y = 0$.
B $x \oplus 1 = 0$.
C $y \oplus 1 = 0$.
D $x = 0$.
E $y = 0$.
F $x \oplus 2 = 0$.
G $y \oplus 2 = 0$.
H $x \oplus y \oplus 2 = 0$.
I $x \oplus y \oplus 1 = 0$.
J $(2 \otimes x) \oplus y = 0$.
K $(2 \otimes x) \oplus y \oplus 1 = 0$.
L $(2 \otimes x) \oplus y \oplus 2 = 0$.

D2

APPLYING MATHEMATICS

For each problem, write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem.

For problems 1 through 9, the universe for each variable is Z.

1 The average depth of the Atlantic Ocean is 1168 ft. less than the average depth of the Pacific Ocean. The average depth of the Pacific Ocean is 14,048 ft. What is the average depth of the Atlantic Ocean?

2 The length of a side of an equilateral triangle is $12\frac{7}{8}$ ft. What is the perimeter of the triangle in feet?

3 One of the world's fastest trains travelled 41 mi. in 29 min. At this rate, how many miles did the train travel in 1 min.? (Tenths)

4 On a test containing 24 questions, Catherine answered $87\frac{1}{2}\%$ of the questions correctly. How many questions did she answer correctly?

5 Last year Miss Larson saved \$1125. Her savings amounted to 22.5% of her salary. What was her salary last year?

6 The regular price of a refrigerator was \$340. During a sale, the price was reduced to \$272. The discount was what per cent of the regular price?

7 Mr. White has both cherry and apple trees in his orchard. He has $\frac{2}{3}$ as many cherry as apple trees. If he had 10 more apple trees, he would have 55. How many trees of each kind, and how many trees in all, does Mr. White have in his orchard?

8 One of the runways at a certain airport is in the shape of a rectangle that is 6855 ft. long and 200 ft. wide. What is the area of the runway in square feet?

9 How many coins have a total value of \$1.50, if all the coins are quarters or if all the coins are dimes?

For problems 10 and 11, the universe for each variable is N.

10 Mrs. James has 40 teacups. She bought more than 23 but fewer than 30 of these teacups in foreign countries. She received the others as gifts. How many teacups did Mrs. James receive as gifts?

11 Forty-seven more girls than boys are enrolled in Lake High School. If there were 16 more boys, there would be 500 boys. How many girls and how many boys are enrolled?

For problems 12 through 15, the universe for each variable is C.

12 Four times as many children as adults attended a pet show. 476 children attended the show. How many adults and how many persons in all attended the pet show?

13 The Haynes Realty Company bought 200 light bulbs. After some, but not all, of these light bulbs had been given to tenants, there were at most 12 light bulbs left. How many light bulbs were given to tenants?

14 Ralph has 3 kinds of coins. He has 3 times as many dimes as nickels. He has the same number of dimes as quarters. He has more than 5, but fewer than 10, quarters. How many coins of each kind does Ralph have?

15 Certain rate pairs are equivalent to 2/5. The first component of each of these rate pairs is equal to or less than 10. What are the rate pairs?

16 What numbers greater than 43.14 can be added to 120.56 so that each sum is less than 180? The universe for each variable is R_a.

17 $\angle A°$ is $\frac{1}{3}$ of $\angle B°$. $\angle B°$ is $\frac{1}{2}$ of $\angle C°$. $\angle C° = 90$. What is $\angle A°$? What is $\angle B°$? The universe for each variable is the set of members of R_a that are between 0 and 180.

CHECKING UP

If you have trouble with these tests, you should refer to lessons 135, 136, and 137.

Test 126

Each statement expressed in exercises 1 through 6 is an example of a property of R. Name the property.

1 $(\bar{}\frac{1}{2} \cdot \frac{2}{3})\frac{5}{4} = \bar{}\frac{1}{2}(\frac{2}{3} \cdot \frac{5}{4})$.

2 $\frac{5}{7}(\frac{1}{8} + \bar{}\frac{1}{3}) = \frac{5}{7}(\frac{1}{8}) + \frac{5}{7}(\bar{}\frac{1}{3})$.

3 $\bar{}\frac{13}{15} + \frac{7}{8} = \frac{7}{8} + \bar{}\frac{13}{15}$.

4 $\bar{}\frac{5}{2}(\bar{}\frac{5}{6}) \in R$.

5 $\frac{17}{3}(\frac{5}{9}) = \frac{5}{9}(\frac{17}{3})$.

6 $(\bar{}\frac{9}{10} + \bar{}\frac{3}{4}) + \bar{}\frac{3}{10} = \bar{}\frac{9}{10} + (\bar{}\frac{3}{4} + \bar{}\frac{3}{10})$.

A set is named in each of exercises 7 through 12. Tell whether or not the set, together with the operations of addition and multiplication, forms a number system.

7 The set of positive integers

8 The set of negative integers

9 The set of negative rational numbers

10 The union of the set of negative integers and {0}

11 The set of non-negative rational numbers

12 The set of integers

Test 127

Tabulate the solution set of each condition expressed in exercises 13 through 20. Instead of computing to find the solutions, use your knowledge of the properties of R. U = R.

13 $\frac{4}{11} + x = \bar{}\frac{5}{8} + \frac{4}{11}$.

14 $\bar{}7(\bar{}3 \cdot 18) = (\bar{}7 \cdot x)18$.

15 $+\frac{6}{13}(x) = \bar{}\frac{4}{5}(+\frac{6}{13})$.

16 $\frac{5}{7}(x) + \frac{5}{7}(\bar{}\frac{2}{3}) = \frac{5}{7}(\frac{2}{3} + \bar{}\frac{2}{3})$.

17 $\frac{3}{4} + (\frac{8}{15} + \frac{9}{20}) = (\frac{3}{4} + \frac{8}{15}) + x$.

18 $\frac{9}{10}(x + \frac{7}{2}) = \frac{9}{10}(\bar{}\frac{3}{8}) + \frac{9}{10}(\frac{7}{2})$.

19 $\bar{}5(x \cdot \frac{13}{4}) = (\bar{}5 \cdot \frac{2}{7})\frac{13}{4}$.

20 $(\frac{2}{7})x + (\frac{5}{12})x = (\frac{2}{7} + \frac{5}{12})\bar{}\frac{7}{19}$.

The identity elements

You know that, in R_a, the sum of 0 and any given number is the given number. You also know that the product of 1 and any given number is the given number. Remember that zero is the identity element for addition of rational numbers of arithmetic, and that 1 is the identity element for multiplication of rational numbers of arithmetic. In this lesson you will study the identity element for addition and the identity element for multiplication of rational numbers.

A Look at D1. Does sentence A express a true statement? To what rational number has $\bar{}\frac{4}{5}$ been added? What rational number is the sum of 0 and $\bar{}\frac{4}{5}$?

B Is each of the other statements expressed in D1 a true statement? What is the sum of 0 and $\frac{8}{3}$? What is the sum of 0 and $\frac{5}{6}$? What is the sum of 0 and $\bar{}7$?

You can see that, if you add any one of the rational numbers $\bar{}\frac{4}{5}$, $\frac{8}{3}$, $\frac{5}{6}$, or $\bar{}7$ to the rational number 0, the sum that you obtain is the same as the rational number that you added. Now you will use variables to help you decide if the sum of 0 and any given rational number is the given number.

D1

A $0 + \bar{}\frac{4}{5} = \bar{}\frac{4}{5}$.

B $0 + \frac{8}{3} = \frac{8}{3}$.

C $0 + \frac{5}{6} = \frac{5}{6}$.

D $0 + \bar{}7 = \bar{}7$.

C Look at D2. What basic pair indicates the rational number 0? Think of (c, d) as any rational number. What is the universe for c and d?

D For each c and d, do you obtain a true statement from condition E? How do you know?

E What is the sum of 0 and any rational number of arithmetic? For each c and d, do you obtain a true statement from condition F? Explain your answers.

F For each c and d, do you obtain a true statement from condition G? Upon what property does your answer depend?

You have learned that the sum of 0 and any rational number is the given rational number. We say that the rational number 0 is the *identity element for addition of rational numbers. The identity-element property of addition of rational numbers* is expressed below.

The universe for x *is R. For each* x, $0 + x = x$.

G What condition is expressed in D3? For each x, is $x + 0 = x$? Upon what property of addition of rational numbers does your answer depend?

H Do you think there is a number, other than zero, that can be added to a given rational number to obtain the given rational number?

Decide whether each of the following sentences expresses a true statement or a false statement.

I $0 + {}^{-}\frac{5}{3} = {}^{-}\frac{5}{3}$. K ${}^{-}8 + 0 = 0$. M $0 + {}^{-}1 = {}^{-}1$.
J $\frac{2}{7} + 0 = 0$. L $\frac{1}{4} + 0 = \frac{1}{4}$. N ${}^{-}\frac{4}{2} + 0 = 0$.

Now you will decide if there is an identity element for multiplication of rational numbers.

A Look at D4. Is statement H true? Is the product of 1 and ${}^{-}\frac{2}{3}$ the same as ${}^{-}\frac{2}{3}$?

B Study sentences I, J, and K in D4. Is each statement true? In each statement, is the product of the rational number 1 and the given rational number the same as the given number?

Next you will decide if the product of 1 and any given rational number is the same as the given rational number. Think of (c, d) as any rational number. The universe for c and d is R_a.

C Look at D5. Is (1, 0) the basic pair for the rational number 1? How do you know that, for each c and d, you obtain a true statement from condition L?

D How do you know that, for each c and d, you obtain a true statement from condition M?

E For each c and d, do you obtain a true statement from condition N? Upon what property does your answer depend?

Universe for c and $d = R_a$.
E $(0, 0) + (c, d) = (0 + c, 0 + d)$.
F $(0 + c, 0 + d) = (c, d)$.
G $(0, 0) + (c, d) = (c, d)$.

D2

$U = R$.
$x + 0 = x$.

D3

H $1 \times {}^{-}\frac{2}{3} = {}^{-}\frac{2}{3}$.
I $1 \times \frac{5}{8} = \frac{5}{8}$.
J $1 \times \frac{9}{4} = \frac{9}{4}$.
K $1 \times {}^{-}6 = {}^{-}6$.

D4

Universe for c and $d = R_a$.
L $(1, 0) \times (c, d) = (c + 0, d + 0)$.
M $(c + 0, d + 0) = (c, d)$.
N $(1, 0) \times (c, d) = (c, d)$.

D5

You have learned that the product of 1 and any given rational number is the given rational number. We say that the number 1 is the *identity element for multiplication of rational numbers. The identity-element property of multiplication of rational numbers* is expressed below.

The universe for x *is R. For each* x, $1 \cdot x = x$.

F What condition is expressed in D6? For each x, is $x \cdot 1 = x$? Upon what property of multiplication of rational numbers does your answer depend?

G Do you think there is a number, other than 1, by which a given number can be multiplied to obtain the given number as the product?

Decide whether each of the following sentences expresses a true statement or a false statement.

H $\frac{9}{10} \times 1 = 1$. J $\frac{-4}{3} \times 1 = \frac{-4}{3}$. L $-2 \times 1 = 1$.

I $1 \times \frac{-7}{8} = \frac{-7}{8}$. K $-1 \times \frac{5}{8} = \frac{5}{8}$. M $1 \times \frac{1}{8} = \frac{1}{8}$.

You know that the product of 0 and any rational number of arithmetic is 0. In lesson 132 we used a zero property, along with other properties, to suggest what the product of two rational numbers should be. Now that you have a definition for the product of two rational numbers, you will decide whether or not the rational numbers actually do have a zero property.

N Look at D7. Is each statement expressed in D7 true? For each statement, what is the product of 0 and the given number?

Now you will use variables to decide if the product of 0 and any given rational number is 0. Think of (c, d) as any rational number. The universe for c and d is R_a.

O Look at D8. For each c and d, do you obtain a true statement from condition R? How do you know?

P How do you know that, for each c and d, you obtain a true statement from condition S?

U = R.

$x \cdot 1 = x$.

D6

O $0 \times \frac{-4}{3} = 0$.

P $0 \times \frac{2}{5} = 0$.

Q $0 \times -15 = 0$.

D7

Universe for c and $d = R_a$.

R $(0, 0) \times (c, d) = (0 + 0, 0 + 0)$.

S $(0 + 0, 0 + 0) = (0, 0)$.

T $(0, 0) \times (c, d) = (0, 0)$.

D8

Q For each c and d, do you obtain a true statement from condition T? Upon what property does your answer depend?

You have learned that the product of 0 and any rational number is 0. This property is the *zero property of multiplication of rational numbers.* The property is expressed below.

The universe for x *is R. For each* x, $0 \cdot x = 0$.

R For each x, is $x \cdot 0 = 0$? Upon what property of multiplication of rational numbers does your answer depend?

In this lesson you studied properties of addition and multiplication of rational numbers that involve zero and one.

On your own

For each condition expressed in exercises 1 through 9, decide if you obtain a true statement for each replacement of the variable. U = R.

1 $x = 1 \cdot x$. 4 $x \cdot 0 = 0$. 7 $x = 0 + x$.

2 $x + 0 = 0$. 5 $x + x = 0$. 8 $1 = x \cdot 1$.

3 $1 + x = x$. 6 $1 \cdot x = x$. 9 $0 \cdot x = x$.

For each of exercises 10 through 25, tell whether the sentence expresses a true statement or a false statement.

10 $0 \cdot {}^{-}\frac{3}{2} = {}^{-}\frac{3}{2}$.
11 $\frac{1}{5} + 0 = \frac{1}{5}$.
12 $\frac{5}{6} = 1 \cdot \frac{5}{6}$.
13 ${}^{-}9 \cdot 1 = {}^{-}1$.
14 $\frac{7}{10} = 0 + \frac{7}{10}$.
15 $0 = {}^{-}\frac{5}{9} \cdot 0$.
16 $1 \cdot 0 = 1$.
17 ${}^{-}\frac{2}{3} + 0 = 0$.
18 $1 \cdot {}^{-}8 = {}^{-}8$.
19 $0 + {}^{-}1 = 0$.
20 $0 + \frac{7}{2} = \frac{7}{2}$.
21 ${}^{-}4 = {}^{-}4 \cdot 1$.
22 $\frac{8}{3} + 1 = \frac{8}{3}$.
23 $0 \cdot {}^{-}\frac{7}{4} = 0$.
24 ${}^{-}14 + 0 = {}^{-}14$.
25 $1 \cdot \frac{9}{5} = 1$.

KEEPING SKILFUL

For each condition expressed in exercises 1 through 6, tabulate the solution set. For exercises 1 through 3, U = N. For exercises 4 through 6, U = N × N.

1 $n > 25 \wedge n < 685 - 651$.
2 $17 + v = 85 \vee 204 \div 3 = v$.
3 $\sim (375 > c + 80)$.
4 $x + y = 125 \wedge x + 52 = 70$.
5 $a + b < 5 \wedge b + 2 = a$.
6 $r + s = 6 \wedge r + 3 > s$.

For each condition expressed below, tabulate the solution set if possible. Otherwise, give a standard description of the solution set. $U = R_a$.

7 $c < 26.12 + 18.14$.
8 $a - 16.2 \geqq 105.7$.
9 $y + \frac{1}{2} > \frac{7}{8} \wedge 4 - y > 2$.
10 $d = \frac{1}{2}(\frac{1}{4}) \vee d < \frac{3}{4}(\frac{1}{6})$.
11 $x - .75 = 2.25 \wedge x + 3.2 = 4$.
12 $\sim(z + \frac{9}{10} \neq 4)$.

Each exercise below involves rational numbers of arithmetic. For each exercise, find the sum or product.

13 $6\frac{1}{2} + 3\frac{9}{16} + 2$
14 $\frac{25}{26} \times \frac{13}{100}$
15 $65\frac{2}{3} + 17\frac{1}{2}$
16 $\frac{1}{4} \times 625$
17 $100.2 + 30.45$
18 $.704 \times 1.89$
19 $.4 + .3 + 1.14$
20 30.4×1.75

The additive inverse and subtraction

You know that the sum of 0 and any given rational number is the given number. The rational number 0 is the identity element for addition of rational numbers. In this lesson you will study pairs of rational numbers that have 0 as their sum.

A Look at D1. Which of the sentences express true statements?

The sum of ${}^{+}\frac{7}{2}$ and ${}^{-}\frac{7}{2}$ is 0. When the sum of two numbers is 0, each of the numbers is the *additive inverse* of the other.

B What number is the additive inverse of ${}^{+}\frac{7}{2}$? Of ${}^{-}\frac{7}{2}$?

C Look at sentences B and C in D1. How do you know that ${}^{-}23$ is the additive inverse of ${}^{+}23$? What number is the additive inverse of ${}^{-}23$? Of ${}^{+}5$? Of ${}^{-}5$?

D Look at sentence D. How do you know that 0 is the additive inverse of 0?

E What number is the additive inverse of ${}^{-}7$? How do you know?

F Is (0, 7) the basic pair for ${}^{-}7$? Name the basic pair for ${}^{+}7$. Which component of the basic pair for ${}^{+}7$ is the same as the first component of the basic pair for ${}^{-}7$? Which component of the basic pair for ${}^{+}7$ is the same as the second component of the basic pair for ${}^{-}7$?

G What number is the additive inverse of ${}^{+}\frac{3}{4}$?

H Name the basic pair for ${}^{+}\frac{3}{4}$. For ${}^{-}\frac{3}{4}$. How are the components of these basic pairs related?

You can see that, to obtain the basic pair for the additive inverse of a given rational

A $^{+}\frac{7}{2} + {}^{-}\frac{7}{2} = 0.$ C $^{+}5 + {}^{-}5 = 0.$

B $^{+}23 + {}^{-}23 = 0.$ D $0 + 0 = 0.$

D1

Universe for $a = R_a$.

$(a, 0) + (0, a) = (a, a).$

E $(2, 0) + (0, 2) = (2, 2).$

F $(\frac{13}{2}, 0) + (0, \frac{13}{2}) = (\frac{13}{2}, \frac{13}{2}).$

G $(\frac{4}{5}, 0) + (0, \frac{4}{5}) = (\frac{4}{5}, \frac{4}{5}).$

D2

"The additive inverse of $^{-}(^{+}3)$ positive three"

D3

---

ad di tive in verse (ad′ə tiv in vėrs′ or in′vėrs). If the sum of two numbers is 0, each number is the additive inverse of the other. For each x and y, if $x + y = 0$, then each of the numbers, x and y, is the additive inverse of the other. The universe for x and y is R.

number, you can interchange the components of the basic pair for the given number.

Name the basic pair for the additive inverse of each rational number expressed below.

I $(\frac{4}{3}, 0)$ J $(5, 0)$ K $(0, \frac{2}{5})$ L $(0, 16)$

M What is the basic pair for 0? What is the basic pair for the additive inverse of 0?

N What condition is expressed in D2? What is the universe for a? To obtain statement E from the condition, a was replaced by 2. Is statement E true? Explain why $(2, 0)$ and $(0, 2)$ are additive inverses of each other.

O What replacement was made for a in $(a, 0) + (0, a) = (a, a)$ to obtain each of the other statements expressed in D2? Is each of these statements true?

P What is the additive inverse of $(0, \frac{13}{2})$?

Q For each a, is $(a, 0) + (0, a) = (a, a)$? Is $(a, 0) + (0, a) = 0$? Remember that the universe is R_a. Explain your answers.

R For each a, what is the additive inverse of $(a, 0)$? Of $(0, a)$?

Now you know how to find the additive inverse of any rational number. The property that enables you to do this is the *additive-inverse property of rational numbers*. The property is expressed below.

The universe for a *is* R_a. *For each* a, *the additive inverse of* (a, 0) *is* (0, a). *The additive inverse of* (0, a) *is* (a, 0).

S Does each rational number have an additive inverse? Explain your answer.

T D3 shows how to write and read a symbol for the additive inverse of $^{+}3$. How do you know that $^{-}(^{+}3)$ and $^{-}3$ are the same number? That $^{-}(^{-}\frac{2}{3})$ and $^{+}\frac{2}{3}$ are the same number?

The symbol below at the right is read "the additive inverse of x." Since we use this symbol to express the additive inverse of a variable, it is not necessary to use parentheses. The symbol ^{-}x expresses the same idea as the symbol $^{-}(x)$.

$-x$

U How do you know that, for each x, $x + {}^{-}x = 0$? U = R.

Now you will see how you can use the idea of the additive inverse to help you find the difference of two rational numbers.

A You have learned that subtraction is related to addition. For example, $7 - 3 = 4$ also means $7 = 3 + 4$. Does $7 - 4 = 3$ also mean $7 = 4 + 3$?

B Look at D4. How do you know that condition H has the same solution as condition I?

You know that the difference of $^{-}\frac{2}{3}$ and $^{+}\frac{1}{2}$ is the solution of $^{-}\frac{2}{3} = {}^{+}\frac{1}{2} + x$. You can use what you know about the additive inverse of a rational number to help you find this solution.

C Look at D5. Condition I, expressed in D4, is expressed again in D5. In condition J, $(^{+}\frac{1}{2} + x)$ has been added to what number?

From condition I you know that you want to find a replacement for x so that $^{-}\frac{2}{3}$ is the same rational number as $(^{+}\frac{1}{2} + x)$. Therefore, using the well-defined property of addition of rational numbers, you know that $^{-}(^{+}\frac{1}{2}) + {}^{-}\frac{2}{3}$ is the same sum as $^{-}(^{+}\frac{1}{2}) + (^{+}\frac{1}{2} + x)$. The replacement for x that satisfies condition J will also satisfy condition I.

REMINDER

The sum of two rational numbers remains the same when one or both of the numbers used in forming the sum are replaced by equal rational numbers.

See lesson 131, page 178.

D You know that $^{-}\frac{1}{2} = {}^{-}(^{+}\frac{1}{2})$. How do you know that condition K has the same solution as condition J?

E How do you know that condition L has the same solution as condition K?

F How do you know that condition M has the same solution as condition L?

G How do you know that condition N has the same solution as condition M?

H How do you know that condition O has the same solution as condition N?

Now look at D6. Condition H was expressed in D4 and condition O was expressed in D5. Condition O was obtained from condition H by relating subtraction to addition and then using the procedure followed in D5. Thus, $^{-}\frac{2}{3} - {}^{+}\frac{1}{2} = x$ and $^{-}\frac{2}{3} + {}^{-}\frac{1}{2} = x$ have the same solution, and statement P is true.

I You know that $^{-}\frac{1}{2}$ is the additive inverse of $^{+}\frac{1}{2}$. Also, from statement P, you know that the difference of $^{-}\frac{2}{3}$ and $^{+}\frac{1}{2}$ is the same number as the sum of $^{-}\frac{2}{3}$ and $^{-}\frac{1}{2}$. Is the difference of $^{-}\frac{2}{3}$ and $^{+}\frac{1}{2}$ the same number as the sum of $^{-}\frac{2}{3}$ and the additive inverse of $^{+}\frac{1}{2}$? What is a standard name of $^{-}\frac{2}{3} - {}^{+}\frac{1}{2}$?

In any statement like $^{-}\frac{2}{3} - {}^{+}\frac{1}{2} = {}^{-}\frac{7}{6}$, the number $^{-}\frac{2}{3}$ is the *minuend*, $^{+}\frac{1}{2}$ is the *subtrahend*, and $^{-}\frac{7}{6}$ is the *difference*. In $a - b = c$, the minuend is a, the subtrahend is b, and the difference is c. The universe for a, b, and c is R.

U = R.

H $^{-}\frac{2}{3} - {}^{+}\frac{1}{2} = x.$

I $^{-}\frac{2}{3} = {}^{+}\frac{1}{2} + x.$

D4

U = R.

I $^{-}\frac{2}{3} = {}^{+}\frac{1}{2} + x.$

J $^{-}(^{+}\frac{1}{2}) + {}^{-}\frac{2}{3} = {}^{-}(^{+}\frac{1}{2}) + (^{+}\frac{1}{2} + x).$

K $^{-}\frac{1}{2} + {}^{-}\frac{2}{3} = {}^{-}\frac{1}{2} + (^{+}\frac{1}{2} + x).$

L $^{-}\frac{1}{2} + {}^{-}\frac{2}{3} = (^{-}\frac{1}{2} + {}^{+}\frac{1}{2}) + x.$

M $^{-}\frac{1}{2} + {}^{-}\frac{2}{3} = 0 + x.$

N $^{-}\frac{1}{2} + {}^{-}\frac{2}{3} = x.$

O $^{-}\frac{2}{3} + {}^{-}\frac{1}{2} = x.$

D5

U = R.

H $^{-}\frac{2}{3} - {}^{+}\frac{1}{2} = x.$

O $^{-}\frac{2}{3} + {}^{-}\frac{1}{2} = x.$

P $^{-}\frac{2}{3} - {}^{+}\frac{1}{2} = {}^{-}\frac{2}{3} + {}^{-}\frac{1}{2}.$

D6

J Follow the steps given in D4, D5, and D6 to show that $+3 - {}^{-}\frac{7}{8} = +3 + {}^{+}\frac{7}{8}$.

From the preceding exercises, you see that the difference of two rational numbers is equal to the sum of the minuend and the additive inverse of the subtrahend. This is the *difference property of rational numbers*. The property is expressed below.

The universe for x *and* y *is R. For each* x *and* y, $x - y = x + {}^{-}y$.

K Follow the steps given in D4, D5, and D6 to prove that, for each x and y, $x - y = x + {}^{-}y$. The three sentences below will help you. The universe for x, y, and z, is R.

$x - y = z$. $x = y + z$. ${}^{-}y + x = {}^{-}y + (y + z)$.

L Suppose you want to find the difference of $^{-}2$ and $^{-}3$. Which number is the minuend? Which number is the subtrahend? What number is the additive inverse of the subtrahend?

M What two numbers do you add to find the difference of $^{-}2$ and $^{-}3$? What is the difference of $^{-}2$ and $^{-}3$?

N What two numbers do you add to find the difference of $^{+}\frac{1}{3}$ and $^{+}\frac{2}{5}$? Is one number the minuend? Is the other number the additive inverse of the subtrahend? What is the difference of $^{+}\frac{1}{3}$ and $^{+}\frac{2}{5}$?

O Describe in words how you can find the difference of $+6$ and $^{-}\frac{5}{2}$. Describe how you can find the difference of $^{-}\frac{1}{4}$ and $^{-}5$. Describe how you can find the difference of any two rational numbers x and y. The universe for x and y is R.

Now that you know how to find the difference of two rational numbers, you will study some of the properties of subtraction of rational numbers.

P Is N closed under subtraction? Is R_a closed under subtraction? Give three examples to explain each of your answers.

Q How do you know that the set of rational numbers is closed under subtraction? Remember that $x - y = x + {}^{-}y$. The universe for x and y is R.

R Does subtraction of rational numbers have the commutative property? Give three examples to explain your answer.

S Use the following sentence to prove that subtraction of rational numbers does not have the associative property:

$(+3 - {}^{-}\frac{7}{8}) - {}^{-}5 = +3 - ({}^{-}\frac{7}{8} - {}^{-}5)$.

From exercises Q, R, and S, you see that, while subtraction of rational numbers is closed, it does not have either the commutative or the associative property.

In this lesson you learned what is meant by the additive inverse of a rational number, and you learned how to find the difference of two rational numbers. You also learned that the set of rational numbers is closed under subtraction.

On your own

For each of exercises 1 through 8, decide if the two numbers named are additive inverses of each other. Explain your answers.

1 $^{-}\frac{3}{7}, {}^{-}\frac{7}{3}$ **2** $+4, {}^{-}4$ **3** $0, 0$ **4** $^{+}\frac{6}{5}, {}^{-}\frac{5}{6}$ **5** $+64, {}^{-}64$ **6** $+1, 0$ **7** $^{+}\frac{2}{3}, {}^{+}\frac{1}{3}$ **8** $+3, {}^{-}\frac{1}{3}$

For each number named below, tell what number you can add to it to obtain the sum 0.

9 $^{+}\frac{7}{4}$ **10** $^{-}\frac{4}{1750}$ **11** 0 **12** $^{+}\frac{3}{8}$ **13** $^{-}27$ **14** $+429$ **15** $^{-}\frac{11}{12}$ **16** $+39$

Find the difference of the components of each ordered pair of rational numbers named in exercises 17 through 34. The first component of each pair is the minuend.

17 $(^{+}\frac{2}{3}, {}^{+}\frac{3}{2})$ **18** $(^{-}15, {}^{-}29)$ **19** $(^{+}\frac{4}{7}, {}^{-}3)$ **20** $(+6, {}^{-}11)$ **21** $(^{+}\frac{7}{8}, {}^{+}\frac{3}{4})$ **22** $(^{-}12, {}^{+}\frac{6}{7})$ **23** $(^{+}\frac{11}{8}, {}^{-}7)$ **24** $(^{-}6, +23)$ **25** $(^{-}\frac{13}{12}, {}^{-}\frac{5}{6})$ **26** $(^{-}\frac{1}{10}, {}^{-}\frac{5}{4})$ **27** $(^{-}6, +49)$ **28** $(^{+}\frac{9}{5}, {}^{+}\frac{3}{10})$ **29** $(^{-}\frac{5}{9}, {}^{+}\frac{4}{3})$ **30** $(0, {}^{-}396)$ **31** $(0, {}^{+}\frac{5}{8})$ **32** $(^{-}\frac{6}{19}, {}^{+}\frac{4}{3})$ **33** $(^{+}\frac{5}{7}, {}^{+}\frac{1}{3})$ **34** $(^{-}12, {}^{-}86)$

The multiplicative inverse and division

You know that the product of $^{+}1$ and any given rational number is the given number. Because this is true, $^{+}1$ is the identity element for multiplication of rational numbers. In this lesson you will first study pairs of rational numbers that have $^{+}1$ as their product. Then you will learn how to find the quotient of two rational numbers.

A The products of four pairs of numbers are expressed in D1. Name each of these products.

You know that the product of $^{+}\frac{2}{3}$ and $^{+}\frac{3}{2}$ is $^{+}1$. When the product of two numbers is $^{+}1$, each of the numbers is the *multiplicative inverse* of the other.

B What number is the multiplicative inverse of $^{+}\frac{2}{3}$? Of $^{+}\frac{3}{2}$?

C Look at sentences B and C. Why is $^{-}\frac{5}{4}$ the multiplicative inverse of $^{-}\frac{4}{5}$? What is the multiplicative inverse of $^{-}\frac{5}{4}$? Of $^{+}6$? Of $^{+}\frac{1}{6}$?

D Look at sentence D. How do you know that $^{+}1$ is the multiplicative inverse of $^{+}1$?

The multiplicative inverse of a number is often called the *reciprocal* of the number. From now on, we will usually refer to the multiplicative inverse as the reciprocal.

E What number is the reciprocal of $^{+}\frac{2}{3}$? Of $^{-}\frac{5}{4}$? Of $^{+}\frac{1}{6}$?

F How do you know that $^{+}\frac{11}{4}$ and $^{-}\frac{4}{11}$ are not reciprocals of each other? How do you know that $^{-}\frac{15}{7}$ and $^{+}\frac{7}{15}$ are not reciprocals?

G How do you know that the reciprocal of a positive rational number is a positive rational number? How do you know that the reciprocal of a negative rational number is a negative rational number?

H What is the reciprocal of $^{+}1$? Why are $^{+}1$ and $^{-}1$ not reciprocals of each other?

I How do you know that the rational number 0 does not have a reciprocal? Is 0 the reciprocal of any rational number?

J You know that $^{+}\frac{2}{3}$ and $^{+}\frac{3}{2}$ are reciprocals. What is the basic pair for $^{+}\frac{2}{3}$? For $^{+}\frac{3}{2}$?

In *Book 1* you learned that, if the product of two members of R_a is 1, then each of the numbers is the reciprocal of the other. You also learned that, if $a \in Z$, then the reciprocal of a is $1 \div a$.

Look at the two expressions at the right. Each expression names the reciprocal of a. Read each expression "1 divided by a." Both expressions mean the same as the expression $1 \div a$.

$$\frac{1}{a} \qquad 1/a$$

K How do you know that $1/\frac{2}{3}$ is the reciprocal of $\frac{2}{3}$? Remember that $\frac{2}{3}$ is a rational number of arithmetic. Is $1/\frac{2}{3} = \frac{3}{2}$?

L What is the first component of the basic pair for $^{+}\frac{2}{3}$? What is the first component of the basic pair for $^{+}\frac{3}{2}$? Are these first components reciprocals of each other? Explain your answer.

A $^{+}\frac{2}{3} \times {}^{+}\frac{3}{2} = {}^{+}1$. **C** $^{+}6 \times {}^{+}\frac{1}{6} = {}^{+}1$.

B $^{-}\frac{4}{5} \times {}^{-}\frac{5}{4} = {}^{+}1$. **D** $^{+}1 \times {}^{+}1 = {}^{+}1$.

D1

multiplicative inverse (mul'tə plə kā'tiv). If the product of two numbers is $^{+}1$, each number is the multiplicative inverse of the other. For each x and y, if $x \cdot y = {}^{+}1$, then each of the numbers, x and y, is the multiplicative inverse of the other. The universe for x and y is R.

M What is the reciprocal of $^{-}\frac{7}{8}$? What is the second component of the basic pair for $^{-}\frac{7}{8}$? What is the second component of the basic pair for the reciprocal of $^{-}\frac{7}{8}$?

N Are the two second components that you named for exercise M reciprocals of each other?

O Name the basic pairs for $^{+}3$ and $^{+}\frac{1}{3}$.

P Are $^{+}3$ and $^{+}\frac{1}{3}$ reciprocals? Compare the first components of the basic pairs that you named for exercise O. What can you say about the two first components?

Now you know that, if two positive rational numbers are reciprocals of each other, the first components of the basic pairs for these numbers are reciprocals. If two negative rational numbers are reciprocals, the second components of the basic pairs for these numbers are reciprocals.

Name the basic pair for the reciprocal of each of the rational numbers expressed below.

Q $(\frac{4}{3}, 0)$ R $(0, 5)$ S $(0, \frac{8}{15})$ T $(\frac{7}{6}, 0)$

Next you will study an important property of rational numbers.

A Look at sentence E in D2. What is the universe for the variable? For each a, is $(a, 0) \times (\frac{1}{a}, 0) = (a \cdot \frac{1}{a}, 0)$? Is $(a \cdot \frac{1}{a}, 0) = (1, 0)$? Is $(a, 0) \times (\frac{1}{a}, 0) = (1, 0)$? Explain each answer.

B What is a standard name of $(1, 0)$? For each a, what is the reciprocal of $(a, 0)$? What is the reciprocal of $(\frac{1}{a}, 0)$? U = Z.

C What is the basic pair for $^{+}\frac{18}{3}$? What is the basic pair for the reciprocal of $^{+}\frac{18}{3}$? What is a standard name of the reciprocal of $^{+}\frac{18}{3}$?

D What rational number is the reciprocal of $^{+}\frac{16}{5}$? Of $^{+}\frac{5}{16}$? Of $^{+}\frac{1}{9}$? Of $^{+}9$?

E Now look at sentence F in D2. What is the universe for b? For each b, what is the reciprocal of $(0, b)$? What is the reciprocal of $(0, \frac{1}{b})$? How do you know?

U = Z.

E $(a, 0) \times (\frac{1}{a}, 0) = (1, 0)$.

F $(0, b) \times (0, \frac{1}{b}) = (1, 0)$.

D2

U = R.

G $^{+}\frac{8}{5} \div {}^{-}\frac{2}{5} = y$.

H $^{+}\frac{8}{5} = y \cdot {}^{-}\frac{2}{5}$.

I $(^{+}\frac{8}{5})(^{-}\frac{5}{2}) = (y \cdot {}^{-}\frac{2}{5})(^{-}\frac{5}{2})$.

J $(^{+}\frac{8}{5})(^{-}\frac{5}{2}) = y(^{-}\frac{2}{5} \cdot {}^{-}\frac{5}{2})$.

K $(^{+}\frac{8}{5})(^{-}\frac{5}{2}) = y(1)$.

L $(^{+}\frac{8}{5})(^{-}\frac{5}{2}) = y$.

D3

F What is the basic pair for $^{-}\frac{4}{3}$? What is the basic pair for the reciprocal of $^{-}\frac{4}{3}$? What is a standard name of the reciprocal of $^{-}\frac{4}{3}$?

The property that you have been studying is the *reciprocal property of rational numbers.* The property is stated below.

The universe for a *and* b *is Z. For each* a, $(a, 0)$ *and* $(\frac{1}{a}, 0)$ *are reciprocals of each other. For each* b, $(0, b)$ *and* $(0, \frac{1}{b})$ *are reciprocals of each other.*

G Does each rational number that is not zero have a reciprocal? How do you know?

In the work that follows, you will learn to find the quotient of two rational numbers.

H You know that division is related to multiplication. For example, $35 \div 7 = 5$ also means $35 = 5 \cdot 7$. Does $35 \div 5 = 7$ also mean $35 = 7 \cdot 5$?

I Look at sentence G in D3. The number $^{+}\frac{8}{5}$ is the dividend. What number is the divisor?

J Look at sentence H. How do you know that condition G has the same solution as condition H?

You know that the solution of $^{+}\frac{8}{5} = y \cdot {}^{-}\frac{2}{5}$ is the quotient of $^{+}\frac{8}{5}$ and $^{-}\frac{2}{5}$. You can use what you know about the reciprocal of a rational number to help you find this solution.

K Look again at D3. In condition I, $(y \cdot {}^{-}\frac{2}{5})$ and what number have been multiplied? What is the reciprocal of $^{-}\frac{2}{5}$?

From condition H, you know that you want to find a replacement for y so that $^{+}\frac{8}{5}$ is the same rational number as $y \cdot {}^{-}\frac{2}{5}$. Because of the well-defined property of multiplication of rational numbers, $(^{+}\frac{8}{5})(^{-}\frac{5}{2})$ is the same product as $(y \cdot {}^{-}\frac{2}{5})(^{-}\frac{5}{2})$. The replacement for y that satisfies condition I also satisfies condition H.

REMINDER

The product of two rational numbers remains the same when either or both of the numbers used in forming the product are replaced by equal rational numbers.

See lesson 133, page 185.

L Look again at D3. How do you know that condition J has the same solution as condition I?

M How do you know that condition K has the same solution as condition J?

N How do you know that condition L has the same solution as condition K?

Now study D4. Sentences G and L are repeated in D4. Condition L was obtained from condition G by relating division to multiplication and then using the procedure followed in D3. Thus, $^{+}\frac{8}{5} \div {}^{-}\frac{2}{5} = y$ and $(^{+}\frac{8}{5})(^{-}\frac{5}{2}) = y$ have the same solution, and statement M is true.

O From statement M, you know that the quotient of $^{+}\frac{8}{5}$ and $^{-}\frac{2}{5}$ is the same as the product of $^{+}\frac{8}{5}$ and the reciprocal of $^{-}\frac{2}{5}$. What is a standard name of $^{+}\frac{8}{5} \div {}^{-}\frac{2}{5}$?

D3

$U = R.$

G $^{+}\frac{8}{5} \div {}^{-}\frac{2}{5} = y.$

H $^{+}\frac{8}{5} = y \cdot {}^{-}\frac{2}{5}.$

I $(^{+}\frac{8}{5})(^{-}\frac{5}{2}) = (y \cdot {}^{-}\frac{2}{5})(^{-}\frac{5}{2}).$

J $(^{+}\frac{8}{5})(^{-}\frac{5}{2}) = y(^{-}\frac{2}{5} \cdot {}^{-}\frac{5}{2}).$

K $(^{+}\frac{8}{5})(^{-}\frac{5}{2}) = y(1).$

L $(^{+}\frac{8}{5})(^{-}\frac{5}{2}) = y.$

D4

G $^{+}\frac{8}{5} \div {}^{-}\frac{2}{5} = y.$

L $(^{+}\frac{8}{5})(^{-}\frac{5}{2}) = y.$

M $^{+}\frac{8}{5} \div {}^{-}\frac{2}{5} = (^{+}\frac{8}{5})(^{-}\frac{5}{2}).$

D5

N $^{-}\frac{2}{7} \cdot \frac{+1}{-\frac{2}{7}} = +1.$

O $(^{+}\frac{8}{9}) \cdot +1 / {}^{+}\frac{8}{9} = +1.$

From the preceding work, you can see that the quotient of two rational numbers is equal to the product of the dividend and the reciprocal of the divisor.

P Follow the steps given in D3 and D4 to show that $^{-}\frac{10}{7} \div {}^{-}\frac{15}{14} = {}^{-}\frac{10}{7} \times {}^{-}\frac{14}{15}$. What is a standard name of $^{-}\frac{10}{7} \div {}^{-}\frac{15}{14}$?

Find the quotient of the two numbers named in each exercise below. In each exercise, the first number named is the dividend, and the second number named is the divisor.

Q $^{+}\frac{4}{5}, {}^{+}\frac{16}{15}$

R $^{-}\frac{1}{6}, {}^{+}\frac{11}{12}$

S $^{-}\frac{5}{2}, {}^{-}\frac{23}{22}$

T $^{-}\frac{4}{19}, {}^{+}\frac{4}{19}$

U $^{-}\frac{7}{8}, {}^{-}\frac{21}{9}$

V $+33, {}^{+}\frac{1}{11}$

W $^{-}55, {}^{-}\frac{1}{5}$

X $^{+}\frac{13}{16}, {}^{+}\frac{39}{32}$

Y $^{+}\frac{45}{7}, {}^{-}\frac{9}{28}$

So far in this lesson, you have used the reciprocal property of rational numbers to help

you find the quotient of two rational numbers. Next you will learn another way of stating the reciprocal property. Then we will state the quotient property of rational numbers.

A Look at sentence N in D5. What symbol is used to express $+1 \div {}^{-}\frac{2}{7}$? How do you know that sentence N expresses a true statement? Is $+1/{}^{-}\frac{2}{7}$ the reciprocal of ${}^{-}\frac{2}{7}$?

B In sentence O, what symbol is used to express $+1 \div {}^{+}\frac{8}{9}$? Does sentence O express a true statement? What is the reciprocal of ${}^{+}\frac{8}{9}$?

C Is $+1/+150$ the reciprocal of $+150$? What is the reciprocal of $+1/+150$? Of $+1/{}^{+}\frac{7}{9}$?

Now we can express the reciprocal property of rational numbers in a new way. To do this, we use as the universe the set of rational numbers that are not zero. That is, we use the set of *non-zero rational numbers*. This set contains every rational number except 0. We will use the symbol at the right as a name of this set.

Z_r

D Is $+2 \in Z_r$? Is ${}^{-}\frac{2}{3} \in Z_r$? Is $0 \in Z_r$?

The new way of expressing the reciprocal property of rational numbers is stated below.

The universe for x *is* Z_r. *For each* x, *the reciprocal of* x *is* $1/x$.

We will use this idea, that the reciprocal of the non-zero rational number x is $1/x$, in expressing the *quotient property of rational numbers*. The property is expressed below.

The universe for x *is R. The universe for* y *is* Z_r. *For each* x *and* y, $x \div y = x\,(1/y)$.

E Does each member of Z_r have a reciprocal?

F Use the quotient property of rational numbers to prove that the set of non-zero rational numbers is closed under division. Use Z_r as the universe for both x and y.

G You know there is no rational number that is the reciprocal of 0. Use this fact and what you know about division of rational numbers to explain why division by 0 is impossible.

H What is the quotient of 0 and any rational number except 0?

I Does division of rational numbers have the commutative property? Give three examples to explain your answer.

J Does division of rational numbers have the associative property? Use the statement expressed below to explain your answer.

$({}^{-}\frac{5}{8} \div {}^{+}\frac{15}{4}) \div {}^{-}\frac{7}{12} = {}^{-}\frac{5}{8} \div ({}^{+}\frac{15}{4} \div {}^{-}\frac{7}{12})$.

In this lesson you studied the reciprocal property of rational numbers. You learned to find the quotient of two rational numbers, and you learned that Z_r is closed under division.

On your own

For each of exercises 1 through 8, decide whether or not the numbers named are reciprocals of each other. Explain each answer.

1 ${}^{+}\frac{3}{7}, {}^{+}\frac{7}{3}$ 3 ${}^{-}\frac{7}{10}, {}^{+}\frac{7}{10}$ 5 $^{-}14, {}^{-}\frac{1}{14}$ 7 $+1, {}^{-}1$

2 $^{-}1, {}^{-}1$ 4 ${}^{+}\frac{15}{8}, {}^{-}\frac{8}{15}$ 6 ${}^{-}\frac{16}{21}, {}^{+}\frac{21}{16}$ 8 $+1, +1$

For each of exercises 9 through 20, give the reciprocal of the number named.

9 $^{-}10$ 12 ${}^{-}\frac{6}{5}$ 15 ${}^{+}\frac{67}{70}$ 18 $+200$

10 $+17$ 13 ${}^{-}\frac{2}{11}$ 16 ${}^{-}\frac{23}{24}$ 19 $^{-}159$

11 ${}^{+}\frac{5}{8}$ 14 ${}^{+}\frac{4}{13}$ 17 ${}^{-}\frac{19}{12}$ 20 ${}^{-}\frac{99}{100}$

Find the quotient of the rational numbers named in each of exercises 21 through 29. The first number named is the dividend.

21 $+8, {}^{-}\frac{8}{7}$ 24 $+25, {}^{+}\frac{30}{6}$ 27 ${}^{-}\frac{26}{27}, +52$

22 ${}^{-}\frac{3}{7}, {}^{-}\frac{9}{14}$ 25 $^{-}18, {}^{+}\frac{5}{8}$ 28 ${}^{-}\frac{4}{5}, {}^{-}65$

23 ${}^{+}\frac{1}{10}, {}^{+}\frac{4}{5}$ 26 $^{-}14, {}^{-}\frac{7}{4}$ 29 ${}^{+}\frac{37}{40}, {}^{-}37$

Tell whether each of the sentences below expresses a true statement or a false statement.

30 ${}^{+}\frac{5}{3} \cdot \frac{+1}{+\frac{5}{3}} = +1.$

31 ${}^{-}\frac{6}{7} \cdot \frac{+1}{-\frac{7}{6}} = +1.$

32 $\frac{+1}{+8} \cdot {}^{-}8 = +1.$

33 ${}^{-}\frac{9}{10} \cdot \frac{+1}{-\frac{18}{20}} = +1.$

34 $\frac{+1}{+\frac{4}{2}} \cdot +2 = +1.$

35 ${}^{+}\frac{5}{16} \cdot \frac{+1}{+\frac{16}{5}} = +1.$

Using mixed numerals and decimals for rational numbers

You have already learned how to find the sum, the product, the difference, and the quotient of two rational numbers. In this lesson you will compute with mixed numerals and decimals for rational numbers.

In *Book 1* you learned how to compute with mixed numerals and decimal numerals for rational numbers of arithmetic. Now you will see that the computation you use for rational numbers is the same as the computation you use for rational numbers of arithmetic. The only difference is that you must decide whether you should obtain a numeral that expresses a positive rational number or a numeral that expresses a negative rational number.

First you will learn how to find mixed numerals for rational numbers.

A Look at sentence A in D1. Is $^{-}\frac{1}{4} + {}^{-}\frac{7}{8} = {}^{-}\frac{9}{8}$? Name the basic pair for $^{-}\frac{9}{8}$.

B The mixed numeral $1\frac{1}{8}$ can be used to name $\frac{9}{8}$. Is $(0, \frac{9}{8}) = (0, 1\frac{1}{8})$? Is $^{-}\frac{9}{8} = {}^{-}1\frac{1}{8}$?

C How do you know that statement A is true?

D Read sentence B in D1. Is $^{+}\frac{3}{5} - {}^{-}\frac{1}{2} = {}^{+}\frac{11}{10}$? Name the basic pair for $^{+}\frac{11}{10}$.

E What mixed numeral can you use to name $\frac{11}{10}$? Is $(\frac{11}{10}, 0) = (1\frac{1}{10}, 0)$? How do you know that $^{+}\frac{11}{10} = {}^{+}1\frac{1}{10}$?

F How do you know that statement B is true?

G Look at sentence C in D1. Is $^{+}\frac{5}{2} \times {}^{+}\frac{9}{10} = {}^{+}\frac{45}{20}$? What is the basic pair for $^{+}\frac{45}{20}$? Is $(\frac{45}{20}, 0) = (2\frac{5}{20}, 0)$? Is $(2\frac{5}{20}, 0) = (2\frac{1}{4}, 0)$?

H How do you know that statement C is true?

A $^{-}\frac{1}{4} + {}^{-}\frac{7}{8} = {}^{-}1\frac{1}{8}$. **C** $^{+}\frac{5}{2} \times {}^{+}\frac{9}{10} = {}^{+}2\frac{1}{4}$.
B $^{+}\frac{3}{5} - {}^{-}\frac{1}{2} = {}^{+}1\frac{1}{10}$. **D** $^{-}\frac{5}{6} \div {}^{+}\frac{3}{4} = {}^{-}1\frac{1}{9}$.

D1

E $^{+}5\frac{1}{2} + {}^{-}2\frac{1}{4} = {}^{+}3\frac{1}{4}$.
F $^{+}3\frac{4}{5} - {}^{+}6\frac{3}{10} = {}^{+}3\frac{4}{5} + {}^{-}6\frac{3}{10}$.
G $^{+}3\frac{4}{5} + {}^{-}6\frac{3}{10} = {}^{+}3\frac{4}{5} + {}^{-}5\frac{13}{10}$.
H $^{+}3\frac{4}{5} + {}^{-}5\frac{13}{10} = {}^{-}2\frac{1}{2}$.

D2

I $^{+}1\frac{2}{5} \times {}^{-}5\frac{1}{4} = {}^{+}\frac{7}{5} \times {}^{-}\frac{21}{4}$.
J $^{+}\frac{7}{5} \times {}^{-}\frac{21}{4} = {}^{-}7\frac{7}{20}$.
K $^{-}5\frac{2}{3} \div {}^{-}3\frac{1}{2} = {}^{-}\frac{17}{3} \div {}^{-}\frac{7}{2}$.
L $^{-}\frac{17}{3} \div {}^{-}\frac{7}{2} = {}^{-}\frac{17}{3} \times {}^{-}\frac{2}{7}$.
M $^{-}\frac{17}{3} \times {}^{-}\frac{2}{7} = {}^{+}1\frac{13}{21}$.

D3

I Now read sentence D in D1. Is $^{-}\frac{5}{6} \div {}^{+}\frac{3}{4} = {}^{-}\frac{5}{6} \times {}^{+}\frac{4}{3}$? Is $^{-}\frac{5}{6} \times {}^{+}\frac{4}{3} = {}^{-}\frac{20}{18}$?

J How do you know that $^{-}\frac{20}{18} = {}^{-}1\frac{1}{9}$? Is statement D true?

Next you will learn how to compute with mixed numerals for rational numbers.

K Study sentence E in D2. Is $^{+}5\frac{1}{2} = {}^{+}5 + {}^{+}\frac{1}{2}$? Is $^{-}2\frac{1}{4} = {}^{-}2 + {}^{-}\frac{1}{4}$? What is the sum of $^{+}5$ and $^{-}2$? What is the sum of $^{+}\frac{1}{2}$ and $^{-}\frac{1}{4}$?

L Is statement E true? How do you know that the sum of $^{+}5\frac{1}{2}$ and $^{-}2\frac{1}{4}$ is a positive number?

M Study sentence F in D2. Is statement F true? Upon what property does your answer depend?

N Now look at sentence G. How do you know that $^{-}6\frac{3}{10} = {}^{-}5\frac{13}{10}$? Is statement G true?

O Read sentence H in D2. What is the sum of $^{+}3$ and $^{-}5$? What is the sum of $^{+}\frac{4}{5}$ and $^{-}\frac{13}{10}$? Is statement H true?

P What is the difference of $+3\frac{4}{5}$ and $+6\frac{3}{10}$? How do you know that the difference is a negative number?

Q Now study sentence I in D3. How do you know that statement I is true?

R Read sentence J. What is the product of $+\frac{7}{5}$ and $-\frac{21}{4}$? Is $-\frac{147}{20} = -7\frac{7}{20}$? Is statement J true?

S What is the product of $+1\frac{2}{5}$ and $-5\frac{1}{4}$? How do you know that the product of $+1\frac{2}{5}$ and $-5\frac{1}{4}$ is a negative number?

T Read sentence K in D3. How do you know that statement K is true?

U Is statement L true? Upon what property does your answer depend?

V Now look at sentence M. What is the product of $-\frac{17}{3}$ and $-\frac{2}{7}$? Is $+\frac{34}{21} = +1\frac{13}{21}$? Is statement M true?

W What is the quotient of $-5\frac{2}{3}$ and $-3\frac{1}{2}$? How do you know that the quotient is positive?

In *Book 1* you learned how to use decimal numerals to name rational numbers of arithmetic. Then you computed with decimals. Now you will learn how to use decimal numerals to name rational numbers.

A Look at sentence A in D4. Is $(\frac{5}{10}, 0)$ the basic pair for $+\frac{5}{10}$? What decimal can you use to name the first component of $(\frac{5}{10}, 0)$?

B Is $(\frac{5}{10}, 0) = (.5, 0)$? Are $(\frac{5}{10}, 0)$ and $(.5, 0)$ the same rational number? How do you know that statement A is true?

C Now look at sentence B. Name the basic pair for $+1\frac{75}{100}$. What decimal can you use to name the first component of this basic pair?

D Is $(1\frac{75}{100}, 0) = (1.75, 0)$? Are $(1\frac{75}{100}, 0)$ and $(1.75, 0)$ the same rational number? How do you know that statement B is true?

E Read sentence C in D4. Name the basic pair for $-\frac{3}{1000}$. What decimal can you use to name the second component of this basic pair?

F How do you know that statement C is true?

A $+\frac{5}{10} = +.5$. **D** $-2\frac{1}{5} = -2.2$.

B $+1\frac{75}{100} = +1.75$. **E** $-7\frac{4}{25} = -7.16$.

C $-\frac{3}{1000} = -.003$. **F** $+\frac{6}{10000} = +.0006$.

D4

G $+42.6 + +15.8$

$$\begin{array}{r} 42.6 \\ +\ 15.8 \\ \hline 58.4 \end{array}$$

$+58.4$

H $+8.025 + -3.420$

$$\begin{array}{r} 8.025 \\ -\ 3.420 \\ \hline 4.605 \end{array}$$

$+4.605$

D5

G Explain why each of the other sentences in D4 expresses a true statement.

Next you will learn how to find the sum and the difference of rational numbers that are expressed by decimals.

H Example G in D5 shows how to find the sum of $+42.6$ and $+15.8$. Does the numeral 42.6 express a number in tenths? Does the numeral 15.8 express a number in tenths?

I In which way is the sum of 42.6 and 15.8 expressed, in tenths, in hundredths, or in thousandths? What is the sum of 42.6 and 15.8?

J Is the sum of two positive rational numbers positive, or is it negative? What is the sum of $+42.6$ and $+15.8$?

K Example H shows how to find the sum of a positive rational number and a negative rational number. You first find the difference of which two numbers?

L Does each of the numerals 8.025 and 3.420 express a number in thousandths? What is the difference of 8.025 and 3.420?

M Why is the sum of $+8.025$ and -3.420 a positive number? What is the sum?

N Now look at sentence I in D6. The difference of $^-7.35$ and $^+16.6$ is the same as the sum of what two numbers?

O Example J shows how to find the sum of $^-7.35$ and $^-16.6$. Why is the numeral 16.60 used instead of the numeral 16.6?

P What is the sum of 7.35 and 16.6? Why is the sum of $^-7.35$ and $^-16.6$ a negative number? What is this sum? What is the difference of $^-7.35$ and $^+16.6$?

Q Read sentence K in D6. The sum of what two numbers is the same as the difference of $^-23.043$ and $^-6.323$?

R Example L shows how to find the sum of $^-23.043$ and $^+6.323$. To find this sum, do you add 6.323 to 23.043, or do you subtract 6.323 from 23.043?

S What is the difference of 23.043 and 6.323? Why is $^-23.043 + {}^+6.323$ a negative number? What is the difference of $^-23.043$ and $^-6.323$?

For the two numbers named in each exercise, first find a decimal to express the sum. Then find a decimal to express the difference. Use the first number as the minuend.

T $^+2.001, {}^+.972$

U $^-25.3, {}^+16.85$

V $^+7.79, {}^-22.38$

W $^-6.3326, {}^-4.724$

In the following exercises, you will learn how to find the product and the quotient of rational numbers that are expressed by decimals.

A Look at D7. The product of what two numbers is expressed in sentence M? How do you know that statement M is true?

B Now look at sentence N. Is statement N true? Why is the product $^-\frac{71820}{10000}$, and not $^+\frac{71820}{10000}$?

C Example O in D7 shows how to use decimals to find the product of $^+6.84$ and $^-1.05$. Does the decimal 6.84 express a number in hundredths? Does the decimal 1.05 express a number in hundredths?

I $^-7.35 - {}^+16.6 = {}^-7.35 + {}^-16.6.$

J
$$\begin{array}{r} 7.35 \\ 16.60 \\ \hline 23.95 \\ \hline {}^-23.95 \end{array}$$

K $^-23.043 - {}^-6.323 = {}^-23.043 + {}^+6.323.$

L
$$\begin{array}{r} 23.043 \\ 6.323 \\ \hline 16.720 \\ \hline {}^-16.720 \end{array}$$

D6

M $^+6.84 \times {}^-1.05 = \frac{^+684}{100} \times \frac{^-105}{100}.$

N $\frac{^+684}{100} \times \frac{^-105}{100} = \frac{^-71820}{10000}.$

O
$$\begin{array}{r} 6.84 \\ 1.05 \\ \hline 3420 \\ 68400 \\ \hline 7.1820 \\ \hline {}^-7.1820 \end{array}$$

P $^-.16 \times {}^-.5 = \frac{^-16}{100} \times \frac{^-5}{10}.$

Q
$$\begin{array}{r} .16 \\ .5 \\ \hline .080 \\ \hline {}^+.080 \end{array}$$

D7

D Why is the product of 6.84 and 1.05 expressed in ten-thousandths? Name the product of 6.84 and 1.05.

E Is the product of a positive rational number and a negative rational number positive, or is it negative? Name the product of $^+6.84$ and $^-1.05$.

F Look at sentence P in D7. How do you know that statement P is true? Is $\frac{-16}{100} \times \frac{-5}{10} = \frac{+80}{1000}$?

G Why should the product of .16 and .5 be expressed in thousandths? What is the product of .16 and .5?

H Is the product of two negative rational numbers positive, or is it negative? What is the product of $^-.16$ and $^-.5$?

Next you will learn how to find the quotient of two rational numbers that are expressed by decimals. In *Book 1* you learned that, no matter what numbers are being divided, you can multiply both the dividend and the divisor by the same non-zero number without affecting the quotient. You will use this idea to help you compute with decimals for rational numbers.

I Read sentence R in D8. The quotient of what two numbers is expressed at the left of the symbol for equality?

As you know, it is easier to find the quotient of .165 and 1.1 if you express the divisor in ones rather than in tenths.

J Look again at sentence R. How can you obtain +11 from +1.1? How can you obtain $^-1.65$ from $^-.165$? Is statement R true?

K Now look at sentence S. Is $^-1.65 = \frac{-165}{100}$? What is the reciprocal of +11? How do you know that statement S is true?

L Look at sentence T. What number is the quotient of $\frac{-165}{100}$ and +11? What number is the quotient of $^-1.65$ and +11? Of $^-.165$ and +1.1? How do these quotients compare?

Example U shows another way to find the quotient of $^-.165$ and +1.1. Study example U.

M Is the quotient of .165 and 1.1 the same as the quotient of 1.65 and 11? In example U, the dividend is expressed in hundredths. How is the quotient expressed? What is the quotient of 1.65 and 11? Of .165 and 1.1?

N Which is the quotient of a negative number and a positive number, a negative number or a positive number? What is the quotient of $^-.165$ and +1.1?

R $^-.165 \div {}^+1.1 = {}^-1.65 \div {}^+11.$

S $^-1.65 \div {}^+11 = \frac{-165}{100} \times \frac{+1}{11}.$

T $\frac{-\cancel{165}^{15}}{100} \times \frac{+1}{\cancel{11}_{1}} = \frac{-15}{100}.$

U
$$\begin{array}{r|l} 11\overline{)1.65} & 10 \\ \underline{110} & \\ 55 & 5 \\ \underline{55} & \\ & .15 \end{array}$$

$^-.15$

D8

V $^+.23658 \div {}^+6.22 = {}^+23.658 \div {}^+622.$

W
$$\begin{array}{r|l} 622\overline{)23.658} & 30 \\ \underline{18660} & \\ 4998 & 8 \\ \underline{4976} & \\ 22 & .038 \end{array}$$

$^+.038$

D9

O Read sentence V in D9. How can you obtain the quotient $^+23.658 \div {}^+622$ from the quotient $^+.23658 \div {}^+6.22$? Is statement V true?

P Example W shows how to find the quotient of +23.658 and +622. Is the last remainder less than $\frac{1}{2}$ times the divisor? To the nearer thousandth, is the quotient .038, or is it .039?

Q Is the quotient of two positive numbers positive, or is it negative? What is the quotient of $^+.23658$ and $^+6.22$ to the nearer thousandth?

X $+54.3 \div {}^{-}.32 = +5430 \div {}^{-}32.$

Y

$$
\begin{array}{r|l}
32)5430.0 & 1000 \\
32000 & \\
\hline
22300 & 600 \\
19200 & \\
\hline
3100 & 90 \\
2880 & \\
\hline
220 & 6 \\
192 & \\
\hline
28 & 169.6
\end{array}
$$

$^{-}169.7$

D10

R Now look at sentence X in D10. How can you obtain the quotient expressed at the right of the symbol for equality from $+54.3 \div {}^{-}.32$? Is statement X true?

S Look at example Y. Suppose that you want to find the quotient of 5430 and 32 to the nearer tenth. How many digits should there be to the right of the decimal point in the numeral that expresses the dividend?

T How do you know that the quotient of 5430.0 and 32 is the same as the quotient of 54.3 and .32?

U Look again at example Y. Is the last remainder greater than $\frac{1}{2}$ times the divisor? Why is the quotient, to the nearer tenth, 169.7, and not 169.6? What is the quotient of $+54.3$ and $^{-}.32$ to the nearer tenth?

V Use decimals to find the quotient of $^{-}.372$ and $^{-}2.4$ to the nearer hundredth. By what number do you first multiply both the dividend and the divisor? What decimal do you use to express the dividend?

W When you divide 3.72 by 24, the last remainder is 12. Is this remainder equal to $\frac{1}{2}$ times the divisor?

If the last remainder is equal to $\frac{1}{2}$ times the divisor, you should round the quotient to the nearer *even* hundredth (or tenth, or whatever the case may be).

X What is the quotient of $^{-}.372$ and $^{-}2.4$ to the nearer hundredth?

In this lesson you learned how to use mixed numerals and decimal numerals to name rational numbers. You also learned how to compute with mixed numerals and decimals.

On your own

Use a mixed numeral to name each rational number expressed in exercises 1 through 8.

1 $+\frac{16}{5}$ 3 $^{-}\frac{32}{3}$ 5 $^{-}\frac{3}{2}$ 7 $+\frac{18}{12}$

2 $+\frac{27}{8}$ 4 $^{-}\frac{12}{7}$ 6 $+\frac{10}{4}$ 8 $^{-}\frac{9}{6}$

Use a decimal to name each rational number expressed in exercises 9 through 16.

9 $^{-}\frac{3}{5}$ 11 $^{-}\frac{3}{4}$ 13 $+\frac{1}{25}$ 15 $^{-}\frac{2}{1000}$

10 $+\frac{11}{20}$ 12 $^{-}\frac{16}{10}$ 14 $^{-}\frac{175}{100}$ 16 $+\frac{3}{50}$

For each pair of rational numbers named in exercises 17 through 24, first find the sum of the two numbers. Then find the difference of the two numbers. Use the first number of each pair as the minuend.

17 $+8\frac{3}{4}$, $+16\frac{2}{3}$

18 $+1.06$, $^{-}3.2$

19 $^{-}25\frac{1}{5}$, $^{-}7\frac{4}{5}$

20 $^{-}3$, $+2.78$

21 $^{-}6.002$, $^{-}7.3354$

22 $+12\frac{1}{2}$, $^{-}4\frac{7}{8}$

23 $+3.65$, $+.4$

24 $^{-}8.427$, $+8.028$

For each pair of rational numbers named in exercises 25 through 32, first find the product of the two numbers. Then find the quotient. Use the first number of each pair as the dividend. For exercises 25, 27, 29, 30, and 32, find the quotient to the nearer hundredth.

25 $^{-}4.36$, $^{-}.007$

26 $+15\frac{1}{2}$, $+8\frac{1}{3}$

27 $^{-}.015$, $^{-}.13$

28 $^{-}12\frac{2}{5}$, $^{-}7\frac{3}{4}$

29 $+5.25$, $+3.2$

30 $^{-}11.2$, $+6.85$

31 $^{-}11\frac{3}{8}$, $+2\frac{5}{6}$

32 $+13.9$, $^{-}1.09$

Repeating decimals as names of rational numbers

Now that you know how to compute with decimals, you will learn how to find a decimal numeral for any rational number.

A Suppose that you want to find a decimal for $^{-}\frac{5}{16}$. Name the basic pair for $^{-}\frac{5}{16}$. To what set do the components of this basic pair belong? Which component is the non-zero component?

In *Book 1* you learned that you can think of $\frac{5}{16}$ as the quotient of 5 and 16. You will use this idea to help you find a decimal for $^{-}\frac{5}{16}$.

B Look at D1. Example A shows the first step in dividing 5 by 16. What decimal expresses the dividend in the first step? Name the first digit in the decimal for the quotient. Name the remainder that is obtained in the first step.

Since you did not obtain a remainder of 0 in the first step of the division process, you know that the process can be continued. Example B shows the second step. Notice the digits (in white) that were inserted after the first step so that the process could be continued.

C How is the dividend expressed in the second step? Name the second digit in the decimal for the quotient. Name the remainder obtained in the second step.

Example C in D1 shows the third step in dividing 5 by 16. Notice that more digits for 0 have been inserted for this step.

D How is the dividend expressed in the third step? What is the third digit in the decimal for the quotient? What remainder is obtained in the third step?

E Example D shows the fourth step in the division process. What is the fourth digit in the decimal for the quotient? What remainder is obtained in the fourth step?

Since the remainder obtained in the fourth step is 0, you know that you do not need to continue the steps. You have found a decimal for $\frac{5}{16}$.

F What decimal expresses the quotient of 5 and 16? Is $\frac{5}{16} = .3125$? Is $(0, \frac{5}{16}) = (0, .3125)$?

G How do you know that $^{-}\frac{5}{16} = {}^{-}.3125$?

A

$$\begin{array}{r|l} 16\overline{)5.0} & 3 \\ \underline{48} & \\ 2 & \end{array}$$

B

$$\begin{array}{r|l} 16\overline{)5.0} & 3 \\ \underline{48} & \\ 2 & 1 \\ \underline{16} & \\ 4 & \end{array}$$

C

$$\begin{array}{r|l} 16\overline{)5.00} & 30 \\ \underline{480} & \\ 20 & 1 \\ \underline{16} & \\ 4 & 2 \\ \underline{32} & \\ 8 & \end{array}$$

D

$$\begin{array}{r|l} 16\overline{)5.000} & 300 \\ \underline{4800} & \\ 200 & 10 \\ \underline{160} & \\ 40 & 2 \\ \underline{32} & \\ 8 & 5 \\ \underline{80} & \\ 0 & .3125 \end{array}$$

D1

H In lesson 88 of *Book 1*, you learned that, if the remainder 0 is obtained in the division process, the quotient can be expressed by a *terminating decimal.* Is the numeral $^{-}.3125$ a terminating decimal?

I The work in D2 shows how to find a decimal for $^{+}\frac{2}{3}$. What decimal expresses the dividend in example E? What is the first digit in the decimal for the quotient? What remainder is obtained in the first step?

J In the first step in example E, you divide 20 by 3. In the second step, what number do you divide by 3?

K What do you observe about the first and second digits in the decimal for the quotient? How do you know that you could continue the division process beyond the second step?

L Imagine that you have completed five steps. In each step, what digit do you obtain in the decimal for the quotient? In each step, what remainder do you obtain?

M In finding the quotient of 2 and 3, will you ever obtain a remainder of 0? Will there be a last digit in the decimal for $\frac{2}{3}$? What is the quotient to the nearer thousandth?

The decimal for $\frac{2}{3}$ is the *repeating decimal* shown at the right. The bar above the digit 6 tells you that 6 repeats on and on. 6 is the *repetend.* Read the symbol "point six, repetend six."

$.\overline{6}$

E

$$\begin{array}{r|l} 3\overline{)2.0} & 6 \\ 1\,8 & \\ \hline 2 & 6 \\ 1\,8 & \\ \hline 2 & .6\,6 \end{array}$$

D2

REMINDER

A repeating decimal is a decimal in which digits repeat in a pattern. The pattern repeats without end. The digits that repeat in a pattern form the repetend.

See lesson 88, page 384, Book 1.

N How do you know that $^{+}\frac{2}{3} = {}^{+}.\overline{6}$? What is the repetend in the decimal for $^{+}\frac{2}{3}$?

Suppose that, at some step of the division process, you obtain 0 as a remainder. You know then that the decimal for the quotient is a terminating decimal and that your work is completed. But if you do not obtain a remainder of 0, you should continue the division process until the remainder you obtain is the same as the dividend you had before any zeros were inserted or the same as a remainder you obtained at an earlier step. When the dividend or a remainder repeats, then you know that the digits in the decimal for the quotient will also begin to repeat. In such a case, you know that the decimal for the quotient is a repeating decimal.

Use the method shown in D1 and D2 to find a decimal for each rational number named below.

O $^{+}\frac{3}{8}$ P $^{-}\frac{1}{6}$ Q $^{-}\frac{5}{2}$ R $^{+}\frac{1}{3}$ S $^{+}\frac{11}{40}$

Now you know what it means to obtain a terminating decimal or a repeating decimal for a rational number. Next you will learn how to decide which rational numbers can be expressed by terminating decimals.

First, you will review some ideas that you studied in *Book 1* concerning factors and powers of numbers. Remember that a factor is one of the numbers used to form a product.

F $100 = 10^2$.
G $125 = 5^3$.
H $10 = 2^1 \cdot 5^1$.
I $100 = 2^2 \cdot 5^2$.
J $6 = 2^1 \cdot 3^1$.
K $9 = 3^2$.

D3

L $\frac{3}{10} = \frac{3}{10^1}$.
M $\frac{23}{1000} = \frac{23}{10^3}$.
N $\frac{3}{8} = \frac{375}{10^3}$.
O $\frac{27}{90} = \frac{3}{10^1}$.
P $\frac{7}{50} = \frac{14}{10^2}$.
Q $\frac{9}{20} = \frac{45}{10^2}$.

D4

A Look at sentence F in D3. 100 is the second power of what number? Is statement F true?

REMINDER

Numbers like 10^2 are powers. The base of 10^2 is 10, and the exponent is 2. The exponent tells how many times the base is used as a factor. $10^2 = 10 \times 10$ is a true statement.

See lesson 53, page 227, Book 1.

B Now read sentence G in D3. 125 is the third power of what number?

C Look at sentence H. How do you know that 2 and 5 are prime numbers? How do you know that $2^1 \cdot 5^1 = 2 \cdot 5$? Is $2^1 \cdot 5^1$ a complete factorization of 10?

REMINDER

A prime number has exactly two factors. A complete factorization of a number is that number expressed as the product of prime numbers.

See lesson 69, page 296, and lesson 70, page 300, Book 1.

D Read sentence I in D3. What is the complete factorization of 100?

E Now read sentences J and K. What is the complete factorization of 6? Of 9?

You will use the ideas you have just reviewed to help you decide which rational numbers can be expressed by terminating decimals.

F Look at D4. All the numbers expressed in D4 are rational numbers of arithmetic. How do you know that statement L is true? What decimal expresses $\frac{3}{10}$?

G How do you know that statement M is true? What decimal expresses $\frac{23}{1000}$?

H Look at sentence N. How can you obtain $\frac{375}{10^3}$ from $\frac{3}{8}$? What decimal expresses $\frac{3}{8}$?

I Look at sentence O. How can you obtain $\frac{3}{10^1}$ from $\frac{27}{90}$? What decimal expresses $\frac{27}{90}$?

J Study sentences P and Q. What decimal expresses $\frac{7}{50}$? What decimal expresses $\frac{9}{20}$?

K Think about the rational numbers of arithmetic that can be indicated by fractions whose denominators are powers of 10. Can each of these numbers be expressed by a terminating decimal?

L What are the prime factors of 10^2? Of 10^3? Of 10^5? Is each prime factor that you named a factor of 10? Name the prime factors of 10.

M Give a complete factorization of the denominator of each of the following fractions: $\frac{7}{10}$, $\frac{31}{100}$, $\frac{5}{8}$, $\frac{7}{25}$, and $\frac{3}{40}$.

N Are the prime factors that you named for exercise M the only prime factors of 10? Can each of the rational numbers named in exercise M be expressed by a terminating decimal?

O What is the prime factor of the denominator of $\frac{1}{3}$? What is the prime factor of the denominator of $\frac{2}{11}$? What is the prime factor of the denominator of $\frac{1}{5}$? Which of the rational numbers $\frac{1}{3}$, $\frac{2}{11}$, and $\frac{1}{5}$ can be expressed by a terminating decimal?

P Is $\frac{2}{1} = 2.0$? Is $\frac{65}{1} = 65.0$? What terminating decimal can you use to express $\frac{1}{1}$? To express $\frac{10}{1}$? To express $\frac{562}{1}$?

Q Think about the rational numbers of arithmetic that are indicated by basic fractions whose denominators are 1. Can each of these numbers be expressed by a terminating decimal? Explain your answer.

From the examples in exercises F through Q, you should see that, if the denominator of a basic fraction is 1, or if each prime factor of the denominator is a factor of 10, then the number indicated by the fraction can be expressed by a terminating decimal.

R Think about the fraction $\frac{3}{24}$. The prime factors of 24 are 2 and 3. The decimal for $\frac{3}{24}$ is .125. Explain why $\frac{3}{24}$ can be expressed by a terminating decimal even though 24 has a prime factor other than 2 or 5.

Without using the division process, decide whether or not each rational number named below can be expressed by a terminating decimal. Explain your answers.

S $^{+}\frac{1}{12}$ T $^{-}\frac{9}{15}$ U $^{-}\frac{5}{9}$ V $^{+}\frac{4}{16}$ W $^{-}\frac{6}{30}$

X Use a decimal to express $^{-}\frac{1}{2}$. Use a common fraction numeral to express $^{-}.5000$. Suppose that you wrote 50 zero digits after the digit 5 in the decimal $^{-}.5$. Would the decimal you obtain still name $^{-}\frac{1}{2}$? How do you know that $^{-}\frac{1}{2} = {}^{-}.50\overline{0}$?

Y Express $^{+}\frac{4}{5}$ by using a decimal with ten digits for zero. Is $^{+}\frac{4}{5} = {}^{+}.80\overline{0}$?

Z Each terminating decimal is also a repeating decimal. What is the repetend of each repeating decimal that is also a terminating decimal?

In the preceding exercises, you learned how to determine which rational numbers can be expressed by terminating decimals. You also learned why all terminating decimals are repeating decimals.

Now you will see that every rational number can be expressed by a repeating decimal. The work in D5 shows the steps in finding a decimal for $\frac{1}{7}$. Digits for 0 have been inserted wherever necessary at each step.

Study D5. In the first step, 10 is divided by 7. Therefore, the first digit in the numeral for the quotient is 1. The first remainder is 3.

A What is the sixth remainder that was obtained? How does the sixth remainder compare with the dividend in the first step?

B Look at the numeral for the quotient. In this numeral, how does the seventh digit compare with the first digit?

C How do the first remainder and the seventh remainder compare?

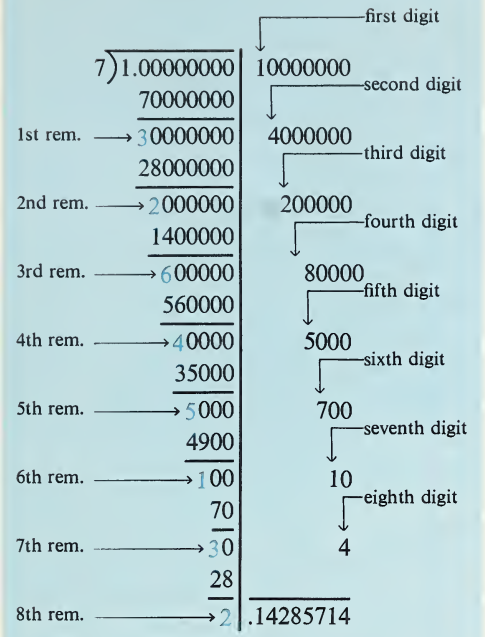
D In the numeral for the quotient, how do the second digit and the eighth digit compare?

E How do the second and eighth remainders compare?

F If the division process were continued for one more step, what would be the ninth digit in the numeral for the quotient? What would the ninth remainder be?

From the work in D5, you can see that the remainders begin to repeat after the sixth step. For this reason, each step of the division process will repeat, and the digits in the numeral for the quotient will also repeat. If you continued the division process, you would see that these digits, 142857, form the repetend of the decimal for $\frac{1}{7}$. The repeating decimal $.\overline{142857}$ names the rational number $\frac{1}{7}$.

G When you divide a number by 7, how do you know that the remainder will never be greater than 6? How do you know that you cannot obtain more than 7 different remainders when you divide a number by 7? How do you know that, if you divide 1 by 7, the remainder will never be 0?



D5

H Name the different remainders that you obtain when you divide 1 by 7.

I How many digits are in the repetend of the decimal for $\frac{1}{7}$? How does the number of digits in the repetend 142857 compare with the number of different remainders that you obtain when you divide 1 by 7?

J When you divide a number by 11, you cannot obtain more than 11 different remainders. Name the possible remainders.

K Since $\frac{5}{11}$ cannot be expressed by a terminating decimal, you cannot obtain a remainder of 0 when you divide 5 by 11. Thus, without computing, you know that you cannot obtain more than 10 different remainders when you divide 5 by 11. This means that the repetend of the decimal for $\frac{5}{11}$ cannot have more than 10 digits. Find the decimal for $\frac{5}{11}$. What is the repetend? What is the number of digits in the repetend?

L How do you know that you cannot obtain more than 17 different remainders when you divide a number by 17?

M How do you know that, when you divide 5 by 17, you will not obtain a remainder of 0? How do you know that the repetend of the decimal for $\frac{5}{17}$ does not contain more than 16 digits?

N Without computing, tell what could be the greatest number of digits in the repetend of the decimal for $\frac{7}{27}$. In the repetend of the decimal for $+\frac{6}{121}$. In the repetend of the decimal for $-\frac{87}{6238}$.

O Explain why every rational number can be expressed by a repeating decimal.

In this lesson you learned how to find a decimal to name a rational number and how to decide which rational numbers can be expressed by terminating decimals. You also learned that, if you include decimals with repetend 0, every rational number can be expressed by a repeating decimal.

On your own

Give the repetend of the decimal for each number named in exercises 1 through 5.

1 $\frac{7}{4}$ **2** $\frac{5}{3}$ **3** $\frac{3}{16}$ **4** $\frac{1}{33}$ **5** $\frac{5}{24}$

First, tell whether the decimal for each rational number named below is a terminating decimal or a repeating decimal that does not terminate. If the decimal is a repeating decimal that does not terminate, tell the greatest number of digits that could be in the repetend. Then express the number by a decimal.

6 $\frac{5}{8}$ **9** $\frac{4}{25}$ **12** $-\frac{9}{20}$ **15** $\frac{8}{9}$ **18** $\frac{3}{500}$

7 $-\frac{6}{11}$ **10** $\frac{33}{3}$ **13** $\frac{15}{4}$ **16** $-\frac{5}{3}$ **19** $\frac{7}{24}$

8 $\frac{7}{3}$ **11** $-\frac{3}{13}$ **14** $-\frac{11}{16}$ **17** $\frac{32}{80}$ **20** $-\frac{76}{125}$

KEEPING SKILFUL

For each of exercises 1 through 6, tell what basic pair indicates the rational number named.

1 $^{+}\frac{2}{3}$ 3 $^{+}\frac{7}{12}$ 5 $^{+}\frac{25}{30}$

2 $^{-}\frac{5}{6}$ 4 $^{-}12$ 6 0

Give a standard name of each rational number named in exercises 7 through 15.

7 $(\frac{3}{4}, \frac{1}{2})$ 10 $(10, \frac{5}{4})$ 13 $(\frac{9}{20}, \frac{9}{20})$

8 $(0, \frac{4}{5})$ 11 $(\frac{28}{3}, 0)$ 14 $(\frac{14}{4}, \frac{17}{3})$

9 $(\frac{9}{4}, 2)$ 12 $(\frac{21}{10}, \frac{21}{5})$ 15 $(\frac{29}{30}, \frac{3}{4})$

For each of exercises 16 through 21, tell what rational number you can add to the given number to obtain the sum 0.

16 $^{-}\frac{35}{36}$ 18 $^{+}\frac{3}{7}$ 20 $^{-}72$

17 $^{+}39$ 19 $^{-}\frac{51}{10}$ 21 $^{+}\frac{100}{80}$

For each of exercises 22 through 30, find the sum of the pair of numbers named.

22 $^{+}314, ^{+}49$ 25 $^{-}\frac{91}{2}, ^{+}20$ 28 $^{-}\frac{1}{5}, ^{+}\frac{7}{21}$

23 $^{-}\frac{81}{85}, ^{-}\frac{3}{5}$ 26 $^{+}9, ^{-}2$ 29 $0, ^{+}\frac{19}{20}$

24 $^{+}75, ^{-}\frac{100}{3}$ 27 $\frac{0}{2}, ^{-}\frac{29}{9}$ 30 $^{-}\frac{3}{35}, ^{+}\frac{61}{70}$

For each of exercises 31 through 39, find the product of the pair of numbers named.

31 $^{+}\frac{24}{4}, ^{+}\frac{7}{5}$ 34 $0, ^{-}\frac{4}{25}$ 37 $^{-}1, ^{+}\frac{39}{6}$

32 $^{+}\frac{9}{4}, ^{-}\frac{17}{4}$ 35 $^{-}8, ^{-}\frac{3}{2}$ 38 $^{-}12, ^{-}13$

33 $^{+}\frac{93}{100}, ^{-}\frac{20}{31}$ 36 $^{-}\frac{32}{11}, ^{+}\frac{21}{16}$ 39 $^{-}\frac{10}{11}, ^{-}4$

Find the difference of the numbers named in each of exercises 40 through 48. The first number named is the minuend.

40 $^{+}6, ^{+}\frac{3}{5}$ 43 $^{+}80, 0$ 46 $^{-}362, ^{+}25$

41 $\frac{0}{9}, ^{-}\frac{14}{5}$ 44 $^{-}\frac{74}{3}, ^{-}2$ 47 $\frac{0}{12}, ^{+}65$

42 $^{+}\frac{17}{2}, ^{-}\frac{25}{3}$ 45 $^{+}79, ^{-}76$ 48 $^{-}\frac{25}{88}, ^{-}\frac{1}{11}$

Find the quotient of the numbers named in each of exercises 49 through 57. The first number named is the dividend.

49 $^{+}8, ^{+}\frac{4}{5}$ 52 $^{-}\frac{5}{12}, ^{+}1$ 55 $^{-}\frac{15}{2}, ^{-}\frac{5}{9}$

50 $^{-}\frac{3}{10}, ^{+}\frac{9}{8}$ 53 $^{+}\frac{7}{8}, ^{-}\frac{13}{4}$ 56 $^{+}\frac{1}{3}, ^{-}7$

51 $^{-}\frac{7}{6}, ^{-}\frac{9}{15}$ 54 $^{+}1, ^{+}\frac{3}{5}$ 57 $^{-}4, ^{+}13$

CHECKING UP

If you have trouble with these two tests, you should refer to lessons 138 through 142.

Test 128

In each of exercises 1 through 4, a property of rational numbers is expressed. For each exercise, give the name of the property. The universe for x is R. The universe for y is Z_r.

1 $0 + x = x.$ 3 $1 \cdot x = x.$

2 $0 \cdot x = 0.$ 4 $x \div y = x\,(1/y).$

For each of exercises 5 through 9, first give the additive inverse of each rational number named. Then give the multiplicative inverse of each number.

5 $^{+}9$ 6 $^{-}\frac{3}{8}$ 7 $^{+}2\frac{4}{5}$ 8 $^{-}\frac{18}{11}$ 9 $^{+}\frac{41}{3}$

Test 129

For each of exercises 10 through 14, use a mixed numeral to express the rational number named.

10 $^{+}\frac{17}{3}$ 11 $^{-}\frac{34}{5}$ 12 $^{-}\frac{11}{6}$ 13 $^{+}\frac{53}{2}$ 14 $^{+}\frac{85}{7}$

For each of exercises 15 through 19, use a decimal to express the rational number named.

15 $^{+}\frac{7}{10}$ 16 $^{-}\frac{6}{25}$ 17 $^{+}\frac{73}{40}$ 18 $^{-}\frac{29}{11}$ 19 $^{-}\frac{48}{7}$

For each pair of rational numbers named in exercises 20 through 23, first find the sum of the two numbers. Then find the difference of the two numbers. Use the first number of each pair as the minuend.

20 $^{+}4\frac{1}{3}, ^{+}2\frac{1}{2}$ 22 $^{+}.0068, ^{-}2.39$

21 $^{-}7\frac{3}{5}, ^{+}10\frac{2}{3}$ 23 $^{-}12.7, ^{-}305.004$

For each pair of rational numbers named in exercises 24 through 31, first find the product of the two numbers. Then find the quotient of the two numbers. Use the first number of each pair as the dividend. For exercises 28 through 31, find the quotient to the nearer hundredth.

24 $^{-}14\frac{2}{3}, ^{+}3\frac{3}{4}$ 28 $^{-}304.72, ^{-}52.8$

25 $^{+}9\frac{3}{8}, ^{+}15\frac{5}{6}$ 29 $^{+}.4792, ^{-}.37$

26 $^{-}7\frac{1}{2}, ^{-}12\frac{5}{9}$ 30 $^{-}3.007, ^{+}.52$

27 $^{+}20\frac{4}{7}, ^{-}5\frac{1}{3}$ 31 $^{+}8.75, ^{+}.29$

End-of-block tests on properties of rational numbers and on computation

Positive integral powers and scientific notation

It is convenient to have a short form for expressing such numbers as 70,000,000,000 and .000002. In *Book 1* you learned how scientific notation can be used to express certain natural numbers. You learned, for example, that the numeral $7(10^{10})$ is scientific notation for 70,000,000,000.

In this lesson you will first learn to express rational numbers by using exponential notation. Then you will learn to express rational numbers in scientific notation. Finally, you will use your knowledge of exponents to find the product and the quotient of two rational numbers.

A Look at D1. The decimal numeral in the ? chart expresses a rational number. How do you know that the decimal numeral expresses a positive rational number?

B The names of the periods in the decimal numeral are shown above the numeral. Which digits in the numeral form the billions' period?

C Each of the positions, or places, within a period also has a special name. The names of the positions are shown below the digits in the numeral. The digit 3 expresses the number three hundred billion. What number does the digit 6 express? What number does the digit 5 express?

D In the numeral in D1, which digits form the millions' period? In this period, what number does the digit 9 express? What number is expressed by the digit 7? By the digit 1?

billions			millions			thousands					
3	6	5	9	7	1	4	0	8	2	4	7
hundred billions	ten billions	billions	hundred millions	ten millions	millions	hundred thousands	ten thousands	thousands	hundreds	tens	ones

D1

A $10^3 = 10 \times 10 \times 10$.

B $(\frac{3}{5})^3 = \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}$.

C $(\frac{-2}{3})^4 = \frac{-2}{3} \times \frac{-2}{3} \times \frac{-2}{3} \times \frac{-2}{3}$.

D2

E Which digits form the thousands' period? In this period, what number is expressed by each of the digits? Read the numeral in D1.

F In the numeral in D1, the digit 8 expresses 8(1000). How many times is 10 used as a factor to obtain the product 1000?

G Now read sentence A in D2. You know that ? the numeral 10^3 expresses a number in exponential notation. What is the exponent of 10^3? What is the base of 10^3? 10^3 is what power of 10? Is $10^3 = 1000$?

H In 10^3, the exponent 3 is a positive rational number. Is 3 a positive integer? Is 3 an element of I_p?

REMINDER

The set of positive integers is $\{+1, +2, +3, \ldots\}$. The symbol I_p is a name of this set.

See lesson 129, page 168.

An exponent that is a positive integer is a *positive integral exponent*. A power that has a positive integer for an exponent is a *positive integral power*.

I Read sentence B in D2. Name the positive integral exponent of $(\frac{3}{5})^3$. Name the base of $(\frac{3}{5})^3$. How do you know that $(\frac{3}{5})^3$ is a positive integral power?

J In $(\frac{3}{5})^3$, how many times is $\frac{3}{5}$ used as a factor? Is statement B true?

K Explain why $\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}=(\frac{3}{5}\times\frac{3}{5})\times\frac{3}{5}$. What is a standard name of $\frac{3}{5}\times\frac{3}{5}$?

L Is $\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}=\frac{9}{25}\times\frac{3}{5}$?

M What is a standard name of $\frac{9}{25}\times\frac{3}{5}$? What is a standard name of $\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}$?

N Is $(\frac{3}{5})^3=\frac{27}{125}$? What power of $\frac{3}{5}$ is $\frac{27}{125}$?

O Now look at sentence C in D2. Does it express a true statement? Name the exponent of $(\frac{-2}{3})^4$. Name the base of $(\frac{-2}{3})^4$. How do you know that $(\frac{-2}{3})^4$ is a positive integral power?

P Is $\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}=(\frac{-2}{3}\times\frac{-2}{3})\times\frac{-2}{3}\times\frac{-2}{3}$? Explain your answer. What is a standard name of $\frac{-2}{3}\times\frac{-2}{3}$?

Q Is $(\frac{-2}{3})^4=(\frac{4}{9}\times\frac{-2}{3})\times\frac{-2}{3}$? What is a standard name of $\frac{4}{9}\times\frac{-2}{3}$?

R Is $(\frac{-2}{3})^4=\frac{-8}{27}\times\frac{-2}{3}$? What is a standard name of $\frac{-8}{27}\times\frac{-2}{3}$? What is a standard name of $(\frac{-2}{3})^4$?

Give a standard name of the power expressed in each of exercises S through Z.

S $(\frac{-1}{4})^2$ U $(\frac{-2}{5})^3$ W $(\frac{3}{8})^2$ Y $(-7)^3$

T $(+4)^2$ V $(1)^7$ X $(\frac{-3}{4})^4$ Z $(\frac{+1}{2})^5$

Next you will use your knowledge of positive integral powers to find products of powers that have the same base.

A Look at numeral D in D3. What is the exponent of 10^3? Of 10^4? Do 10^3 and 10^4 have the same base? Name the base.

B How do you know that numeral E names the same number as numeral D?

A $10^3 = 10\times10\times10$.

B $(\frac{3}{5})^3=\frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}$.

C $(\frac{-2}{3})^4=\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}\times\frac{-2}{3}$.

D2

D $10^3\times10^4$

E $(10\times10\times10)\times(10\times10\times10\times10)$

F $10\times10\times10\times10\times10\times10\times10$

G 10^7

D3

H $(\frac{3}{4})^2\cdot(\frac{3}{4})^3=(\frac{3}{4})^5$.

I $(-4)^3\cdot(-4)^3=(-4)^6$.

D4

positive integral power (in′tə grəl). A power that has a positive integral exponent. Suppose that, in x^n, x is replaced by any member of R, and n is replaced by any member of I_p. Then x^n is a positive integral power of x. The replacement for n is the positive integral exponent of x^n. The replacement for x is the base of x^n. The number obtained from x^n is the nth power of x.

C What property of multiplication of rational numbers tells you that numeral F names the same number as numeral E?

D How many times is 10 used as a factor in the number expressed by numeral F? In 10^7? Does numeral G express the same number as numeral F? Is 10^7 the product of 10^3 and 10^4?

E Is the sum of the exponents of 10^3 and 10^4 the same number as the exponent of 10^7?

F What is a standard name of 10^7? What is a standard name of $10^3\times10^4$?

G Now read the sentences in D4. Use steps like those in D3 to show why each sentence in D4 expresses a true statement.

You have been studying a special method of finding the product of two positive integral powers that have the same base. You can use this method because of the *product property of positive integral powers*. The property is expressed below.

The universe for x *is R. The universe for* m *and* n *is* I_p. *For each* x, m, *and* n, $x^m \cdot x^n = x^{m+n}$.

Tell whether each of the following sentences expresses a true statement or a false statement.

H $10^6 \cdot 10^3 = 10^3$.

I $(^-2)^3 \cdot (^-2)^5 = (^-2)^8$.

J $(\frac{1}{3})^4 \times (\frac{1}{3})^4 = (\frac{1}{3})^{16}$.

K $(^-5)^6 \times (^-5)^3 = (^-5)^9$.

Now you will use your knowledge of positive integral powers to find quotients of powers that have the same base.

A Look at D5. Numeral J expresses the quotient of what two rational numbers?

B What is the exponent of 10^5? Of 10^3? Does the dividend, 10^5, have the same base as the divisor, 10^3? Name this base.

C Now look at numeral K. Does numeral K name the same number as numeral J? How do you know?

D The greatest common factor of two given numbers is the greatest number that is a factor of both numbers. Look again at numeral K. Name the greatest common factor of $10 \times 10 \times 10 \times 10 \times 10$ and $10 \times 10 \times 10$. Is $1000 = 10^3$?

E How does numeral L show that both the dividend and the divisor have been divided by 1000?

F Does numeral M name the same number as numeral L? Is 10^2 the quotient of 10^5 and 10^3?

G Is the difference of the exponent of 10^5 and the exponent of 10^3 the same number as the exponent of 10^2?

H What is a standard name of 10^2? What is a standard name of the quotient of 10^5 and 10^3?

J $\dfrac{10^5}{10^3}$

K $\dfrac{10 \times 10 \times 10 \times 10 \times 10}{10 \times 10 \times 10}$

L $\dfrac{\cancel{10}^{1} \times \cancel{10}^{1} \times \cancel{10}^{1} \times 10 \times 10}{\cancel{10} \times \cancel{10} \times \cancel{10}}$

M 10^2

D5

N $\dfrac{(\frac{5}{8})^6}{(\frac{5}{8})^2} = (\frac{5}{8})^4$.

O $\dfrac{14^5}{14^2} = 14^3$.

P $\dfrac{(^-3)^8}{(^-3)^5} = (^-3)^3$.

D6

I Now study sentence N in D6. Which numeral expresses the dividend? Which numeral expresses the divisor? Do $(\frac{5}{8})^6$ and $(\frac{5}{8})^2$ have the same base? Which is greater, the exponent of the dividend or the exponent of the divisor?

J Study sentences O and P. In each case, do the dividend and the divisor have a common base? Is the exponent of the dividend greater than the exponent of the divisor?

K Use steps like those in D5 to show why each statement expressed in D6 is true.

You have been studying a special method of finding the quotient of two powers that have positive integral exponents and the same base. You can use this method because of the *quotient property of positive integral powers*. The property is expressed on the next page.

The universe for x *is* Z_r. *The universe for* m *and* n *is* I_p. *For each* x, m, *and* n, *if* m *is greater than* n, *then* $x^m \div x^n = x^{m-n}$.

Tell whether each of the following sentences expresses a true statement or a false statement.

L $(^{+}\frac{5}{6})^7 \div (^{+}\frac{5}{6})^4 = (^{+}\frac{5}{6})^3$.

M $(^{-}\frac{2}{3})^9 \div (^{-}\frac{2}{3})^3 = (^{-}\frac{2}{3})^3$.

N $(^{-}20)^5 \div (^{-}20)^4 = (^{-}20)^1$.

O $\frac{3^{10}}{3^5} = 3^2$.

P $\frac{18^4}{18^2} = 9^2$.

In *Book 1* you learned that scientific notation is a kind of exponential notation that makes use of powers of 10. You will recall that, in scientific notation, a number is expressed as the product of a number from 1 to 10 and a power of 10. Remember that the set of numbers from 1 to 10 contains 1, but not 10.

A Look at sentence A in D7. By what power of 10 can you divide 600 to obtain 6? How do you know that the numeral 6×10^2 expresses 600 in scientific notation?

B Look at sentence B. By what power of 10 can you divide 1256 to obtain a number from 1 to 10? What is the quotient of 1256 and 10^3? Is $1256 = 1.256 \times 10^3$?

C Does the numeral 1.256×10^3 express 1256 in scientific notation? Explain your answer.

D Read sentence C in D7. By what power of 10 can you divide 43,285 to obtain a number from 1 to 10? Is $43{,}285 \div 10^4 = 4.3285$? By what power of 10 can you multiply 4.3285 to obtain 43,285?

E Does the numeral 4.3285×10^4 express 43,285 in scientific notation?

Use scientific notation to express each of the numbers named below.

F 80,000,000 G 14,000 H 18,700 I 24,369,000 J 17 K 322

Now you will learn how to use scientific notation to find the product of two numbers.

A $600 = 6 \times 10^2$.
B $1256 = 1.256 \times 10^3$.
C $43{,}285 = 4.3285 \times 10^4$.

D7

D $45{,}000 \times 5100$
E $(4.5 \times 10^4) \times (5.1 \times 10^3)$
F $(4.5 \times 5.1) \times (10^4 \times 10^3)$
G 22.95×10^7

D8

H $780{,}000 \div 3900$
I $(7.8 \times 10^5) \div (3.9 \times 10^3)$
J $\frac{7.8 \times 10^5}{3.9 \times 10^3}$
K 2×10^2

D9

L Look at D8. Numeral D expresses the product of what two numbers?

M How do you know that numeral E names the same number as numeral D?

N Does numeral F name the same number as numeral E? Upon what properties does your answer depend?

O How do you know that numeral G expresses the same number as numeral F?

P What is a standard name of 22.95×10^7? Of $45{,}000 \times 5100$?

Use steps like those in D8 to show why the sentences in exercises Q, R, and S express true statements.

Q $30 \times 6725 = 20.175 \times 10^4$.
R $492 \times 65{,}000{,}000 = 31.980 \times 10^9$.
S $6200 \times 328{,}000 = 20.336 \times 10^8$.

T What is a standard name of 20.175×10^4? Of 31.980×10^9? Of 20.336×10^8?

Next you will learn how to use scientific notation to find the quotient of two numbers.

U Look at D9. Numeral H expresses the quotient of what two numbers?

V How do you know that numeral I names the same number as numeral H?

W How do you know that numeral J names the same number as numeral I?

X Look at numeral K. Is 2 the quotient of 7.8 and 3.9? Is 10^2 the quotient of 10^5 and 10^3? Is $780{,}000 \div 3900 = 2 \times 10^2$?

Y Is the quotient of 780,000 and 3900 equal to the product of the following two numbers: the number that is the quotient of 7.8 and 3.9 and the number that is the quotient of 10^5 and 10^3?

Z What is a standard name of 2×10^2? What is a standard name of $780{,}000 \div 3900$?

In this lesson you learned what is meant by a positive integral power. You studied the product property and the quotient property of positive integral powers, and you used these properties to help you find the products and quotients of rational numbers.

On your own

Name the exponent and the base of each power expressed in exercises 1 through 5. Then give a standard name of the power.

1 10^6 **2** $(\frac{3}{4})^3$ **3** $(^-4)^5$ **4** $(^-\frac{2}{11})^4$ **5** $(\frac{1}{10})^7$

For each of exercises 6 through 11, first use exponential notation to express the product or the quotient named. Then give a standard name of the product or the quotient.

6 $(^-\frac{3}{8})^4 \cdot (^-\frac{3}{8})^5$

7 $(\frac{2}{3})^5 \div (\frac{2}{3})^2$

8 $12^9 \div 12^3$

9 $5^1 \times 5^3 \times 5^3$

10 $(^-2)^3 \cdot (^-2)^2 \cdot (^-2)^4$

11 $(7^4 \div 7^2) \times 7^3$

Use scientific notation to express each number named in exercises 12 through 17.

12 22,000	**14** 6,924,000	**16** 83
13 500	**15** 345,900	**17** 2964

For each of exercises 18 through 21, use scientific notation to express the numbers named. Then find the product of the two numbers and give a standard name of the product.

18 8000	**19** 51,000	**20** 256	**21** 45
60,000	700,000	23	1230

For each of exercises 22 through 28, use scientific notation to express the numbers named. Then find the quotient of the two numbers and give a standard name of the quotient. Use the first number as the dividend.

22 2,900,000,000,000	**24** 582,000,000	
1,450,000	1,940,000	
23 78,000,000	**25** 67,500,000	
13,000	1,350,000	
26 34,000	**27** 800,000	**28** 12,000
170	1600	3600

KEEPING SKILFUL

For each of exercises 1 through 12, use a decimal to express the rational number named.

1 $\frac{3}{10}$	**4** $\frac{16}{3}$	**7** $\frac{65}{4}$	**10** $\frac{2}{11}$
2 $^-\frac{22}{25}$	**5** $\frac{3}{80}$	**8** $^-\frac{37}{9}$	**11** $^-\frac{3}{16}$
3 $^-\frac{51}{8}$	**6** $^-\frac{1}{500}$	**9** $\frac{17}{8}$	**12** $^-\frac{11}{1000}$

For each of exercises 13 through 24, give a standard name of the number named in the exercise.

13 10^3	**16** $(^-\frac{1}{4})^4$	**19** $(^-\frac{1}{16})^3$	**22** $(\frac{1}{10})^5$
14 $(\frac{1}{2})^3$	**17** $(^-10)^5$	**20** 5^2	**23** $(^-65)^2$
15 $(^-6)^4$	**18** 8^4	**21** 1^9	**24** $(\frac{1}{8})^4$

Each exercise below involves rational numbers. For each exercise, find the sum, difference, product, or quotient.

25 $10.7 + 8.401$	**31** $^-2.11 \times ^-2.11$
26 $13.4 - 3.2$	**32** $^-6.451 + ^-3.02$
27 $3.06 + ^-21.32$	**33** $^-24.631 - ^-18.12$
28 $16.3 \times .41$	**34** $^-1.956 \div .03$
29 $1.3 \div 2$	**35** $.011 \times ^-3.4$
30 $43.4 - ^-16.34$	**36** $^-7.488 \div ^-2.4$

Negative integral powers and the zero power

In lesson 143, you worked with positive integral powers. In this lesson you will learn that the integer 0 and the negative integers are also used as exponents.

A How do you know that $3^2 \times 3^3 = 3^{2+3}$? Is $3^{2+3} = 3^5$? Use standard names to show that $3^2 \times 3^3 = 3^5$.

B How do you know that $9^4 \div 9^1 = 9^{4-1}$? Is $9^{4-1} = 9^3$? Use standard names to show that $9^4 \div 9^1 = 9^3$.

C Look at D1. What must you know before you can decide whether sentences A and B express true statements or false statements?

Before you can understand the statements expressed in D1, you must know the meaning of the symbol at the right. You can read this symbol "the zero power of ten." You can also read the symbol "ten to the zero power," or "ten to the zero."

10^0

D The exponent of 10^0 is zero. Is $0 \in I_p$? Is $0 \in I$? What is the exponent of 10^3? Is $3 \in I_p$?

E Name the base of 10^0. Name the base of 10^3. Do 10^0 and 10^3 have the same base?

F 10^3 means that 10 is used as a factor three times. It does not make sense to think of using a number as a factor zero times. Why?

We would like the product property of powers to apply to powers that have zero as an exponent as well as to positive integral powers. We would like to define 10^0 in such a way that the product of 10^0 and 10^3 is equal to 10^{0+3}.

A $10^0 \cdot 10^3 = 10^{0+3}$.
B $10^0 \cdot 10^3 = 10^3$.

D1

C $(\bar{\frac{1}{2}})^0 \cdot (\bar{\frac{1}{2}})^4 = (\bar{\frac{1}{2}})^{0+4}$.
D $(\bar{\frac{1}{2}})^0 \cdot (\bar{\frac{1}{2}})^4 = (\bar{\frac{1}{2}})^4$.

D2

zero power. The zero power of any non-zero rational number is 1. The universe for x is Z_r. For each x, $x^0 = 1$.

G Study sentences A and B in D1. Suppose that we define 10^0 so that statement A is true. Explain why statement B is also true.

H What is the identity element for multiplication of rational numbers? Is $1 \cdot 10^3 = 10^3$?

The number 1 is the *only* identity element for multiplication of rational numbers. If we want $10^0 \cdot 10^3 = 10^3$ to be a true statement, 10^0 must equal 1.

I Study sentences C and D in D2. Explain why $(\bar{\frac{1}{2}})^0$ must be equal to 1 if statement D is to be true. If statement D is true, how do you know that statement C is also true?

We will use variables to develop a definition of a power with 0 as an exponent.

J Suppose that, for each x and n, we define x^0 so that $x^0 \cdot x^n = x^n$. Will $x^0 \cdot x^n$ then be equal to x^{0+n}? How do you know? The universe for x is Z_r, the set of non-zero rational numbers. The universe for n is I_p.

K How must we define x^0 so that, for each x and n, $x^0 \cdot x^n = x^{0+n}$?

In the preceding exercises, you have learned what is meant by a power whose exponent is zero. We call a power whose exponent is zero a *zero power*. The zero power of any non-zero rational number is equal to 1.

Notice that we have not defined 0^0. You will learn the reason for this as you continue your study of mathematics.

L For each x, is $x^0 \cdot x^0 = x^0$? The universe for x is Z_r. Use the definition of the zero power to explain your answer.

So far, you have worked with exponents that are either positive integers or the integer zero. However, negative integers are also used as exponents.

Look at D3. Before you can understand the statements expressed in the display, you must understand the idea expressed by the symbol at the right. You can read this symbol "the negative two power of ten." You can also read the symbol in several other ways: "ten to the negative two power" and "ten to the negative two."

10^{-2}

A Look again at D3. The exponent of 10^{-2} is -2. Is $-2 \in I_p$? Is $-2 \in I_n$? What is the exponent of 10^2? Is $2 \in I_p$?

B Name the base of 10^{-2}. Name the base of 10^2. Do 10^{-2} and 10^2 have the same base?

An exponent that is a negative integer, such as the exponent of 10^{-2}, is a *negative integral exponent*. We would like the product property of powers to apply to powers that have negative integral exponents also. This means that we would like to define 10^{-2} in such a way that the product of 10^{-2} and 10^2 is 10^{-2+2}.

C Look at sentence E in D3. How do you know that the exponents -2 and 2 are additive inverses of each other? What is the sum of -2 and 2?

D Compare sentences E, F, and G. Is $10^{-2+2} = 10^0$? Explain why $10^0 = 1$.

E If we define 10^{-2} so that statement E is true, is statement G also true? Explain your answer.

D3

E $10^{-2} \cdot 10^2 = 10^{-2+2}$.
F $10^{-2} \cdot 10^2 = 10^0$.
G $10^{-2} \cdot 10^2 = 1$.

D4

H $(\frac{-3}{4})^{-5} \cdot (\frac{-3}{4})^5 = (\frac{-3}{4})^{-5+5}$.

negative integral power. A power that is the reciprocal of a positive integral power. The universe for x is Z_r. The universe for n is I_p. For each x and n, $x^{-n} = 1/x^n$.

F What is the product of two numbers that are reciprocals of each other?

G If 10^{-2} is defined so that statement G is true, why must 10^{-2} be the reciprocal of 10^2?

The reciprocal of 10^2 is $1/10^2$. Therefore, we define 10^{-2} so that $10^{-2} = 1/10^2$.

H Look at D4. If statement H is true, why must $(\frac{-3}{4})^{-5}$ be equal to $1/(\frac{-3}{4})^5$? Use the sentences in D3 to help you explain your answer.

For exercises I and J, the universe for x is Z_r. The universe for n is I_p.

I For each x and n, what is the reciprocal of x^n? How do you know that $-n \in I_n$?

J Suppose that we define x^{-n} in such a way that, for each x and n, $x^{-n} \cdot x^n = x^{-n+n}$. To do this, why must we define x^{-n} to be equal to $1/x^n$?

In the preceding exercises, you have learned what is meant by a power that has a negative integral exponent. We call a power that has a negative integral exponent a *negative integral power*. A negative integral power is the reciprocal of a positive integral power. The exponents of the two powers are the additive inverses of each other. The bases of the two powers are the same.

K For each x and n, how are $1/x^n$ and x^{-n} related? How are x^{-n} and x^n related? The universe for x is Z_r. The universe for n is I_p.

L You now know the definition of a negative integral power. You have learned that 0 does not have a reciprocal. Use this knowledge to explain why a negative power of the number 0 is not defined.

Express each power named below as a reciprocal of a positive integral power. For exercise M, you can write the numeral $1/(^-6)^7$.

M $(^-6)^{-7}$ **O** 52^{-5} **Q** $(^-100)^{-2}$ **S** 43^{-3}
N $(\frac{2}{3})^{-4}$ **P** $(^-\frac{7}{2})^{-3}$ **R** $(\frac{5}{8})^{-1}$ **T** $(^-\frac{9}{10})^{-2}$

The product property of positive integral powers was used to develop definitions of both the zero power and negative integral powers. Now you will decide if there is a product property of powers that applies to all powers with integral exponents.

A Look at D5. How do you know that statement I is true? Is statement J true?

B Read sentence K. What is the greatest common factor of 10^4 and 10^7? How can you obtain $1/10^3$ from $10^4/10^7$? Is statement K true?

C How do you know that sentence L expresses the same statement as sentence K? Is statement L true?

D Is the base of 10^{-3} the same as the base of 10^4 and 10^{-7}? What is the sum of the exponents of 10^4 and 10^{-7}? Is this sum the same number as the exponent of 10^{-3}?

E Is the product of 10^4 and 10^{-7} a power of 10 whose exponent is the sum of the exponents of 10^4 and 10^{-7}?

Look at D6. Each sentence in D6 expresses a true statement about the product of two powers of 10.

F Read sentences M and N in D6. Use the steps given in D5 to explain why statements M and N are true.

I $10^4 \cdot 10^{-7} = 10^4 \cdot \frac{1}{10^7}$.

J $10^4 \cdot 10^{-7} = \frac{10^4}{10^7}$.

K $10^4 \cdot 10^{-7} = \frac{1}{10^3}$.

L $10^4 \cdot 10^{-7} = 10^{-3}$.

D5

M $10^{-3} \cdot 10^{-5} = 10^{-8}$.

N $10^9 \cdot 10^{-2} = 10^7$.

O $10^0 \cdot 10^{-6} = 10^{-6}$.

D6

G Read sentence O in D6. How do you know that $10^0 \cdot 10^{-6} = 1 \cdot 10^{-6}$? Is statement O true? How can you obtain the exponent of 10^{-6} from the exponents of 10^0 and 10^{-6}?

For each of the following exercises, use the steps given in D5 to help you find the product named in the exercise. Use exponential notation to express the product.

H $13^{-4} \times 13^0$ **J** $(^-17)^2 \times (^-17)^{-6}$
I $(\frac{1}{4})^{-5} \times (\frac{1}{4})^{-4}$ **K** $(\frac{5}{6})^{-1} \times (\frac{5}{6})^1$

Now you know how to find the product of any two powers of a non-zero rational number when the powers have integral exponents. The property expressed below is the *product property of integral powers.*

The universe for x *is* Z_r. *The universe for* m *and* n *is I. For each* x, m, *and* n, $x^m \cdot x^n = x^{m+n}$.

For each of exercises L through Q, use the product property stated above to find the product named in the exercise. Use exponential notation to name the product.

L $(^-12)^3 \cdot (^-12)^{-1}$ **O** $(^-4)^0 \cdot (^-4)^2$
M $19^{-7} \times 19^3$ **P** $(^-\frac{11}{13})^{-6} \cdot (^-\frac{11}{13})^4$
N $(\frac{5}{8})^{-6} \cdot (\frac{5}{8})^{-4}$ **Q** $141^{-11} \times 141^8$

Next you will decide if there is a quotient property that applies to all powers with integral exponents. Remember that we are speaking of powers of non-zero rational numbers.

A Look at sentence P in D7. What is the exponent of 10^{-6}? Of 10^{-2}? What number is the base of both 10^{-6} and 10^{-2}?

B How do you know that sentence P expresses a true statement?

C Does sentence Q express a true statement? How do you know?

D How do you know that sentence R expresses a true statement?

E Study sentence S. What is the greatest common factor of 10^2 and 10^6? How can you obtain statement S from statement R? Is statement S true?

F How can you obtain statement T from statement S? Is statement T true?

G What power of 10 is the quotient of 10^{-6} and 10^{-2}?

H Is the base of 10^{-4} the same as the base of 10^{-6} and 10^{-2}? What is the difference of the exponents of 10^{-6} and 10^{-2}? Is the difference of these exponents the exponent of 10^{-4}?

Now you see that the quotient of 10^{-6} and 10^{-2} is a power of 10 whose exponent is the difference of the exponents of 10^{-6} and 10^{-2}.

Look at D8. Each sentence in D8 expresses a true statement about the quotient of two powers of 10.

I Read sentences U and V in D8. Use the steps given in D7 to explain why statements U and V are true.

J Study sentence W. Is $10^0 = 1$? How do you know that statement W is true?

For each of exercises K through N, use the steps given in D7 to find the quotient expressed in the exercise. Use exponential notation to name the quotient.

P $\dfrac{10^{-6}}{10^{-2}} = \dfrac{1}{10^6} \div \dfrac{1}{10^2}$.

Q $\dfrac{10^{-6}}{10^{-2}} = \dfrac{1}{10^6} \cdot 10^2$.

R $\dfrac{10^{-6}}{10^{-2}} = \dfrac{10^2}{10^6}$.

S $\dfrac{10^{-6}}{10^{-2}} = \dfrac{1}{10^4}$.

T $\dfrac{10^{-6}}{10^{-2}} = 10^{-4}$.

D7

U $\dfrac{10^{-3}}{10^5} = 10^{-8}$.

V $\dfrac{10^9}{10^{-6}} = 10^{15}$.

W $\dfrac{10^0}{10^{-4}} = 10^4$.

D8

K $12^3 \div 12^{-4}$

L $(\frac{1}{7})^0 \div (\frac{1}{7})^{-1}$

M $(\frac{3}{4})^{-2} \div (\frac{3}{4})^{-5}$

N $(\frac{-3}{8})^{-7} \div (\frac{-3}{8})^4$

Now you know how to find the quotient of any two powers of a non-zero rational number when the exponents are integers. The *quotient property of integral powers* is expressed below.

The universe for x *is* Z_r. *The universe for* m *and* n *is I. For each* x, m, *and* n, $x^m \div x^n = x^{m-n}$.

For each exercise below, use the quotient property to find the quotient named. Use exponential notation to name the quotient.

O $62^0 \div 62^{-1}$

P $(\frac{-7}{6})^{-6} \div (\frac{-7}{6})^4$

Q $39^{13}/39^{-7}$

R $(-5)^0/(-5)^2$

S $(-21)^{-9} \div (-21)^{-11}$

T $(\frac{6}{11})^{-5} \div (\frac{6}{11})^{-8}$

724,569.39,081

A	7	2	4	5	6	9	3	9	0	8	1
B	10^5	10^4	10^3	10^2	10^1	1	$\frac{1}{10^1}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$
C	10^5	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
D	100,000	10,000	1000	100	10	1	.1	.01	.001	.0001	.00001
	hundred thousands	ten thousands	thousands	hundreds	tens	ones	tenths	hundredths	thousandths	ten-thousandths	hundred-thousandths

D9

As you know, our decimal numeration system is a base-ten system. Now that you know the meaning of the zero power and of negative integral powers, you can see how each digit in a decimal numeral is associated with a power of 10.

A Read the decimal numeral at the top of D9.

B Row B expresses the numbers associated with each digit in the numeral 724,569.39,081. Row C expresses each of these numbers as a power of 10. Express 1 as a power of 10. Express 1000 as a power of 10. Express $\frac{1}{10^2}$ as a power of 10.

For each of exercises C through H, first write the words that you say in reading the numeral. Then write each numeral in expanded form.

C 32.038 **E** .00762 **G** 37,926.345

D 476.25 **F** 1.9 **H** .86259

In this lesson you studied the zero power and negative integral powers. You learned to find the product and the quotient of any two powers of a non-zero rational number when the powers have integral exponents.

On your own

For each of exercises 1 through 9, tell whether the sentence expresses a true statement or a false statement.

1 $(^-13)^0 = 1$.

2 $(\frac{3}{4})^3 = \frac{27}{4}$.

3 $(\frac{5}{8})^{-4} = \frac{1}{(\frac{5}{8})^4}$.

4 $(^-7)^0 = {}^-1$.

5 $(\frac{13}{14})^0 = 0$.

6 $3^{-2} = \frac{1}{9}$.

7 $(\frac{5}{6})^{-1} = (\frac{5}{6})^1$.

8 $6^{-2} = 1 \div 6^2$.

9 $(\frac{2}{3})^{-1} = \frac{1}{\frac{3}{2}}$.

For each exercise below, use exponential notation to express the product named.

10 $21^{-2} \times 21^0$

11 $(^-9)^{-9} \times (^-9)^{10}$

12 $(^-\frac{5}{2})^4 \times (^-\frac{5}{2})^{-6}$

13 $(\frac{2}{3})^7 \times (\frac{2}{3})^7$

14 $12^0 \cdot 12^{-8}$

15 $(^-62)^{-8} \cdot (^-62)^{-12}$

16 $(^-\frac{5}{12})^0 \times (^-\frac{5}{12})^0$

17 $(\frac{13}{7})^{-5} \times (\frac{13}{7})^{-11}$

For each exercise below, use exponential notation to express the quotient named.

18 $\frac{61^{14}}{61^{-14}}$

19 $(^-6)^8 \div (^-6)^0$

20 $(\frac{17}{19})^{-4} \div (\frac{17}{19})^5$

21 $(^-\frac{21}{25})^2 \div (^-\frac{21}{25})^{-3}$

22 $\frac{41^{-9}}{41^{-7}}$

23 $(^-57)^{-6} \div (^-57)^{-10}$

24 $(^-\frac{1}{33})^8 \div (^-\frac{1}{33})^2$

25 $(\frac{8}{15})^0 \div (\frac{8}{15})^{-7}$

For each of exercises 26, 27, and 28, write the numeral in expanded form.

26 6894.75 **27** 701.398 **28** .0010408

Integral powers and scientific notation

You have used scientific notation to express such numbers as 3600 and 2,000,000. In this lesson you will first use your knowledge of negative integral powers of 10 and the zero power of 10 to express numbers like 9.75 and .00048 in scientific notation. Then you will learn to use scientific notation in computation.

A How do you know that the numeral 7.5×10^2 expresses a number in scientific notation? What is a standard name of 7.5×10^2?

B What is a standard name of 5.362×10^1?

C Express 8634.5 in scientific notation. Express 10 in scientific notation.

D Use scientific notation to express 1 million, 2 million, and 5.3 million.

Now you will see how numbers from 1 to 10 can be expressed in scientific notation.

E Look at sentence A in D1. How do you know that the numeral 6×10^0 expresses a number in scientific notation? How do you know that $6 = 6 \times 10^0$?

F Look at sentences B and C. What numeral expresses 3.2 in scientific notation? What is a standard name of 1.5842×10^0?

G Use scientific notation to express 9, 9.7, 9.75, and 9.752.

H Describe a method of expressing any rational number from 1 to 10 in scientific notation.

In the following exercises, you will learn how to use scientific notation to express rational numbers that are between 0 and 1.

A $6 = 6 \times 10^0$.
B $3.2 = 3.2 \times 10^0$.
C $1.5842 = 1.5842 \times 10^0$.

D1

D $.625 = 6.25 \times 10^{-1}$.
E $.03 = 3 \times 10^{-2}$.
F $.0048 = 4.8 \times 10^{-3}$.
G $.0007 = 7 \times 10^{-4}$.

D2

I Read sentence D in D2. How do you know that .625 is between 0 and 1?

J To express .625 in scientific notation, you must express it as the product of a number from 1 to 10 and a power of 10. Is $.625 \times 10^1 = 6.25$? Is 6.25 a number from 1 to 10? How do you know that $.625 = 6.25 \div 10^1$?

K Explain why $6.25 \div 10^1 = 6.25 \times 1/10^1$. Is $6.25 \times 1/10^1 = 6.25 \times 10^{-1}$? How do you know that $.625 = 6.25 \times 10^{-1}$?

L Does the numeral 6.25×10^{-1} express .625 in scientific notation? Explain your answer.

M Now look at sentence E in D2. By what power of 10 should you multiply .03 to obtain a number from 1 to 10? Is $.03 \times 10^2 = 3$? How do you know that $.03 = 3 \div 10^2$?

N Is $3 \div 10^2 = 3 \times 10^{-2}$? How do you know that $.03 = 3 \times 10^{-2}$? What numeral expresses .03 in scientific notation?

O Read sentence F in D2. By what power of 10 should you multiply .0048 to obtain a number from 1 to 10? By what power of 10 should you divide 4.8 to obtain .0048?

P How do you know that $.0048 = 4.8 \times 10^{-3}$? What numeral expresses .0048 in scientific notation?

Q Study sentence G. Explain how 7×10^{-4} is obtained.

Use scientific notation to express each of the numbers named below.

R 9.93 T .053 V 325.68
S .000036 U 16.8 W .14285

In the work that follows, you will learn how to use scientific notation to find the product of two numbers.

A Look at D3. The product of what two numbers is expressed by numeral H?

B Now look at numeral I. How is 3,000,000 expressed in numeral I? How is 26,000 expressed? Does numeral I express the same number as numeral H?

C What properties can you use to prove that numeral J expresses the same number as numeral I? Give the steps in your proof. Name the property that you use for each step.

D How do you know that numeral K expresses the same number as numeral J?

E What numeral expresses the product of 3,000,000 and 26,000 in scientific notation? What is a standard name of the product?

F Look at numerals L and M in D4. How is 75 expressed in numeral M? How is .016 expressed? Does numeral M express the same number as numeral L?

G What properties can you use to prove that numeral N expresses the same number as numeral M?

H Explain why numeral O expresses the same number as numeral N. How do you know that numeral O is not in scientific notation?

I Is $12 \times 10^{-1} = (1.2 \times 10^{1}) \times 10^{-1}$? How do you know that numeral P expresses the same number as numeral O?

J What numeral expresses the product of 75 and .016 in scientific notation? What is a standard name of the product?

Use scientific notation to find each product expressed in exercises K through R. Express the product in scientific notation. Then give a standard name of the product.

K $.67 \times 3000$ O $1.6 \times .00005$
L $15{,}000 \times 4200$ P $.03 \times .0072$
M $.091 \times 34{,}000$ Q $.45 \times 3100$
N $.004 \times .38$ R $.0017 \times .00039$

Next you will learn how to use scientific notation to find the quotient of two numbers.

A Look at D5. The quotient of what two numbers is expressed by numeral Q?

B How do you know that numeral R expresses the same number as numeral Q?

H $3{,}000{,}000 \times 26{,}000$
I $(3 \times 10^{6}) \times (2.6 \times 10^{4})$
J $(3 \times 2.6) \times (10^{6} \times 10^{4})$
K 7.8×10^{10}

D3

L $75 \times .016$
M $(7.5 \times 10^{1}) \times (1.6 \times 10^{-2})$
N $(7.5 \times 1.6) \times (10^{1} \times 10^{-2})$
O 12×10^{-1}
P 1.2×10^{0}

D4

Q $330{,}000 \div .011$

R $(3.3 \times 10^{5}) \div (1.1 \times 10^{-2})$

S $\dfrac{3.3 \times 10^{5}}{1.1 \times 10^{-2}}$

T $\dfrac{3.3}{1.1} \times \dfrac{10^{5}}{10^{-2}}$

U 3×10^{7}

D5

C How do you know that the quotient expressed by numeral S is the same as the quotient expressed by numeral R?

D Look at numeral T. This numeral expresses the quotient of (3.3×10^5) and (1.1×10^{-2}) as the product of two quotients. What is the quotient of 3.3 and 1.1? What is the quotient of 10^5 and 10^{-2}?

E What numeral expresses the quotient of 330,000 and .011 in scientific notation?

Use scientific notation to find each quotient expressed in exercises F through K. Express the quotient in scientific notation. Then give a standard name of the quotient.

F $.086 \div 4.3$

G $.0095 \div 1.9$

H $\dfrac{72{,}000}{4{,}000{,}000}$

I $16{,}000 \div .2$

J $15.4 \div .07$

K $\dfrac{.086}{4300}$

In this lesson you learned how to express rational numbers between 0 and 10 in scientific notation. You also learned how a knowledge of scientific notation can help you find products and quotients.

On your own

Use scientific notation to express each of the numbers named below.

1 .02
2 785,326
3 4.13
4 .3256
5 .0000069
6 16.84
7 .003
8 25,000
9 .0099

For each of exercises 10 through 16, tell which of the numbers named is greater.

10 5,163,000 or 5.163×10^7

11 2.5×10^{-3} or .025

12 15 or 1.52×10^1

13 3.8×10^0 or 38

14 25.6 or 2.56×10^2

15 .0009 or 9.2×10^{-3}

16 .0007 or 7×10^{-2}

Use scientific notation to find each product expressed in exercises 17 through 20. Express the product in scientific notation. Then give a standard name of the product.

17 $186{,}000 \times 3600$

18 $.0024 \times 960$

19 $120{,}000 \times .00003$

20 $.728 \times .00065$

Use scientific notation to find each quotient expressed below. Express the quotient in scientific notation. Then give a standard name of the quotient.

21 $42{,}000 \div 140$

22 $.35 \div 7000$

23 $26.8 \div 6.7$

24 $.0124 \div 3.1$

APPLYING MATHEMATICS

For each problem, write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem.

For problems 1, 2, and 3, the universe is Z. For problems 4 through 7, the universe for each variable is R_a.

1 Mr. Hanson sold $16\frac{1}{2}$ bu. of tomatoes on Friday. He sold a total of 35 bu. of tomatoes on Friday and Saturday. How many bushels of tomatoes did he sell on Saturday?

2 Jim was less than 55 in. tall on Jan. 1. He has grown $2\frac{1}{2}$ in. since then. Now he is more than 57 in. tall. How tall was he on Jan. 1?

3 In a recent year, a total of 35,067,380 tons of coal and iron ore was mined in Canada. The coal mined that year amounted to 10,627,000 tons. How many tons of iron ore were mined in Canada that year?

4 Miss Grant's telephone bill for August was $2.20 more than her bill for July. Her telephone bill for July was $4.35. What was her telephone bill for August?

5 Last week Alice earned more than $10.50. After she had spent $2.60 of this money, she had less than $9.50 left. How much money did she earn last week?

6 What numbers less than 40 can be added to 102.16 so that each sum is greater than 141.26?

7 The sum of three numbers is 32. The first number is 1 more than the second number. The product of the second number and 2 is 25. What are the three numbers?

For problems 8 through 16, the universe for each variable is Z.

8 The York Federal Savings Bank charges 6% interest on the money it lends. Mr. Sayre borrowed $1950 from this bank. At the end of one year, how much interest did Mr. Sayre owe the bank?

9 Mr. Lopez sold $2560 worth of electrical appliances and received a commission of $192. What rate of commission did Mr. Lopez receive?

10 For a party, the Camera Club needs 72 ice-cream bars. At the Snack Shop, 12 ice-cream bars cost $.95. At Jackson's Grocery, 18 ice-cream bars cost $1.25. What will it cost the Camera Club to buy 72 ice-cream bars at one of these stores?

11 The playing portion of a football field is in the shape of a rectangle and is 110 yd. long and $63\frac{1}{3}$ yd. wide. What is the perimeter of this portion of the field in feet?

12 A certain field is triangular in shape. The length of an altitude of the field is 20 yd. The length of the base is 32 yd. What is the area of the field in square yards?

13 A sign in the window of Robinson's Grocery is in the shape of a parallelogram. The area of the sign is 360 sq. in. The length of an altitude is $1\frac{1}{2}$ ft. What is the length of the base in feet?

THINK Is 144 square inches the same area as 1 square foot?

14 9.61 is what fraction of 48.05?

15 What number is $\frac{4}{5}$ of $60\frac{3}{10}$?

16 .3 is .012 of what number?

CHECKING UP

If you have trouble with these two tests, you can find help in lessons 143, 144, and 145.

Test 130

Give a standard name of each power expressed in exercises 1 through 8.

1 5^3 **2** $(\frac{1}{2})^4$ **3** $(^-1)^6$ **4** $(^-3)^5$ **5** 481^0 **6** 2^{-3} **7** $(^-\frac{3}{2})^5$ **8** $(10)^{-3}$

Give a standard name of each number expressed in exercises 9 through 14.

9 3.72×10^3 **10** 4.8×10^0 **11** 7.06×10^5 **12** 2.175×10^{-4} **13** 1.003×10^6 **14** 9.89×10^{-5}

Use scientific notation to express each number named in exercises 15 through 20.

15 6,000,000 **16** 3100 **17** 4.5 **18** .57 **19** .00464 **20** .000104

For exercises 21, 22, and 23, write each numeral in base-ten expanded form.

21 6145.03 **22** .190728 **23** 60,891.207

Test 131

Find the product or the quotient named in each of exercises 24 through 29. Use exponential notation to express the product or the quotient.

24 $(\frac{5}{6})^5 \times (\frac{5}{6})^4$ **25** $(^-12)^8 \cdot (^-12)^{-7}$ **26** $(^-\frac{7}{8})^{-4} \times (^-\frac{7}{8})^0$ **27** $28^{-12} \div 28^{-19}$ **28** $(^-6)^5 \div (^-6)^1$ **29** $10^{-3} \div 10^4$

Find each product expressed in exercises 30 through 33. Use scientific notation in your computation. Then express the product in scientific notation.

30 $800 \times 31{,}000$ **31** $24{,}000{,}000 \times 4000$ **32** $.006 \times 45{,}000$ **33** $.08 \times 120{,}000$

Find each quotient expressed in exercises 34 through 37. Use scientific notation in your computation. Then express the quotient in scientific notation.

34 $40 \div 8{,}000{,}000$ **35** $1443 \div .003$ **36** $54{,}000 \div .18$ **37** $2.7 \div 9000$

End-of-block tests on integral powers and scientific notation

SPECIAL CHALLENGE

In these exercises you will work with numeration systems that have bases other than ten.

A In a base-five numeration system, the first ten natural numbers can be expressed by the numerals 0, 1, 2, 3, 4, 10, 11, 12, 13, and 14. Why is the numeral 10 used to express five? What base-five numeral expresses ten? Write base-five numerals to express the natural numbers ten through twenty.

B Suppose that the numerals for the first five natural numbers are 0, 1, 2, 10, 11. What is the base of this numeration system?

C Convert the base-three numeral 10212 to a base-ten numeral.

D What base-three numeral expresses twenty-five? Expresses thirty-two?

E The base-three numeral $\frac{10}{11}$ expresses the same rational number as the base-ten numeral $\frac{3}{4}$. Convert the base-three numeral $\frac{22}{120}$ to a base-ten numeral.

F What base-three numeral names the same number as the base-ten numeral $\frac{7}{8}$? As the base-ten numeral $\frac{15}{17}$?

When you worked with a base-twelve numeration system in *Book 1*, you used special digits for ten and eleven. Remember that the digit $\mathcal{X}$ (dek) expresses "ten." The digit $\mathcal{E}$ (el) expresses "eleven."

G Convert the base-twelve numerals $2\mathcal{X}3$ and $\frac{1\mathcal{E}}{23}$ to base-ten numerals.

H Convert the base-ten numerals 13, 35, 172, and $\frac{22}{47}$ to base-twelve numerals.

From the reduction property of rational numbers of arithmetic, you know that $\frac{a}{b} = \frac{ka}{kb}$. The universe for a is N. The universe for b and k is C.

I The numerals in this exercise are base-twelve numerals. Is $\frac{3}{9} = \frac{40}{100}$? Is $\frac{7}{\mathcal{X}} = \frac{24}{34}$? Explain your answers.

$\frac{2}{5} = .\overline{4972}.$

$$
\begin{array}{r|l}
5\overline{)2.0000} & 4000 \\
\quad 18000 & \\
\hline
\quad\quad 4000 & \quad 900 \\
\quad\quad 3900 & \\
\hline
\quad\quad\quad 300 & \quad\quad 70 \\
\quad\quad\quad 2\mathcal{E}0 & \\
\hline
\quad\quad\quad\quad 10 & \quad\quad\quad 2 \\
\quad\quad\quad\quad \mathcal{X} & \\
\hline
\quad\quad\quad\quad \not{2} & .4972
\end{array}
$$

We will call base-twelve numerals like 23, $4.\mathcal{E}$, .2, and $.\mathcal{X}3$ *duodecimals*. The duodecimal 4.2 names the same rational number as the base-twelve numeral $\frac{42}{10}$ or the base-ten numeral $\frac{50}{12}$. We will refer to base-twelve numerals like $\frac{42}{10}$ as base-twelve common fraction numerals.

J What base-twelve common fraction numeral names the same number as the duodecimal $3.\mathcal{X}$? As the duodecimal $.02\mathcal{E}5$?

K How can you tell whether or not a rational number that is indicated by a fraction can be expressed by a terminating decimal?

L What decimal expresses the number $\frac{1}{3}$? Is this decimal a terminating decimal?

M What duodecimal expresses the number $\frac{1}{3}$? Is it a terminating duodecimal?

The display above shows how to convert the base-twelve numeral $\frac{2}{5}$ to a duodecimal.

N Study the computation shown in the display. How do you know that $\frac{2}{5}$ cannot be expressed by a terminating duodecimal?

Convert each of the base-twelve numerals in exercises O through T to a duodecimal.

O $\frac{1}{2}$ **P** $\frac{3}{4}$ **Q** $\frac{3}{\mathcal{X}}$ **R** $\frac{2}{7}$ **S** $\frac{5}{6}$ **T** $\frac{7}{9}$

U Explain how to tell when a rational number that is indicated by a fraction can be expressed by a terminating duodecimal.

$U = R_p$.

A $.6/1 \sim 4.2/x$.

B $.6x = 4.2$.

C $\frac{1}{.6}(.6x) = \frac{1}{.6}(4.2)$.

D $(\frac{1}{.6} \times .6)x = \frac{1}{.6}(4.2)$.

E $1x = \frac{1}{.6}(4.2)$.

F $x = 7$.

D1

$U = R$.

G $m - 3 = {}^{-}1$.

H $m + {}^{-}3 = {}^{-}1$.

I $(m + {}^{-}3) + ({}^{+}3) = {}^{-}1 + ({}^{+}3)$.

J $m + ({}^{-}3 + {}^{+}3) = {}^{-}1 + {}^{+}3$.

K $m + 0 = {}^{-}1 + {}^{+}3$.

L $m = {}^{+}2$.

D2

Equivalent conditions

You have used various methods of finding solution sets of conditions when the universe is the set of natural numbers or the set of rational numbers of arithmetic. First, you used the method of replacement to help you find solution sets of conditions. Then you used graphs. Finally, you learned to find solution sets by relating addition and subtraction or by relating multiplication and division.

In this lesson you will learn how to find solution sets by using both a multiplicative inverse and an additive inverse, along with certain properties of multiplication and addition of rational numbers. You will apply these ideas throughout your future study of mathematics.

A Sentence A in D1 expresses what condition for equivalence? What is the universe for x?

To find the solution set of $.6/1 \sim 4.2/x$, you will first obtain a condition whose solution set is easier to determine than the solution set of $.6/1 \sim 4.2/x$. You will use certain definitions and properties of multiplication of rational numbers to find a condition that enables you to think of $1x$, or x, rather than of $4.2/x$.

B You can use the definition of equivalent ordered pairs, which you used when the universe was N or R_a, to help you find a rational number that satisfies a condition for equivalence. How does this definition help you to obtain condition B from condition A?

C Does the x that satisfies $.6x = 4.2$ also satisfy $.6/1 \sim 4.2/x$? Notice that we have said "the x that satisfies" instead of "the replacement for x that satisfies." From now on in this book, we will usually use the shorter expression.

The solution set of condition A is the same as the solution set of condition B. Therefore, $.6/1 \sim 4.2/x$ and $.6x = 4.2$ are *equivalent conditions*. Any two or more conditions that have the same solution set are equivalent.

You obtained condition B from condition A by using a definition. Since it may not be easy to find the solution set of $.6x = 4.2$ immediately, you may wish to obtain other equivalent conditions by using definitions and properties that you have already studied. Each condition that you obtain in this way will be equivalent to both condition A and condition B.

equivalent conditions. Conditions that have the same solution set.

D Look at sentence C. What is the reciprocal of .6? Is the reciprocal of .6 the same as the multiplicative inverse of .6? By what number is $.6x$ multiplied?

E Look again at sentence C. How does the ? well-defined property of multiplication, together with condition B, tell you that the product expressed at the left of the symbol for equality in sentence C is the same as the product expressed at the right of the symbol?

F Are condition B and condition C equivalent? Are condition A and condition C equivalent?

G What property of multiplication of rational numbers can you use to obtain condition D from condition C? Are condition C and condition D equivalent?

H What definition can you use to obtain condition E from condition D? Are condition D and condition E equivalent?

I Look again at sentence E. What is a standard name of the number expressed at the right of the symbol for equality?

J How do you know that condition F is equivalent to condition E?

K What number satisfies $x = 7$? How do you know that condition A and condition F are equivalent? Tabulate $\{x | .6/1 \sim 4.2/x\}$.

From the preceding exercises, you learned how to use the reciprocal of a number and certain properties of multiplication of rational numbers to help you find the solution set of a condition for equivalence. Now you will learn how to use the additive inverse of a number and certain properties of addition of rational numbers to help you obtain the solution set of another kind of condition.

A Look at sentence G in D2. What difference is expressed at the left of the symbol for equality? In the work that follows, you are going to use several properties of rational numbers to obtain a condition that enables you to think of m, rather than of $m - 3$.

You know that some of the properties that you are going to use to help you find the solution set of condition G are properties of addition. You will first use the difference property of rational numbers to obtain a condition that is equivalent to condition G and that involves addition instead of subtraction.

B Is $m - 3 = m + {}^{-}({}^{+}3)$? Is $m - 3 = m + {}^{-}3$? Explain your answers.

C How do you know that condition G and ? condition H are equivalent?

D Think again about condition H. Because the sum of any rational number and 0 is the same as the given rational number, it seems sensible to try to obtain $m + 0$ from $m + {}^{-}3$. What number can you add to ${}^{-}3$ to obtain 0? Is this number the additive inverse of ${}^{-}3$?

Now look at sentence I in D2. From condition H, you know that you want to find an m so that $m + {}^{-}3$ is equal to ${}^{-}1$. According to the well-defined property of addition, you can replace $(m + {}^{-}3)$ in $(m + {}^{-}3) + ({}^{+}3)$ by ${}^{-}1$ without affecting the sum. The sum ${}^{-}1 + ({}^{+}3)$ is the same as the sum $(m + {}^{-}3) + ({}^{+}3)$. The m that satisfies condition I also satisfies condition H.

E You know that the sum of ${}^{-}3$ and ${}^{+}3$ is 0. To obtain a condition that is equivalent to condition I and that involves $m + 0$, you must regroup the numbers in $(m + {}^{-}3) + ({}^{+}3)$. What property of addition of rational numbers can you use to obtain condition J from condition I?

F Does the m that satisfies condition J also satisfy condition K? How do you know? Are condition J and condition K equivalent?

G Compare sentences K and L in D2. For each m, is $m + 0 = m$? Upon what property of addition of rational numbers does your answer depend? What is the sum of ${}^{-}1$ and ${}^{+}3$? Are condition K and condition L equivalent?

U = R.

G $m - 3 = {}^{-}1$.

H $m + {}^{-}3 = {}^{-}1$.

I $(m + {}^{-}3) + ({}^{+}3) = {}^{-}1 + ({}^{+}3)$.

J $m + ({}^{-}3 + {}^{+}3) = {}^{-}1 + {}^{+}3$.

K $m + 0 = {}^{-}1 + {}^{+}3$.

L $m = {}^{+}2$.

D2

U = R.

M $\frac{3}{2} - x = 5$.

N $\frac{3}{2} + {}^{-}x = 5$.

O $({}^{-}\frac{3}{2}) + (\frac{3}{2} + {}^{-}x) = ({}^{-}\frac{3}{2}) + 5$.

P $({}^{-}\frac{3}{2} + \frac{3}{2}) + {}^{-}x = {}^{-}\frac{3}{2} + 5$.

Q $0 + {}^{-}x = {}^{-}\frac{3}{2} + 5$.

R ${}^{-}x = \frac{7}{2}$.

S $x = {}^{-}\frac{7}{2}$.

D3

H What number satisfies condition L?

I Tabulate $\{m | m - 3 = {}^{-}1\}$.

Notice what you have done. From the condition $m - 3 = {}^{-}1$, you obtained a *chain* of conditions. The last condition, condition L, makes it easy to think about m. You used only definitions and properties that you have learned to obtain the chain of conditions; therefore, each condition in the chain is equivalent to each of the other conditions.

Now you will use the additive inverse of a number and certain properties of rational numbers to obtain the solution set of condition M in D3. In other words, you will first find a condition that is equivalent to $\frac{3}{2} - x = 5$, but that enables you to think of x, rather than of $\frac{3}{2} - x$.

J Condition M involves the difference of $\frac{3}{2}$ and x. Is $\frac{3}{2} - x = \frac{3}{2} + {}^{-}x$? How do you know?

K How do you know that condition M and condition N are equivalent?

L Think again about condition N. Remember that you want to find a condition that makes it easy to think of x, rather than of $\frac{3}{2} + {}^{-}x$. Why will it be helpful to obtain a condition that involves $0 + {}^{-}x$, rather than $\frac{3}{2} + {}^{-}x$?

M What number can you add to $\frac{3}{2}$ to obtain 0? Is this number the additive inverse of $\frac{3}{2}$? Now you can write sentence O.

N Compare conditions N and O. From condition N, you know that you want to find an x so that $(\frac{3}{2} + {}^{-}x)$ is the same rational number as 5. Use this fact and the well-defined property of addition of rational numbers to explain why conditions N and O are equivalent.

O Look again at sentence O. Why should you regroup the numbers in $({}^{-}\frac{3}{2}) + (\frac{3}{2} + {}^{-}x)$? What property of addition of rational numbers can you use to obtain condition P from condition O?

P Does the x that satisfies condition P also satisfy condition Q? How do you know? Are condition P and condition Q equivalent?

Q Compare conditions Q and R. For each x, is $0 + {}^{-}x = {}^{-}x$? Upon what property of addition of rational numbers does your answer depend? What is the sum of ${}^{-}\frac{3}{2}$ and 5?

R Compare conditions R and S expressed in D3. From condition R, you know that the additive inverse of x is equal to $\frac{7}{2}$. This also means that x is equal to the additive inverse of $\frac{7}{2}$. What is the additive inverse of $\frac{7}{2}$? What number satisfies condition S?

S Tabulate $\{x | \frac{3}{2} - x = 5\}$.

Next you will use an additive inverse and a reciprocal, along with properties of addition and multiplication of rational numbers, to help you find the solution set of a condition.

A Look at sentence A in D4. What is the universe for y? You are going to obtain the solution set of condition A by first finding an

$U = R$.

A $5y + 2 = {}^{-}13$.

B $(5y + 2) + {}^{-}2 = {}^{-}13 + {}^{-}2$.

C $5y + (2 + {}^{-}2) = {}^{-}13 + {}^{-}2$.

D $5y + 0 = {}^{-}13 + {}^{-}2$.

E $5y = {}^{-}15$.

F $\frac{1}{5}(5y) = \frac{1}{5}({}^{-}15)$.

G $(\frac{1}{5} \cdot 5)y = \frac{1}{5}({}^{-}15)$.

H $1y = \frac{1}{5}({}^{-}15)$.

I $y = {}^{-}3$.

D4

equivalent condition that enables you to think of y, rather than $5y + 2$.

B Look at sentence B. What number has been added to $(5y + 2)$? Is this number the additive inverse of 2?

C Explain why condition A and condition B are equivalent.

D Are condition B and condition C equivalent? Upon what property of addition of rational numbers does your answer depend?

E Explain how to obtain condition D from condition C. Explain how to obtain condition E from condition D.

F Study sentence F in D4. By what number has $5y$ been multiplied? Is this number the reciprocal of 5?

G Use the well-defined property of multiplication of rational numbers to explain how you can obtain condition F from condition E.

H What property of multiplication of rational numbers do you use to obtain condition G from condition F? Explain how you can obtain condition H from condition G. Explain how you can obtain condition I from condition H.

I What number satisfies $y = {}^{-}3$? Are $y = {}^{-}3$ and $5y + 2 = {}^{-}13$ equivalent conditions?

J Tabulate $\{y \mid 5y + 2 = {}^{-}13\}$.

Notice what you have done. To find the solution set of $5y + 2 = {}^{-}13$, you used definitions and properties to obtain a chain of equivalent conditions. You continued your work until you obtained the condition $y = {}^{-}3$, which has a solution set that is easy to find. Since each condition in the chain of conditions is equivalent to each of the other conditions, you know that $\{{}^{-}3\}$ is the solution set of $5y + 2 = {}^{-}13$, and that it is also the solution set of each of the other conditions in the chain.

So far in this lesson, the last condition in each chain of equivalent conditions has always been a condition like $y = {}^{-}3$. In other words, you have been able to find the solution set of the last condition in the chain with very little effort. The work was continued this far to show how the definitions and properties that you have learned are used to obtain equivalent conditions.

When you are finding solution sets, however, you may stop the process of deriving equivalent conditions as soon as you obtain a condition whose solution set you know. For example, look again at sentence E in D4. If you know immediately that ${}^{-}3$ is the solution of $5y = {}^{-}15$, you need not obtain any more equivalent conditions. Since $5y + 2 = {}^{-}13$ and $5y = {}^{-}15$ are equivalent conditions, $\{{}^{-}3\}$ is the solution set of $5y + 2 = {}^{-}13$ as well as of $5y = {}^{-}15$.

K Tabulate $\{a \mid \frac{1}{3}a + 7 = {}^{-}20\}$. The universe for a is R. Continue the process of obtaining equivalent conditions only as far as is necessary to obtain a condition with a solution set that is obvious to you.

In this lesson you learned that equivalent conditions are conditions that have the same solution set. You also learned how to use definitions and properties to obtain a chain of equivalent

conditions and how to use the last condition in the chain to help you find the solution set of the given condition.

On your own

For each of exercises 1 through 10, tell whether the sentence expresses a true statement or a false statement. U = R.

1 $\{x|x+14=\bar{}19\}=\{\bar{}33\}$.
2 $\{w|.25w=\bar{}3.75\}=\{15\}$.
3 $\{a|a-\frac{2}{5}=\frac{9}{5}\}=\{\frac{7}{5}\}$.
4 $\{x|4.5x=3.6\}=\{.8\}$.
5 $\{y|\bar{}5+y=\frac{\bar{}10}{3}\}=\{\frac{\bar{}5}{3}\}$.
6 $\{x|x+\frac{\bar{}5}{6}=\frac{\bar{}5}{3}\}=\{\frac{\bar{}15}{6}\}$.
7 $\{x|6x+15=\bar{}21\}=\{\bar{}6\}$.
8 $\{c|.49-c=3.82\}=\{\bar{}3.33\}$.
9 $\{x|.4x+5.2=\bar{}6.8\}=\{\bar{}30\}$.
10 $\{m|2m-4=14\}=\{11\}$.

For each of exercises 11 through 16, first express a chain of conditions each of which is equivalent to the given condition. The last condition in the chain should require that x be equal to a certain rational number. Then tell what definition or property you used to obtain each condition in the chain. Finally, tabulate the solution set of the condition expressed in the exercise. U = R.

11 $x+4.6=\bar{}3.9$.
12 $\frac{1}{2}x=\frac{3}{5}$.
13 $x-17=32$.
14 $\frac{\bar{}3}{4}+x=\frac{\bar{}5}{2}$.
15 $6-x=\frac{2}{3}$.
16 $4x+3=\bar{}17$.

Tabulate the solution set of each condition expressed in exercises 17 through 36. U = R.

17 $m+23=\bar{}19$.
18 $b-3.25=\bar{}1.68$.
19 $\bar{}7x=\bar{}56$.
20 $\bar{}14+x=\bar{}9\frac{2}{3}$.
21 $3x=\bar{}19$.
22 $x+\bar{}2\frac{5}{6}=\bar{}1\frac{2}{3}$.
23 $\bar{}6.4+c=8.9$.
24 $\frac{1}{2}-x=\frac{7}{4}$.
25 $x-\frac{\bar{}7}{10}=\frac{\bar{}1}{4}$.
26 $2a+15=37$.
27 $\bar{}5\frac{1}{3}-y=4\frac{3}{4}$.
28 $17-x=\bar{}9$.
29 $.35x=6.65$.
30 $.5w+\bar{}16.75=21.4$.
31 $20+3x=32$.
32 $\frac{1}{2}x+17=\bar{}7$.
33 $3x-.7=2$.
34 $\frac{2}{3}+\frac{5}{4}y=15\frac{2}{3}$.
35 $.54-x=\bar{}.38$.
36 $\bar{}5m-\bar{}2=12$.

Inequalities and compound conditions

You have learned how to use definitions and properties to help you find the solution sets of simple conditions for equality when the universe is a set of rational numbers. In this lesson you will find the solution sets of conditions for inequality and of compound conditions.

A Look at sentence A in D1. You can use the condition for equality $2+x=\bar{}1$ to help you find the solution set of condition A. What is the universe for x? Tabulate $\{x|2+x=\bar{}1\}$.

B Since $\bar{}3$ is the solution of $2+x=\bar{}1$, you can use $\bar{}3$ to help you find the solution set of condition A. If you replace x by $\bar{}3$ in $2+x>\bar{}1$, do you obtain a true statement?

C Is $\bar{}4<\bar{}3$? Is $\bar{}4$ a solution of $2+x>\bar{}1$? Is $\frac{\bar{}9}{2}<\bar{}3$? Is $\frac{\bar{}9}{2}$ a solution of $2+x>\bar{}1$?

D Is any rational number that is less than $\bar{}3$ a solution of $2+x>\bar{}1$?

E Is $\frac{\bar{}5}{2}$ a solution of $2+x>\bar{}1$? Is 0 a solution? Is $+7$ a solution?

F Is every rational number that is greater than $\bar{}3$ a solution of $2+x>\bar{}1$?

G Is $\{x|x>\bar{}3\}$ the solution set of $2+x>\bar{}1$? Is $\{x|x>\bar{}3\}$ an infinite set?

In *Book 1* you learned that there is at least one member of R_a between any two members of R_a. Now you will decide if the set of rational numbers also has this density property.

H You know $\frac{\bar{}5}{2}$ is a member of $\{x|x>\bar{}3\}$. Is $\bar{}2$ a member? Is $\frac{\bar{}5}{2}<\bar{}2$?

I Is $\bar{}3<\frac{\bar{}5}{2}$? You know that $\bar{}3<\frac{\bar{}5}{2}$ and $\frac{\bar{}5}{2}<\bar{}2$. We say that $\frac{\bar{}5}{2}$ is between $\bar{}3$ and $\bar{}2$.

U = R.

A $2 + x > {}^{-}1$.

B $3x - 4 < 2$.

C $4 - x > 5$.

D1

U = R.

D

$\{x | x > {}^{-}3\}$

E

$\{x | x < 2\}$

F

$\{x | x < {}^{-}1\}$

D2

J How do you know that $^{-}\frac{11}{4}$ is a member of $\{x | x > {}^{-}3\}$? Is $^{-}\frac{11}{4}$ between $^{-}3$ and $^{-}\frac{5}{2}$? Explain your answer.

If you continue to study examples like those in exercises I and J, you will see that there is at least one rational number that is between any two given rational numbers. We will accept without proof the *density property of the set of rational numbers*. The property is expressed below.

The universe for x, y, *and* r *is R. For each* x *and* y, *if* $x < y$, *then there is a rational number* r *such that* $x < r \wedge r < y$.

K Use the density property of the set of rational numbers to explain why there is no least member of $\{x | x > {}^{-}3\}$.

L Look again at D1. What condition for equality can you use to help you find the solution set of condition B?

M Tabulate $\{x | 3x - 4 = 2\}$.

N Is 2 a solution of $3x - 4 < 2$? Is any rational number that is greater than 2 a solution?

O Is every rational number that is less than 2 a solution of $3x - 4 < 2$?

P Give a standard description of the solution set of $3x - 4 < 2$.

Q Is $\{x | x < 2\}$ an infinite set? Why can you not tabulate $\{x | x < 2\}$?

R Look at sentence C in D1. What condition for equality can you use to help you find the solution set of $4 - x > 5$?

S Tabulate $\{x | 4 - x = 5\}$.

T Is $^{-}1$ a solution of $4 - x > 5$? Is $^{-}\frac{1}{2}$ a solution? Is $7\frac{1}{2}$ a solution?

U Is any rational number that is greater than $^{-}1$ a solution of $4 - x > 5$?

V Is $^{-}\frac{3}{2}$ a solution of $4 - x > 5$? Is $^{-}4$ a solution? Is $^{-}6\frac{1}{3}$ a solution?

W Is every rational number that is less than $^{-}1$ a solution of $4 - x > 5$?

X Give a standard description of the solution set of $4 - x > 5$.

Y Is $\{x | x < {}^{-}1\}$ an infinite set?

Now you will study graphs of solution sets of conditions that involve rational numbers.

A D2 shows the graphs of the solution sets of the conditions expressed in D1. What is the universe for x?

B Look at graph D. How are the dots in the graph of $\{x | x > {}^{-}3\}$ indicated?

C Does the graph of $\{x | x > {}^{-}3\}$ contain the dot for point $^{-}3$? The dot for point $^{+}25$?

D Look at graph E. Is the dot for point $^{+}2$ contained in the graph of $\{x | x < 2\}$?

E Look at graph F. Is the dot for point $^{-}1$ contained in the graph of $\{x | x < {}^{-}1\}$?

In the work that follows, you will learn how to make graphs of the solution sets of compound conditions.

F What two simple conditions form the compound condition expressed in D3? What connective is used? What is the universe for y?

G What condition for equality can you use to help you find the solution set of $^{-}\frac{3}{4} - y < 4$? Tabulate $\{y | {}^{-}\frac{3}{4} - y = 4\}$.

H Is $\{y | y > {}^{-}\frac{19}{4}\}$ the solution set of $^{-}\frac{3}{4} - y < 4$?

I Now give a standard description of the solution set of $y + \frac{1}{4} < {}^{-}2$.

J How do you know that $\{y | {}^{-}\frac{19}{4} < y < {}^{-}\frac{9}{4}\}$ is the solution set of the compound condition?

You remember that, in lesson 113, when you were using the rational numbers of arithmetic, a set like $\{y | {}^{-}\frac{19}{4} < y < {}^{-}\frac{9}{4}\}$ was called an open interval in R_a. When the universe is R, a set like this is called an open interval in R.

K Look at the graph in D3. How are the dots in the graph of $\{y | {}^{-}\frac{19}{4} < y < {}^{-}\frac{9}{4}\}$ indicated?

L Look at D4. What two simple conditions form the compound condition expressed in D4? What connective is used in the condition?

M How do you know that the solution set of $4.5 + x > 3.6 \vee 7.2 - x = 8.1$ must contain every member of the solution set of each of the simple conditions?

N Give a standard description of the solution set of $4.5 + x > 3.6$. Give a standard description of the solution set of $7.2 - x = 8.1$.

O Explain why $\{x | x \geqq {}^{-}.9\}$ contains the same members of R as $\{x | x > {}^{-}.9 \vee x = {}^{-}.9\}$.

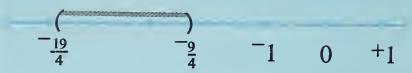
P Is $^{-}.9$ a member of $\{x | x \geqq {}^{-}.9\}$? How is this indicated in the graph in D4?

Q Do the dots for point $^{-}.9$ and all points to the right of point $^{-}.9$ form the graph of $\{x | x \geqq {}^{-}.9\}$? How are these dots indicated?

R Now look at D5. What condition is expressed? What is the universe for the variable?

$U = R.$

$^{-}\frac{3}{4} - y < 4 \wedge y + \frac{1}{4} < {}^{-}2.$

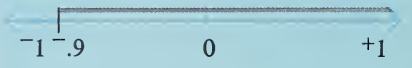


$\{y | {}^{-}\frac{19}{4} < y < {}^{-}\frac{9}{4}\}$

D3

$U = R.$

$4.5 + x > 3.6 \vee 7.2 - x = 8.1.$



$\{x | x \geqq {}^{-}.9\}$

D4

$U = \overline{R}_n.$

$\sim (4m - 7 > 5).$

$\{m | \sim (m > 3)\}$

D5

S The connective "not" in $\sim (4m - 7 > 5)$ means that the solution set of the condition is the set of all members of $\overline{R}_n$ that do *not* satisfy $4m - 7 > 5$. Give a standard description of the solution set of $4m - 7 > 5$.

T Think about the set of non-negative rational numbers each of which is less than or equal to 3. Is each member of this subset of $\overline{R}_n$ a number that does not satisfy $4m - 7 > 5$? How do you know that $\{m | \sim (m > 3)\}$ is the solution set of $\sim (4m - 7 > 5)$?

U Look at the graph in D5. How is the graph of $\{m|\sim(m>3)\}$ indicated? Why are no dots shown for points to the left of point 0?

V Which dot forms the graph of $\{m|m=3\}$? Is $\{m|m=3\}$ a subset of $\{m|\sim(m>3)\}$?

W Which dots form the graph of $\{m|m<3\}$? Is $\{m|m<3\}$ a subset of $\{m|\sim(m>3)\}$?

X Is the graph of $\{m|m\leqq 3\}$ the same as the graph of $\{m|\sim(m>3)\}$? Is $\{m|m\leqq 3\} = \{m|\sim(m>3)\}$? Is $\{m|m\leqq 3\} = \overline{\{m|m>3\}}$?

Now you know how to find solution sets of simple conditions for inequality and of compound conditions in one variable when the universe is R or $\overline{R}_n$. You also know how to make a graph of a set of rational numbers.

On your own

For each of exercises 1 through 7, tell whether the sentence expresses a true statement or a false statement. U = R.

1 $\{z|z+5<{}^-2\} = \{z|z<{}^-7\}$.

2 $\{x|{}^-3+x>7\} = \{x|x>4\}$.

3 $\{x|x-7.8<{}^-9.3\} = \{x|x<{}^-1.5\}$.

4 $\{x|\frac{1}{3}-x>\frac{{}^-5}{4}\} = \{x|x<{}^-1\frac{7}{12}\}$.

5 $\{c|6c-2<10\} = \{c|c<2\}$.

6 $\{x|\frac{5}{6}+x>\frac{7}{3}\wedge x-\frac{3}{4}<{}^-3\} = \{\ \}$.

7 $\{w|\sim(6w+11>35)\} = \{w|\sim(w<4)\}$.

Give a standard description of the solution set of each condition expressed below. U = R.

8 ${}^-12+a<17$.

9 $b-2.6<4.1$.

10 $4-m>{}^-3$.

11 $3x+{}^-5>19$.

12 $\sim(\frac{3}{8}+\frac{1}{3}x<1\frac{2}{3})$.

13 $\sim(4x-11<5)$.

14 $\frac{5}{4}+w>\frac{{}^-3}{2}\wedge\frac{9}{10}-w>\frac{{}^-7}{2}$.

15 $y-4\frac{1}{4}={}^-3\frac{1}{2}\vee y+1\frac{3}{8}>2\frac{1}{8}$.

Make a graph of each solution set described below. For exercises 16 and 17, U = $\overline{R}_n$. For exercises 18 through 22, U = R.

16 $\{x|x<2.5\}$

17 $\{w|w-\frac{1}{4}<1\frac{1}{2}\}$

18 $\{b|\frac{3}{8}-b>2\frac{5}{8}\}$

19 $\{x|7x+8<15\}$

20 $\{x|\sim(x+3.5>4)\}$

21 $\{y|{}^-1<y<\frac{{}^-1}{4}\}$

22 $\{m|2\frac{1}{3}-m<1\frac{5}{6}\vee 2m-1\frac{1}{2}>{}^-4\frac{1}{4}\}$

Conditions for problems involving rational numbers

The problems in this lesson are much like problems that you solved earlier in this book, but they involve rational numbers.

Read the problem in D1. It is the same as a problem that you studied in lesson 126. There you found that neither a natural number nor a rational number of arithmetic can be used to get an answer to this problem. In the exercises that follow, you will develop the condition for the problem. Then you will decide if there is a rational number that satisfies the condition. The universe for each variable is R.

A You know that at noon the temperature in Lake City was 10 degrees above zero. You also know that by 7 P.M. the temperature had dropped 15 degrees. Why is $10-15$ used in the condition?

$10-15$

B Use a as a variable for the temperature in degrees at 7 P.M. Does the sentence at the right express the condition for the problem?

$10-15=a$.

> At noon on Tuesday, the temperature in Lake City was 10 degrees above zero. By 7 P.M., the temperature had dropped 15 degrees. What was the temperature at 7 P.M.?
>
> D1

C Now you can find the solution set of $10 - 15 = a$. How do you obtain $10 + {}^{-}15 = a$ from $10 - 15 = a$? Are $10 - 15 = a$ and $10 + {}^{-}15 = a$ equivalent conditions?

D What number satisfies $10 + {}^{-}15 = a$? Does the number that satisfies $10 + {}^{-}15 = a$ also satisfy $10 - 15 = a$?

E Tabulate $\{a \mid 10 - 15 = a\}$.

You can use ${}^{-}5$ to get the answer to the problem in D1. The temperature in Lake City at 7 P.M. was ${}^{-}5$ degrees, or 5 degrees below zero.

Read the problem in D2. The work that follows will help you develop a condition for equivalence for the problem. The universe for each variable is R_p.

F You know that Mr. Carson traveled at an average rate of 57.6 mi. per hour. The rate pair named at the right represents this rate. $57.6/1$

To what does the first component of 57.6/1 refer? To what does the second component refer?

G What are you asked to find in the problem? How do you know that 153.6 is the first component of the rate pair whose second component you are to find? For what is y a variable? $153.6/y$

H Does the sentence at the right express the condition for the problem? $57.6/1 \sim 153.6/y.$

I Now you can find the solution set of the condition for the problem. How can you obtain $57.6y = 153.6$ from $57.6/1 \sim 153.6/y$?

J What computation can you use to find the number that satisfies $57.6y = 153.6$?

K How do you know that the quotient of 153.6 and 57.6 is $2.\overline{6}$?

L Tabulate $\{y \mid 57.6y = 153.6\}$. Then tabulate $\{y \mid 57.6/1 \sim 153.6/y\}$.

At noon on Tuesday, the temperature in Lake City was 10 degrees above zero. By 7 P.M., the temperature had dropped 15 degrees. What was the temperature at 7 P.M.?

D1

After Mr. Carson had driven 153.6 mi. on an expressway, he discovered that he had been traveling at an average rate of 57.6 mi. per hour. How many hours did it take him to travel the 153.6 mi.? (Tenths)

D2

Look again at D2. The word "tenths," within parentheses, tells you to use a decimal to express the answer to the nearer tenth.

M Give the answer to the problem.

Now read the problem in D3. The universe for each variable is R_p.

N The problem in D3 compares the distance from the earth to the sun with the distance from the earth to the moon. Does the rate pair named below represent this comparison?

$$(9.3 \times 10^7)/(2.4 \times 10^5)$$

O Notice that each component of the rate pair is expressed in scientific notation. To what does the first component refer? To what does the second component refer?

P In this problem, to find how many times as great one distance is as the other means to find how many miles the earth is from the sun for each mile that the earth is from the moon. Use m as a variable for the number of miles the earth is from the sun. Why do you use $m/1$, and not $1/m$? $m/1$

Q The sentence in blue below expresses the condition for the problem. What does the condition tell you about $m/1$?

$$(9.3 \times 10^7)/(2.4 \times 10^5) \sim m/1.$$

R How do you know that the two conditions expressed below are equivalent?

$(9.3 \times 10^7) / (2.4 \times 10^5) \sim m/1.$

$(9.3 \times 10^7) = (2.4 \times 10^5)m.$

S Can you find the number that satisfies $(9.3 \times 10^7) = (2.4 \times 10^5)m$ by finding the quotient of (9.3×10^7) and (2.4×10^5)?

T Find the number that satisfies $(9.3 \times 10^7) = (2.4 \times 10^5)m$. Use scientific notation to help you.

U Is $\{3.875 \times 10^2\}$ the solution set of the condition $(9.3 \times 10^7) = (2.4 \times 10^5)m$? Tabulate $\{m|(9.3 \times 10^7) / (2.4 \times 10^5) \sim m/1\}$.

V What is a standard name of 3.875×10^2?

Now you can give the answer to the problem. The distance from the earth to the sun is about 387.5 times as great as the distance from the earth to the moon.

So far in this lesson you have used simple conditions to solve problems. Now you will use compound conditions for problems.

Read the problem in D4. The universe for each variable is R.

A You know that Swamp City is 5 ft. below sea level. Why can you use the number expressed at the right to represent the elevation of Swamp City in feet? "Elevation" refers to vertical distance above or below sea level. $^{-}5$

B Do you know the elevation of Silver City? Use n as a variable for the elevation of Silver City in feet. n

C Does the sentence at the right express one simple condition for the problem? $n > {}^{-}5.$

The earth is about 9.3×10^7 mi. from the sun and about 2.4×10^5 mi. from the moon. The distance from the earth to the sun is about how many times as great as the distance from the earth to the moon?

D3

Swamp City is 5 ft. below sea level. West City is 3440 ft. above sea level. The elevation of Silver City is greater than the elevation of Swamp City. The difference of the elevation of West City and the elevation of Silver City is greater than 500 ft. What is the elevation of Silver City in feet?

D4

D What is the elevation of West City in feet? Explain why the sentence below at the right expresses another simple condition for the problem. $3440 - n > 500.$

E Does the sentence in blue below express the compound condition for the problem?

$$n > {}^{-}5 \wedge 3440 - n > 500.$$

Now you will find the solution set of the compound condition.

F What connective is used in the condition? What do you know about each member of the solution set of such a condition?

G How do you know that $\{n|n < 2940\}$ is the solution set of $3440 - n > 500$? Is $\{n|n < 2940\}$ a finite set?

H Does each member of $\{n|n < 2940\}$ satisfy $n > {}^{-}5$? Explain your answer.

I Is $\{n|{}^{-}5 < n < 2940\}$ the solution set of $n > {}^{-}5 \wedge 3440 - n > 500$?

You can use $\{n \mid {}^{-}5 < n < 2940\}$ to get the answer to the problem.

The elevation of Silver City is between 5 ft. below sea level and 2940 ft. above sea level.

Now read the problem in D5. The universe for each variable is R_p.

You know that Miss Davis can buy oil that costs either \$.45 per quart or \$.60 per quart. Therefore, you can use a compound condition that includes "or."

J One grade of oil costs \$.45 per quart. Does the rate pair expressed at the right represent this rate? .45/1

K You want to find a rate pair that is equivalent to .45/1. Use n as a variable for the amount that 5 qt. of oil can cost. Why is n the first component of the rate pair you are to find? Name the second component. $n/5$

L Does the sentence at the right express a simple condition for the problem? $.45/1 \sim n/5.$

M You also know that the other grade of oil costs \$.60 per quart. Does the rate pair expressed at the right represent this rate? .60/1

N Explain why the sentence at the right expresses another simple condition for the problem. Why do you use n as a variable? $.60/1 \sim n/5.$

O Does the sentence in blue below express the compound condition for the problem?

$$.45/1 \sim n/5 \vee .60/1 \sim n/5.$$

Now you will find the solution set of the compound condition.

P Is the solution set of the compound condition the intersection, or is it the union, of the solution sets of the simple conditions? Explain your answer.

Miss Davis needs 5 qt. of oil for her car. She has her choice of two grades of oil. One grade costs \$.45 per quart, and the other grade costs \$.60 per quart. How much can 5 qt. of one of these grades of oil cost?

D5

What numbers can be added to ${}^{-}3.2$ so that each sum is not less than 6.4?

D6

Q What number satisfies $.45/1 \sim n/5$? What number satisfies $.60/1 \sim n/5$?

R Tabulate $\{n \mid .45/1 \sim n/5 \vee .60/1 \sim n/5\}$.

S How much can the 5 qt. of oil cost?

The preceding problems in this lesson were concerned with such things as temperature, distance, and money. Now you will study a problem that concerns numbers only.

Read the problem in D6. The universe for each variable is R.

A Use x as a variable for the number that you are to find. From the problem, you know that it is not the case that the sum of ${}^{-}3.2$ and each x is less than 6.4. Why is $\sim({}^{-}3.2 + x < 6.4)$ the condition for the problem?

B You can use ${}^{-}3.2 + x < 6.4$ to help you find the solutions of $\sim({}^{-}3.2 + x < 6.4)$. Does 9.6 satisfy ${}^{-}3.2 + x < 6.4$? Does any rational number greater than 9.6 satisfy ${}^{-}3.2 + x < 6.4$?

C Does $\{x \mid x \geqq 9.6\}$ contain each member of R that does *not* satisfy ${}^{-}3.2 + x < 6.4$?

D Are $\{x \mid x \geqq 9.6\}$ and $\{x \mid x < 9.6\}$ the same set? How do you know that $\{x \mid x \geqq 9.6\}$ is the solution set of $\sim({}^{-}3.2 + x < 6.4)$?

E Is each member of $\{x \mid x \geqq 9.6\}$ an answer to the problem in D6?

In this lesson you used simple and compound conditions in one variable to solve problems involving rational numbers.

On your own

For each problem, first write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem.

For problems 1 through 8, the universe for each variable is R.

1 At 6 A.M. the temperature in Truro was 3 degrees below zero. At noon the temperature was 10 degrees higher than this. What was the temperature at noon?

2 On the first down, a football team gained 8 yd. On the second down, the team lost 5 yd. How many yards did the team gain on the two downs?

THINK When the team lost 5 yd., did it gain $^{-}5$ yd.?

3 One year the lowest temperature recorded in Canada was 81.4° below zero, and the lowest temperature recorded in Quebec was 22.5° higher than this. What was the lowest temperature recorded in Quebec that year?

4 What numbers can be added to $^{-}14\frac{3}{5}$ so that each sum is greater than $15\frac{7}{10}$?

5 What number can be added to the product of 3 and 6.4 so that the sum is 19.7?

6 What numbers can be multiplied by $3\frac{1}{3}$ so that it is not the case that each product is less than $^{-}16\frac{1}{2}$?

7 What numbers less than 20.5 can be subtracted from 2 so that each difference is less than $^{-}10.7$?

8 What numbers can be subtracted from $9\frac{1}{2}$ so that it is not the case that each difference is greater than 4?

For exercises 9 through 12, the universe for each variable is R_p.

9 The crankshaft of a certain engine makes 2.6×10^3 revolutions per minute. At this rate, how many revolutions does the crankshaft make in 60 minutes?

10 A rocket reached a height of 2.52×10^5 ft. Its expected height was 2.8×10^5 ft. The height that the rocket reached was what fraction of its expected height?

11 During a recent year, Canada produced 27% of North America's output of specially processed steel. The total North American output was 5.50×10^6 tons. How many tons of steel did Canada produce that year?

12 Mr. Frank wants to borrow \$600. Mr. Stacy will lend him the money at 6% interest per year. Mr. Rogers will lend him the money at $5\frac{1}{2}$% interest. At the end of one year, how much interest will Mr. Frank have to pay if he borrows the \$600 from one of the two men?

For exercises 13 through 16, the universe for each variable is $\overline{R}_n$.

13 Mrs. Connor had $17\frac{2}{3}$ sq. yd. of carpeting. She used some, but not all, of this carpeting. She then had less than 4 sq. yd. left. How many square yards of carpeting did she use?

14 Mrs. Sands had less than \$170 in her checking account. After she deposited \$60.35, there was more than \$210 in the account. How much did Mrs. Sands have in her checking account before she deposited the \$60.35?

15 Mr. Ames had at most $4\frac{1}{2}$ gal. of paint. He used $3\frac{3}{4}$ gal. of this paint to paint the kitchen. Then he had more than $\frac{1}{8}$ gal. left. How many gallons of paint did he have before he painted the kitchen?

16 Mr. Allen plans to spend 22 days in England and France. He has his choice of spending either $14\frac{1}{2}$ days or 10 days in England. How many days can Mr. Allen spend in France?

CHECKING UP

The small numerals within parentheses tell what pages to turn to for help if you need it.

Test 132

1 Name the basic pair for $^{+}\frac{23}{3}$. (156)

2 Name the absolute difference of the components of $(2\frac{1}{2}, 5\frac{3}{4})$. (157)

3 Write a standard description of $^{-}\frac{3}{8}$. The universe for (x, y) is $R_a \times R_a$. (161)

4 $^{+}\frac{1}{2} \cdot {}^{-}\frac{3}{4} = {}^{+}\frac{1}{2} \cdot {}^{-}\frac{6}{8}$ is an example of a property of multiplication of rational numbers. Name the property. (185)

5 Write a decimal that expresses $^{-}\frac{5}{3}$. (218)

6 Give a standard name of $(^{-}\frac{5}{4})^4$. (224)

7 Use scientific notation to express .0009. (226)

8 What is the sum of $^{-}8$ and $^{+}\frac{9}{4}$? (177)

9 What is the difference of $^{+}7.5$ and $^{-}1.32$? (214)

10 What is the sum of $^{-}5.83$ and $^{-}1.75$? (213)

11 What is the product of $^{+}6.4$ and $^{+}3.9$? (214)

12 What is the quotient of $^{+}\frac{15}{16}$ and $^{-}\frac{3}{4}$? (211)

13 What is the difference of $^{+}4\frac{1}{6}$ and $^{+}4\frac{3}{4}$? (212)

14 What is the quotient of $^{-}1.98$ and $^{-}2.3$? (Hundredths) (215)

15 What is the product of $^{-}2\frac{3}{5}$ and $1\frac{1}{7}$? (212)

16 Make a graph of $\{x \mid 7\frac{1}{4} - x \geqq 2\frac{1}{2}\}$. U = R. (243)

Test 133

For each of exercises 17 through 27, decide what words or symbols best complete the exercise.

17 Zero is the ~~~ component of each basic pair that indicates a negative rational number. (161)

18 In a rational-number line, point $^{+}\frac{2}{7}$ is to the ~~~ of point $^{-}3$. (166)

19 The rational number of arithmetic that is the mate of $^{+}\frac{3}{10}$ is ~~~. (187)

20 The solution of $\frac{168}{13}x = \frac{225}{7} \cdot \frac{168}{13}$ is ~~~. U = R_p. (192)

21 The solution of $(\frac{7}{9})y + (\frac{15}{11})y = (\frac{7}{9} + \frac{15}{11})\frac{7}{11}$ is ~~~. U = R. (197)

22 The additive inverse of $\frac{7}{2}$ is ~~~. (205)

23 The universe for x and y is R. For each x and y, $x - y = x + {}^{-}y$. This is the ~~~ property of rational numbers. (207)

24 The multiplicative inverse of $^{-}\frac{11}{12}$ is ~~~. (208)

25 The repeating decimal for $^{+}\frac{13}{6}$ is ~~~. (218)

26 A standard name of 7.318×10^{-3} is the numeral ~~~. (233)

27 $(\frac{2}{9} \cdot \frac{9}{2})z = \frac{2}{9}(^{-}8)$ is obtained from $\frac{2}{9}(\frac{9}{2}z) = \frac{2}{9}(^{-}8)$ by using the ~~~ property of multiplication of rational numbers. U = R. (195)

Test 134

For exercises 28 through 39, write "T" for each sentence that expresses a true statement. Write "F" for each sentence that expresses a false statement.

28 The number 1 is the identity element for addition of rational numbers. (202)

29 $\frac{1}{2} + (3 \times \frac{5}{6}) = (\frac{1}{2} + 3)(\frac{1}{2} + \frac{5}{6})$. (197)

30 A negative rational number contains one, and only one, basic pair. (161)

31 $0 \in R_p$. (163)

32 Any positive rational number is greater than any negative rational number. (168)

33 The product of any two negative rational numbers is a positive rational number. (185)

34 1.0076×10^0 is scientific notation for 1.0076. (233)

35 R_n, together with the operations of addition and multiplication, forms a number system. (198)

36 R is closed under subtraction. (207)

37 Z_r is closed under division. (211)

38 43^{12} is equal to $43^{-3} \cdot 43^{-4}$. (230)

39 The symbol $^{-}5 - \frac{1}{2}$ expresses the same number as the symbol $^{+}5 + {}^{-}\frac{1}{2}$. (207)

Test 135

From the list given below each exercise, choose the expression that correctly completes the exercise.

40 $R =$ ——. (163)

a $R_p \cup R_n$ **c** $R_p \cup R_n \cup \{0\}$
b $\overline{R}_n \cap R_p$ **d** $R_p \cup \{0\} \cup \overline{R}_n$

41 An element of I_n is ——. (168)

a $^-9$ **b** $^+\frac{14}{3}$ **c** $+13$ **d** $^-\frac{3}{2}$

42 $\overline{R}_n$ is the union of ——. (187)

a R_n and R_p **c** R_n and $\{0\}$
b R_p and $\{0\}$ **d** I_r and R_n

43 $\frac{1}{4}k = \frac{3}{2}$ is obtained from $0 + \frac{1}{4}k = \frac{3}{2}$ by using the —— property of addition of rational numbers. U = R. (202)

a sum **c** additive-inverse
b well-defined **d** identity-element

44 The solution set of $x \cdot 0 = 0$ is ——. U = R. (203)

a R **b** $\{0\}$ **c** R_p **d** $\{\ \}$

45 $^-\frac{7}{8}$ and —— are reciprocals of each other. (208)

a $\frac{^+1}{\frac{8}{7}}$ **b** $\frac{^+1}{^-\frac{7}{8}}$ **c** $\frac{1}{^-\frac{8}{7}}$ **d** $^-\frac{8}{7}$

46 A standard name of $(\frac{4}{5})^3$ is ——. (224)

a $\frac{64}{125}$ **b** $\frac{64}{5}$ **c** $\frac{12}{5}$ **d** $\frac{12}{15}$

47 $\{x \mid {}^-6\frac{2}{3} > x + 2 \vee x - 4 < {}^-5\} =$ ——. U = R. (244)

a $\{x \mid x < {}^-9\}$ **c** $\{x \mid x < {}^-8\frac{2}{3}\}$
b $\{x \mid x < {}^-1\}$ **d** $\{x \mid x < {}^-8\frac{2}{3} \wedge x < {}^-1\}$

48 Expressed in scientific notation, 9380 is ——. (233)

a 938.0×10^1 **c** 9.380×10^3
b 93.80×10^2 **d** $.9380 \times 10^4$

49 $13^9 \div 13^{-3}$ is the same number as ——. (231)

a 13^6 **b** 13^{-27} **c** 13^{-3} **d** 13^{12}

50 $^-3 + 7m = \frac{4}{3}$ and —— are equivalent conditions. U = R. (238)

a $\frac{1}{7}(7m) = \frac{1}{7}(\frac{13}{3})$ **c** $7m = \frac{1}{7}(\frac{4}{3} + {}^-3)$
b $\frac{1}{7}(7m) = \frac{4}{3} + 3$ **d** $\frac{1}{7}m = \frac{3}{4} + \frac{1}{3}$

CHECKING UP

The small numerals within parentheses tell what pages to turn to for help if you need it.

Test 136

The names of nine properties of R are listed below.

Associative property of addition
Associative property of multiplication
Closure property of addition
Commutative property of multiplication
Difference property
Identity-element property of multiplication
Quotient property
Reciprocal property
Zero property of multiplication

In each of exercises 1 through 7, a property is expressed. Choose the name of the property from the list above. The universe for each variable is R.

1 $a + b \in R$. (177)
2 $p(qr) = (pq)r$. (195)
3 $w \div z = w(1/z)$. (211)
4 $m \cdot 1 = m$. (203)
5 $yz = zy$. (192)
6 $(c + d) + e = c + (d + e)$. (195)
7 $t - u = t + {}^-u$. (207)

Test 137

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

8 For any two sets A and B, B is a subset of $A \cap B$. (63)
9 A closed interval in R_a is an infinite set. (85)
10 The intersection of $\odot W$ and a secant of $\odot W$ contains exactly two members. (115)
11 $\{x \mid \sim (x < 5)\} = \{6, 7, 8, \ldots\}$. U = N. (80)
12 If U = $\{0, 5, 9\}$ and A = $\{0, 5\}$, then $\overline{A} = \{9\}$. (78)
13 If U = R_a, then $\frac{7}{4} \in \{p \mid p > \frac{13}{16}\}$. (87)

14 The intersection of adjacent arcs is the empty set. (120)

15 If $U = R_a$, then $7.5 \in \{r \mid r \approx_{.5} 8\}$. (90)

16 The angles in a linear pair are adjacent angles. (29)

17 R_p is closed under subtraction. (207)

18 $\overline{\{x \mid x \geqq 8\}} = \{x \mid x \leqq 8\}$. $U = R_a$. (78)

Test 138

A list of expressions is given below each exercise in this test. From the list choose the expression that correctly completes the exercise.

19 The absolute difference of $4\frac{1}{2}$ and 7 is ——. (88)

a $^-2\frac{1}{2}$ b $^+2\frac{1}{2}$ c $^-11\frac{1}{2}$ d $^+11\frac{1}{2}$

20 If ℓ_1 is tangent to $\odot B$, then $\ell_1 \cap \odot B$ contains exactly ——. (115)

a 0 points c 2 points
b 3 points d 1 point

21 Points F and G are endpoints of an arc of $\odot M$. If the measure in degrees of $\angle FMG$ is 90, the measure in degrees of major $\overset{\frown}{FG}$ is ——. (120)

a 360 b 270 c 180 d 90

22 The symbol that expresses the idea of "parallel to" is ——. (122)

a $\perp$ b $=$ c $\in$ d $\parallel$

23 Every rhombus is *not* a ——. (124)

a quadrilateral c square
b parallelogram d polygon

24 $^-4 <$ —— is a true statement. (167)

a $^-15$ b $^-2$ c $^-\frac{32}{8}$ d $^-\frac{100}{7}$

25 The sum of any two negative rational numbers is a member of ——. (178)

a R_n b $\overline{R}_n$ c Z d R_p

26 If $\angle GHI$ and $\angle JHK$ are opposite angles, and $\angle GHI^\circ = 120$, then $\angle JHK^\circ =$ ——.

a 120 b 60 c 180 d 30 (27)

27 The product of any negative rational number and 0 is a member of ——. (203)

a R_n b $\{0\}$ c Z_r d R_p

28 $\{q \mid q + 2 > 7 \wedge 12 - q > 3\}$ is the same set as ——. $U = R_a$. (67)

a $\{q \mid q > 5\}$ c $\{q \mid q > 5 \wedge q > 9\}$
b $\{q \mid q < 9\}$ d $\{q \mid 5 < q < 9\}$

Test 139

For each problem in this test, first write a sentence that expresses a condition. Tabulate the solution set of the condition, if possible. Otherwise, give a standard description of the solution set. Then give the answer to the problem.

For problems 29 through 32, the universe for each variable is R_p.

29 When Mr. Challand was painting the gutters on his house, he found that he could paint about 35 ft. in one hour. At this rate, about how long would it take him to paint 168 ft.? (Tenths) (245)

30 During a game, Marion High School's basketball team made 28 baskets in 76 attempts. The number of baskets the team made was what fraction of the number of attempts? (Thousandths) 245)

31 The wheel on a certain car makes 8.6×10^2 revolutions per mile. How many revolutions does this wheel make in 9.9×10^3 mi.? (245)

32 Mr. Kahn wants to have a driveway built. A concrete driveway will cost \$520. An asphalt driveway will cost \$290. If the driveway is to be 80 ft. long, how much can he pay per running foot to have it built? (245)

For problems 33, 34, and 35, the universe for each variable is R.

33 What numbers can be multiplied by 2.4 so that each product is greater than the quotient of $^-7.5$ and $^-5$? (245)

34 What numbers can be added to $4\frac{1}{5}$ so that each sum is not less than $^-2\frac{7}{10}$? (245)

35 What numbers less than $8\frac{1}{2}$ can be subtracted from $^-\frac{2}{3}$ so that each difference is at least $^-4\frac{3}{4}$? (245)

Unit 11

The real numbers

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Squares and square roots

You have been studying powers of rational numbers. Now you will study some properties of the second powers of rational numbers.

A Look at sentence A in D1. How many times is 6 used as a factor in 6^2? What is a standard name of 6^2? Is statement A true?

B Look at sentence B. What product is obtained when $\bar{}\frac{2}{3}$ is used twice as a factor? Why is the product a positive rational number? Is $\frac{4}{9}$ the second power of $\bar{}\frac{2}{3}$?

C Look at sentences C and D. What is a standard name of the second power of $\frac{4}{5}$? Of the second power of $\bar{}7$? Are statements C and D true?

The second power of a number is the *square* of the number. 36 is the square of 6. $\frac{4}{9}$ is the square of $\bar{}\frac{2}{3}$.

D What number is the square of $\frac{4}{5}$? Of $\bar{}7$?

E Explain why 81 is the square of $\bar{}9$.

Find the square of each number named below.

F 12 **G** $\bar{}10$ **H** $\bar{}\frac{3}{4}$ **I** $\frac{11}{13}$ **J** $\bar{}\frac{5}{8}$

square of a number. The second power of the number. The universe for x and a is R. If $x=a^2$, then x is the square of a.

A $6^2 = 36$.

B $\frac{4}{9} = (\bar{}\frac{2}{3})^2$.

C $\frac{16}{25} = (\frac{4}{5})^2$.

D $(\bar{}7)^2 = 49$.

D1

Universe for a and b = R.

$a = b$.	$a^2 = b^2$.
E $\frac{2}{3} = \frac{4}{6}$.	G $(\frac{2}{3})^2 = (\frac{4}{6})^2$.
F $\frac{7}{8} = \frac{21}{24}$.	H $(\frac{7}{8})^2 = (\frac{21}{24})^2$.

D2

Universe for a and b = R.

$a = {}^-(b)$.	$a^2 = b^2$.
I $\frac{8}{9} = {}^-(\frac{-8}{9})$.	K $(\frac{8}{9})^2 = (\frac{-8}{9})^2$.
J $20 = {}^-(-20)$.	L $(20)^2 = (-20)^2$.

D3

K Think about your answers for exercises F through J. Is each square greater than zero? Is each square a positive rational number?

L Explain why the square of any positive rational number or of any negative rational number must be greater than zero.

M What number is the square of 0?

N Is $(0)^2 \in \overline{R}_n$? Is $(-4)^2 \in \overline{R}_n$? Is $(\frac{3}{5})^2 \in \overline{R}_n$? How many rational numbers are there whose squares are not members of $\overline{R}_n$? Remember that the symbol $\overline{R}_n$ is a name of the set of non-negative rational numbers.

The fact that the square of each rational number is a member of $\overline{R}_n$ is a property of the squares of rational numbers. The property is expressed below.

The universe for a *is R. For each* a, $a^2 \geqq 0$.

Next you will study some other properties of squares.

A Look at D2. What is the universe for a and b?

B How was statement E obtained from $a = b$? How was statement G obtained from $a^2 = b^2$? Were the same replacements made in $a = b$ and in $a^2 = b^2$ to obtain statements E and G?

C Is statement E true? How do you know that statement G is true?

D What replacements were made for a and b in $a = b$ and $a^2 = b^2$ to obtain statements F and H? Are both these statements true?

E What property tells you that the rational numbers that form a product may be replaced by equal rational numbers without affecting the product?

F How do you know that, for each a and b, if $a = b$, then $a^2 = b^2$?

G Look at D3. How was statement I obtained from $a = {}^-(b)$? How was statement K obtained from $a^2 = b^2$? Were the same replacements made for a and b in both $a = {}^-(b)$ and $a^2 = b^2$ to obtain statements I and K?

H Is statement I true? Is statement K true?

I Notice that, to obtain statement I from $a = {}^-(b)$, a was replaced by $\frac{8}{9}$ and b was replaced by $\frac{-8}{9}$. How are $\frac{8}{9}$ and $\frac{-8}{9}$ related?

J What is the additive inverse of 20? What replacements were made for a and b in $a = {}^-(b)$ and $a^2 = b^2$ to obtain statements J and L? Are both these statements true?

K For each a and b, if b is the additive inverse of a, how do a and ${}^-(b)$ compare? How does a^2 compare with b^2?

In exercises A through K, you developed another property of squares of rational numbers. The property is expressed below.

The universe for a *and* b *is R. For each* a *and* b, *if* $a = b$ *or if* $a = {}^-b$, *then* $a^2 = b^2$. *If* $a^2 = b^2$, *then* $a = b$ *or* $a = {}^-b$.

REMINDER

Either the symbol ${}^-(b)$ or the symbol ${}^-b$ may be used to express the additive inverse of b. U = R.

See lesson 139, page 205.

Universe for a and $b = \overline{R}_n$.

$a < b$.	$a^2 < b^2$.
M $9 < 11$.	O $9^2 < 11^2$.
N $\frac{3}{4} < \frac{7}{8}$.	P $(\frac{3}{4})^2 < (\frac{7}{8})^2$.

D4

Q $(\frac{3}{5})^2 = \frac{9}{25}$.	S $(^-12)^2 = 144$.
R $(\frac{^-7}{4})^2 = \frac{49}{16}$.	T $(\frac{14}{13})^2 = \frac{196}{169}$.

D5

square root of a number (rüt or růt). If a number is the product of two equal factors, each factor is a square root of that number. The universe for x and a is R. For each x and a, if $x^2 = a$, then x is a square root of a.

L If one rational number is equal to a second rational number, what do you know about the squares of the two numbers?

M If one rational number is the additive inverse of a second rational number, what do you know about the squares of the two numbers?

N If the square of one rational number is equal to the square of a second rational number, are the two numbers necessarily equal? Explain your answer.

O Look at D4. Describe the universe for a and b.

P How was statement M obtained from $a < b$? How was statement O obtained from $a^2 < b^2$? Were the same replacements made for a and b in both $a < b$ and $a^2 < b^2$ to obtain statement M and statement O?

Q Is statement M true? How do you know that statement O is true?

R What replacements were made for a and b in $a < b$ and $a^2 < b^2$ to obtain statements N and P? Are both these statements true?

S If one non-negative rational number is less than a second non-negative rational number, how does the square of the first number compare with the square of the second number?

T If the square of one non-negative rational number is less than the square of a second non-negative rational number, how does the first number compare with the second number?

In exercises O through T, you have been working with a property of squares of non-negative rational numbers. The property is expressed below.

The universe for a *and* b *is* $\overline{R}_n$. *For each* a *and* b, *if* $a < b$, *then* $a^2 < b^2$. *If* $a^2 < b^2$, *then* $a < b$.

U Give an example that shows why the universe for a and b is not R in the property expressed above.

You know that the square of a given number is the product obtained when the given number is used twice as a factor. Now you will learn more about these factors.

A Look at sentences Q and R in D5. What is the square of $\frac{3}{5}$? What two equal factors make up $\frac{49}{16}$? How do you know that statements Q and R are true?

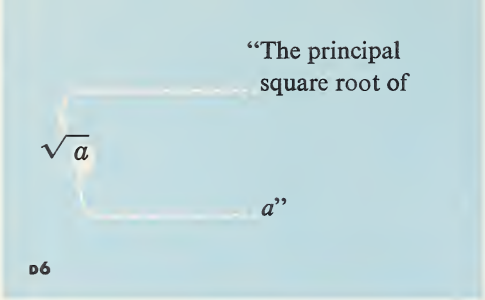
If a number is the product of two equal factors, each factor is a *square root* of the given number. For example, $\frac{3}{5}$ is a square root of $\frac{9}{25}$. $\frac{^-7}{4}$ is a square root of $\frac{49}{16}$.

B Look at sentences S and T in D5. What is a square root of 144? Of $\frac{196}{169}$?

C You know that $\frac{3}{5}$ is a square root of $\frac{9}{25}$. How do you know that $\frac{^-3}{5}$ is also a square root of $\frac{9}{25}$?

D What number other than $\frac{^-7}{4}$ is a square root of $\frac{49}{16}$? What number other than $^-12$ is a square root of 144? Name two square roots of $\frac{196}{169}$.

E Explain why there is no rational number that is a square root of a negative rational number.



D6

For each of exercises F through K, find the additive inverse of the number named. Then find the square of the given number and the square of the additive inverse of that number.

F $\frac{-5}{3}$ G 14 H $\frac{17}{9}$ I $^{-}23$ J $\frac{-25}{32}$ K 48

L Is each square that you just named a positive rational number? Does each one have both a negative and a positive square root?

M Name the square root of 0. Is the square root of 0 a non-negative rational number?

Each positive number has a negative square root and a positive square root. The positive square root of a number is its *principal square root*. 0 is the principal square root of 0.

Mathematicians use the symbol at the right below to express the principal square root of a number. Notice that we read the symbol at the right as "the principal square root of." We use this symbol to express *only* the principal square root of a number. D6 shows how to write and read a symbol for the principal square root of a when the universe for a is $\overline{R}_n$.

$\sqrt{}$

N Is $\sqrt{25} = 5$? Is $\frac{-9}{7}$ the principal square root of $\frac{81}{49}$? What is the principal square root of 0?

O How do you know that $\sqrt{64}$ is not $^{-}8$?

Tell whether each of the sentences below expresses a true statement or a false statement.

P $\sqrt{\frac{100}{81}} = \frac{10}{9}$.

Q $\sqrt{\frac{15}{11}} = \frac{225}{121}$.

R $\sqrt{625} = {}^{-}25$.

S $\sqrt{\frac{64}{49}} = \frac{7}{8}$.

T $\sqrt{\frac{900}{676}} = \frac{30}{26}$.

U $\sqrt{2025} = 45$.

V What is a standard name of $\sqrt{81}$? Of $\sqrt{16}$? Of $\sqrt{100}$?

W Use the definition of square root to explain why, for each a, $(\sqrt{a})^2 = a$. $U = \overline{R}_n$.

When you are working with squares and square roots, it is helpful to have the use of a table of squares and square roots. In this book, the page before the index contains a table that names the integers from 1 through 220, along with the square and the principal square root of each integer. Study this table now.

A Suppose that you want to know the square of 3. Under the letter n in the table, find the blue numeral 3. What numeral is directly to the right of the blue numeral 3? Is the numeral 9 under the symbol n^2? Is 9 the square of 3?

B Suppose that you want to find $\sqrt{3}$. What numeral is directly to the right of the numeral that names the square of 3? Is the numeral 1.732 under the symbol $\sqrt{n}$? The number 1.732 is $\sqrt{3}$ to the nearer thousandth.

In most cases, the numbers named under the symbol $\sqrt{n}$ are only approximately equal to the principal square root of the number named in the same row under the letter n. Each square root is given to the nearer thousandth.

C What is the square of 15? Of 49? Of 99? Of 132? What is the principal square root of each of these numbers? Use the table.

D Use the table to find the square of 25. Explain how you can use the table to find the principal square root of 625.

E What is the principal square root of 961? Of 2916? Of 19,881? Of 28,900?

In this lesson you learned what is meant by the square of a number and a square root of a number. You also studied some of the properties of squares and square roots.

On your own

Use the table of squares and square roots to find the square of the number named in each exercise below. Then find the principal square root of each number named.

1 13	5 74	9 55	13 129	17 106
2 19	6 38	10 140	14 70	18 67
3 176	7 5	11 118	15 191	19 95
4 62	8 200	12 31	16 92	20 212

For each of exercises 21 through 28, use the table to find the principal square root of the number named.

21 289	23 1089	25 4356	27 46,225
22 576	24 33,124	26 24,336	28 14,400

Tell whether each sentence below expresses a true statement or a false statement.

29 $(^{-}15)^2 = 225$.
30 $\sqrt{25} = {}^{-}5$.
31 $\sqrt{14641} = 121$.
32 $(^{-}17)^2 = {}^{-}289$.
33 $(\sqrt{16})^2 = 4$.
34 $(\sqrt{72})^2 = 72$.

Use the properties and ideas that you studied in this lesson to explain why each statement expressed below is true. You should be able to do this without computing.

35 $(810)^2 = (^{-}810)^2$.
36 $(\frac{16}{15})^2 < (\frac{17}{15})^2$.
37 $(\sqrt{7})^2 = 7$.
38 $(\frac{16}{5})^2 = (\frac{32}{10})^2$.
39 $\sqrt{289} < \sqrt{441}$.
40 $(^{-}\frac{2}{5})^2 = (\frac{16}{40})^2$.

KEEPING SKILFUL

For each condition expressed below, tabulate the solution set if possible. Otherwise, give a standard description of the solution set. U = R.

1 $x - 4.25 = {}^{-}1.72$.
2 $a + {}^{-}16 = 25\frac{3}{4}$.
3 $3r + 17.8 > 24.7$.
4 $.65y = 2.665$.
5 $4z + {}^{-}\frac{1}{2} = 5\frac{1}{2}$.
6 $\sim (2h - \frac{4}{5} > 10)$.
7 $t - {}^{-}\frac{7}{10} < 25 \wedge t + \frac{3}{10} > 22$.
8 $c - 6.5 > 4.9 \vee {}^{-}12.4 + c > {}^{-}3.6$.
9 $5.2 - s > 8.6 \wedge 4s + 3.5 > 8.3$.
10 $7 + 5m = 2 \vee {}^{-}2m > 2$.

Determining square roots

In lesson 149 you used a table to find squares and square roots. In this lesson you will learn to find square roots by computation.

Study sentence A in D1. From sentence A, you know that 1 is the square of 1. You also know that $\sqrt{1} = 1$.

A Look at sentence B. 36 is the square of what positive number? What is $\sqrt{36}$?

B Read sentence C. 169 is the square of what positive number? What is $\sqrt{169}$?

C Now look at sentence D. 484 is the square of what positive number? What is $\sqrt{484}$?

D Think about the numbers expressed at the left of the symbols for equality in D1. Is each of these numbers a positive integer? Think about the bases of the powers expressed at the right of the symbols for equality. Is each of these bases a positive integer?

E 625 is the square of what positive integer? What is $\sqrt{625}$? Use the table of squares and square roots to help you.

F 6084 is the square of what positive integer? What is $\sqrt{6084}$?

G Is I_p closed under multiplication? How do you know that the square of every element of I_p is also an element of I_p? Remember that I_p is a name of the set of positive integers.

A $1 = 1^2$.
B $36 = 6^2$.
C $169 = 13^2$.
D $484 = 22^2$.

D1

The square of a positive integer is called a *perfect square*. Every perfect square is a positive integer.

H How do you know that 49 is a perfect square? That 400 is a perfect square?

You can use what you know about the squares of certain integers to help you decide whether or not a given number is a perfect square. Suppose, for example, that you want to know whether or not 110 is a perfect square.

I You know that 100 is a perfect square because $100 = 10^2$ and $10 \in I_p$. You also know that 121 is a perfect square because $121 = 11^2$ and $11 \in I_p$. Is 110 between 10^2 and 11^2?

J If 110 were a perfect square, then 110 would be the square of an integer between 10 and 11. Is there an integer between 10 and 11? Is 110 a perfect square?

Tell which of the numbers named below are perfect squares.

K 64 **L** 121 **M** 136 **N** 225 **O** 54

If you know that a first number is the square of a second number, then you also know that the second number is a square root of the first number. You know that each positive integer has two square roots: a principal, or positive, square root and a negative square root. Unless you are told otherwise, "the square root" of a number will refer to the principal square root.

Of course, not all numbers are perfect squares, and there are times when you need to know the square root of a number that is not a perfect square. In such cases, you need a method for finding square roots. In the exercises that follow, you will learn one way of computing to find the square root of a positive integer that is not a perfect square.

Look at numeral E in D2. You are going to find a number that is approximately equal to $\sqrt{2}$.

E $\sqrt{2}$

F $1 < \sqrt{2} < 2$.

G $1.4 < \sqrt{2} < 1.5$.

H $1.41 < \sqrt{2} < 1.42$.

I $1.414 < \sqrt{2} < 1.415$.

D2

J $\sqrt{2} = 1.41421\ldots$.

D3

perfect square (pèr'fikt). The square of a positive integer. The universe for a and x is I_p. For each a and x if $x = a^2$, then x is a perfect square.

A What is 1^2? What is 2^2? How do you know that 2 is not a perfect square?

B Read sentence F. You know that 2 is between 1^2 and 2^2. Is statement F true?

Because $\sqrt{2}$ is between 1 and 2, you can choose 1.5 as a reasonable estimate of $\sqrt{2}$.

C Is $(1.5)^2 = 2.25$? Is $2 < (1.5)^2$?

D Since $2 < (1.5)^2$, why must it also be true that $\sqrt{2} < 1.5$? Because $\sqrt{2} < 1.5$, you can choose 1.4 as the next estimate of $\sqrt{2}$. What is $(1.4)^2$? Is $(1.4)^2 < 2$?

E Is $1.4 < \sqrt{2}$? How do you know that sentence G in D2 expresses a true statement?

F Why should you choose a number that is greater than 1.4 and less than 1.5 as the next estimate of $\sqrt{2}$?

G Is 1.41 between 1.4 and 1.5? Is $(1.41)^2 = 1.9881$? How do you know that $1.41 < \sqrt{2}$?

H Why is 1.42 a reasonable estimate of $\sqrt{2}$? What is $(1.42)^2$?

I Is $\sqrt{2} < 1.42$? Does sentence H in D2 express a true statement? Explain your answers.

You know that $\sqrt{2}$ is between 1.41 and 1.42. Use 1.415 as your next estimate of $\sqrt{2}$.

J Is $(1.415)^2 = 2.002225$? How do you know that $\sqrt{2} < 1.415$?

K Why does it make sense to choose 1.414 as the next estimate of $\sqrt{2}$? What is $(1.414)^2$? How do you know that sentence I in D2 expresses a true statement?

L Explain how you know that your next estimate of $\sqrt{2}$ should be between 1.414 and 1.415.

Notice that each new estimate of $\sqrt{2}$ has given you a more accurate approximation of $\sqrt{2}$. If you were to continue to estimate $\sqrt{2}$ in the way shown in D2, you would find, using two more steps, that $\sqrt{2}$ is between 1.41421 and 1.41422.

M What is $\sqrt{2}$ to the nearer thousandth? To the nearer ten-thousandth?

Now look at D3. Sentence J expresses a true statement about $\sqrt{2}$. The three dots at the right of the last digit in 1.41421 . . . show that the digits in the decimal for $\sqrt{2}$ go on and on and that the decimal is not a repeating decimal. You should read sentence J in this way: "The principal square root of two is equal to one point four one four two one, and so on."

N Could you continue to estimate $\sqrt{2}$ so that your approximation would be correct to the nearer ten-millionth? To the nearer billionth? To the nearer hundred-billionth?

O If the decimal for $\sqrt{2}$ is not a repeating decimal, then how do you know that $\sqrt{2}$ is not a rational number?

From the work in the preceding exercises, you can see that the method of computing to find the square root of a number by successive approximations requires a great deal of time and effort. There is a more efficient way of computing to find the positive square root of an integer that is not a perfect square. You will study this method now.

K $\sqrt{152}$

L $152 \div 12.5 = 12.160$.

M $(12.160 + 12.5) \div 2 = 12.330$.

N $(12.328 + 12.330) \div 2 = 12.329$.

O $(12.329)^2 = 152.004241$.

D4

A Look at numeral K in D4. 152 is not a perfect square, but it is between two perfect squares, 144 and 169. Is $144 = 12^2$? Is $169 = 13^2$?

B Is 152 between 12^2 and 13^2? How do you know that $\sqrt{152}$ is between 12 and 13?

Because you know that $\sqrt{152}$ is between 12 and 13, you could choose 12.5 or any other number between 12 and 13 as a reasonable estimate of $\sqrt{152}$. If 12.5 *were* the square root of 152, then $(12.5)^2$ would be equal to 152. This means that the quotient of 152 and 12.5 would be 12.5. So, the next step in computing $\sqrt{152}$ is to find the quotient of 152 and 12.5.

C How do you know that the quotient of 152 and 12.5 is the same as the quotient of 1520 and 125?

D Find the quotient of 1520 and 125 to the nearer thousandth. What numeral should you use to express the dividend?

E Does sentence L in D4 express a true statement? What is the quotient of 152 and 12.5 to the nearer thousandth?

F How do you know that 12.5 is not the positive square root of 152?

G Is $152 = 12.160 \times 12.5$? Is $12.160 < 12.5$? How do you know that $\sqrt{152}$ is between 12.160 and 12.5?

Notice that, at first, you discovered that $\sqrt{152}$ is between 12 and 13, so you chose 12.5 as a first estimate of $\sqrt{152}$. Now you know that $\sqrt{152}$ is between 12.160 and 12.5, so you can make a more accurate estimate of $\sqrt{152}$.

K $\sqrt{152}$

L $152 \div 12.5 = 12.160$.

M $(12.160 + 12.5) \div 2 = 12.330$.

N $(12.328 + 12.330) \div 2 = 12.329$.

O $(12.329)^2 = 152.004241$.

D4

H You know that one way to find a number that is between two given numbers is to divide the sum of the given numbers by 2. Does sentence M in D4 express a true statement? Is 12.330 between 12.160 and 12.5?

You now have 12.330 as a new estimate of $\sqrt{152}$. As before, you should find the quotient of 152 and 12.330 to determine how accurate this estimate is.

I Is the quotient of 152 and 12.330 the same as the quotient of 15,200 and 1233? Find the quotient of 15,200 and 1233 to the nearer thousandth.

J Is 12.328 the quotient of 152 and 12.330 to the nearer thousandth? How do you know that $\sqrt{152}$ is between 12.328 and 12.330?

K How do you know that sentence N in D4 expresses a true statement? Is 12.329 between 12.328 and 12.330?

12.329 is approximately equal to $\sqrt{152}$. We say that 12.329 is equal to $\sqrt{152}$ to the nearer thousandth. If you wanted to obtain a more accurate estimate of $\sqrt{152}$, you could do so by finding the quotient of 152 and 12.329. This time, however, you would need to determine the quotient to ten-thousandths or to hundred-thousandths. For most purposes, accuracy to the nearer thousandth is sufficient. Numbers like $\sqrt{2}$ and $\sqrt{152}$ cannot be represented by repeating decimals. Thus, they are not rational numbers. As you continue to study mathematics, you will be able to prove that $\sqrt{2}$ is not a rational number.

L Look again at D4. Is statement O true? This statement should convince you that 12.329 is a reasonably good approximation of $\sqrt{152}$.

M Now suppose that you want to compute $\sqrt{44}$. How do you know that 44 is not a perfect square? $\sqrt{44}$ is between what two consecutive positive integers?

N Is 6.5 a reasonable estimate of $\sqrt{44}$? Why should you next find the quotient of 44 and 6.5?

O How do you know that $\sqrt{44}$ is between 6.5 and 6.769? What is the sum of 6.5 and 6.769? Find the quotient of this sum and 2 to the nearer even thousandth.

P Is 6.634 your next estimate of $\sqrt{44}$? What is the quotient of 44 and 6.634 to the nearer thousandth?

Q How do you know that $\sqrt{44}$ is between 6.632 and 6.634? Is 6.633 the positive square root of 44 to the nearer thousandth?

R What is $(6.633)^2$? Do you think that 6.633 is a reasonably good approximation of $\sqrt{44}$?

Use the method of computation you have just learned to find each of the square roots named below to the nearer thousandth.

S $\sqrt{14}$ T $\sqrt{85}$ U $\sqrt{176}$ V $\sqrt{6}$

In this lesson you learned what a perfect square is. You also learned how to compute to find the positive square root of a number that is not a perfect square.

On your own

Tell which of the numbers named in exercises 1 through 16 are perfect squares. If a number is not a perfect square, compute to find its positive square root to the nearer thousandth.

1 179	5 900	9 1225	13 610
2 196	6 275	10 365	14 211
3 200	7 289	11 6400	15 2500
4 484	8 70	12 256	16 10,000

The real numbers

You studied the set of rational numbers in unit 10. Now you are going to study a set of numbers that includes the rational numbers as a subset.

A Is each statement expressed in D1 true? Is each decimal in D1 a repeating decimal? What is the repetend of each decimal?

B Which of the rational numbers named in D1 can be named by terminating decimals? Remember that, if the digit 0 is the repetend of a repeating decimal for a rational number, then the rational number can be named by a terminating decimal.

Find a repeating decimal for each rational number named below.

C $1\frac{3}{4}$ **E** $^{-}6$ **G** $^{-}15\frac{1}{6}$ **I** $5\frac{1}{7}$ **K** $^{-}\frac{9}{14}$
D $^{-}\frac{2}{3}$ **F** $\frac{7}{11}$ **H** $\frac{3}{5}$ **J** $^{-}\frac{1}{1000}$ **L** $\frac{7}{15}$

You know that a repeating decimal is a decimal in which digits repeat in a pattern, and the pattern repeats without end. Because there is no last digit in a repeating decimal, such a decimal is called a *repeating infinite decimal.* All the decimals that you have worked with so far are examples of repeating infinite decimals.

M In lesson 142 of unit 10, you learned that every rational number can be named by a repeating decimal. Is every repeating decimal also an infinite decimal? Can every rational number be named by an infinite decimal?

Every rational number can be named by a repeating infinite decimal, and every repeating infinite decimal names a rational number.

A $1\frac{1}{2} = 1.5\overline{0}.$ **D** $\frac{1}{11} = .\overline{09}.$
B $^{-}\frac{1}{8} = ^{-}.125\overline{0}.$ **E** $^{-}14 = ^{-}14.\overline{0}.$
C $\frac{7}{3} = 2.\overline{3}.$ **F** $^{-}3\frac{6}{7} = ^{-}3.\overline{857142}.$

D1

G 1.41421 . . .
H 1.7325 . . .
I .101001000100001 . . .
J $^{-}$6.323322333222 . . .

D2

There are, however, certain numbers that can be named by infinite decimals that are not repeating decimals. In the work that follows, you will study such numbers.

A Study the numerals in D2. Numeral G is the ? decimal for $\sqrt{2}$. The square root of 2 is not a rational number. How do you know that numeral G is not a repeating decimal? Is there a last digit in the decimal for $\sqrt{2}$?

B Numeral H names $\sqrt{3}$. The square root of 3 is not a rational number. How do you know that numeral H is not a repeating decimal? Is there a last digit in the decimal for $\sqrt{3}$?

C Now look at numeral I. How do you know that numeral I is not a repeating decimal? How many digits for 0 appear between the first and second digits for 1? How many digits for 0 appear between the second and third digits for 1? Between the third and fourth digits for 1? Between the fourth and fifth digits for 1?

D Notice that, even though numeral I is not a ? repeating decimal, the occurrence of the digits in the decimal seems to follow a certain rule. According to this rule, how many digits for 0 will there be between the fifth and sixth digits for 1 in numeral I? Now write numeral I with 30 digits to the right of the decimal point.

E Is there a last digit in numeral I?

F Look at numeral J in D2. Describe the first two digits to the right of the decimal point. Describe the next four digits to the right of the decimal point. Describe the next six digits to the right of the decimal point.

G Look again at numeral J. Use the rule that is indicated by the first twelve digits to the right of the decimal point to write the next eight digits. How do you know that numeral J is not a repeating decimal? Is there a last digit in numeral J?

The numerals in D2 are infinite decimals because, in each case, the digits in the decimal go on and on without end. However, these infinite decimals have no repetend. For this reason such decimals are called *nonrepeating infinite decimals.*

Tell which of the decimals below are repeating infinite decimals and which are nonrepeating infinite decimals.

H $^{-}2.1\overline{6}$ **J** $15.65\overline{0}$ **L** 2.2361 . . .

I $.\overline{142857}$ **K** $^{-}3.\overline{09}$ **M** $^{-}$.2468101214 . . .

All the numbers that you have considered so far in this lesson are *real numbers.* All numbers that can be named by repeating infinite decimals are real numbers. All numbers that can be named by nonrepeating infinite decimals are also real numbers.

Numbers that can be expressed by nonrepeating infinite decimals are *irrational* (i rash′ən əl) *numbers.* Notice that the word "irrational" is formed by attaching the prefix *ir-*, which means "not," to the word "rational." Thus, an irrational number is *not* a rational number.

N How do you know that an irrational number is a real number? Is the set of real numbers the union of the set of rational numbers and the set of irrational numbers? If the universe is the set of real numbers, is the set of irrational numbers the complement of the set of rational numbers?

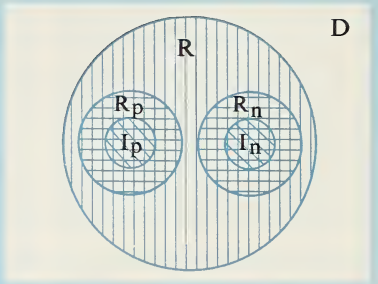
G 1.41421 . . .

H 1.7325 . . .

I .101001000100001 . . .

J $^{-}6.323322333222$ . . .

D2



D3

re al number (rē′əl). A number that can be expressed by an infinite decimal. Certain real numbers can be expressed by repeating infinite decimals. Other real numbers, such as $\sqrt{2}$, can be expressed by nonrepeating infinite decimals.

Since the set of irrational numbers is the complement of the set of rational numbers, we will use the symbol at the right to name the set of irrational numbers. This symbol is read "the set of irrational numbers."

$\overline{R}$

O Look again at exercises H through M. Which of the numbers named are elements of R? Are elements of $\overline{R}$?

Numbers like $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{6}$ are irrational numbers. Later on, you will study a method of proving that the square root of any positive integer that is not a perfect square is an irrational number.

P Do you think that $\overline{R}$ is an infinite set? Explain your answer.

We will use the symbol at the right below as a name of the set of real numbers. This symbol is read "the set of real numbers."

D

Q Is $D = R \cup \overline{R}$? How do you know that D is an infinite set?

R Look at the Venn diagram in D3. Which portion of the diagram represents the set of real numbers?

S Which portion of the Venn diagram represents $\overline{R}$?

T How does the diagram indicate that R_p and R_n are disjoint sets? How does the diagram indicate that R_p and R_n are subsets of R? Are R_p and R_n subsets of $\overline{R}$? Are R_p and R_n subsets of D?

U Name three sets represented in D3 that have the set of positive integers as a subset.

V Name three sets represented in D3 that have the set of negative integers as a subset.

Next you will learn that the real numbers can be put in one-to-one correspondence with the points in a line.

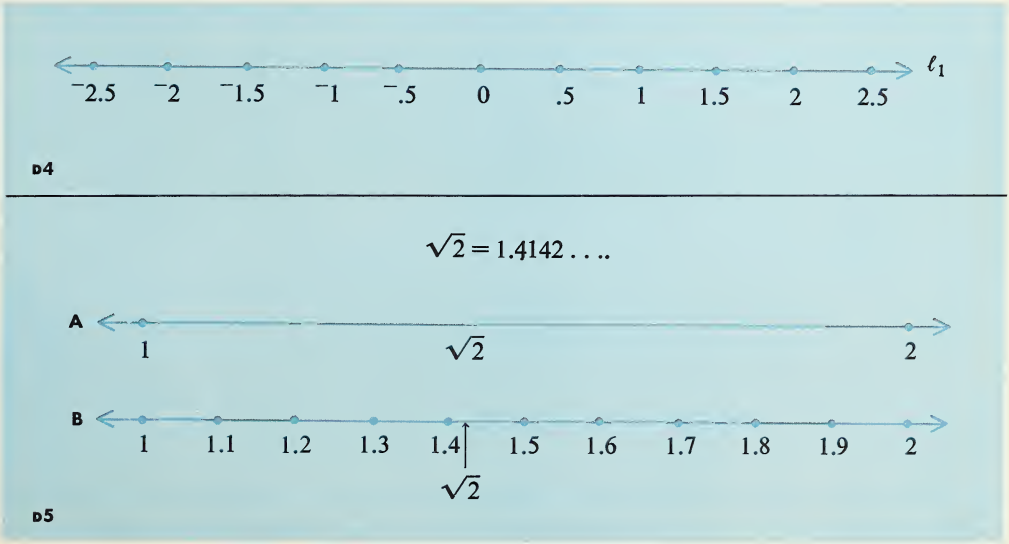
A Look at the picture of ℓ_1 in D4. Is ℓ_1 an infinite set of points? Is R an infinite set of numbers? Is it possible to associate every element of R with a point in ℓ_1?

Look at D5. You will now see how you can determine the approximate location of the point in ℓ_1 that is associated with $\sqrt{2}$. Pictures A and B in D5 represent only the subset of ℓ_1 that contains point 1, point 2, and all points between 1 and 2.

B How do you know that $1 < \sqrt{2} < 2$? How does picture A show that point $\sqrt{2}$ is between point 1 and point 2?

C Is $1.4 < \sqrt{2} < 1.5$? How does picture B show that point $\sqrt{2}$ is between point 1.4 and point 1.5?

D Is $1.41 < \sqrt{2} < 1.42$? How do you know that the point in ℓ_1 that is associated with $\sqrt{2}$ is between point 1.41 and point 1.42?



Now look at D6. Picture C represents only the subset of ℓ_1 that contains point 1.4 and point 1.5 and all points between 1.4 and 1.5.

E You know that $1.41 < \sqrt{2} < 1.42$ is a true statement. How does picture C show that point $\sqrt{2}$ is between point 1.41 and point 1.42?

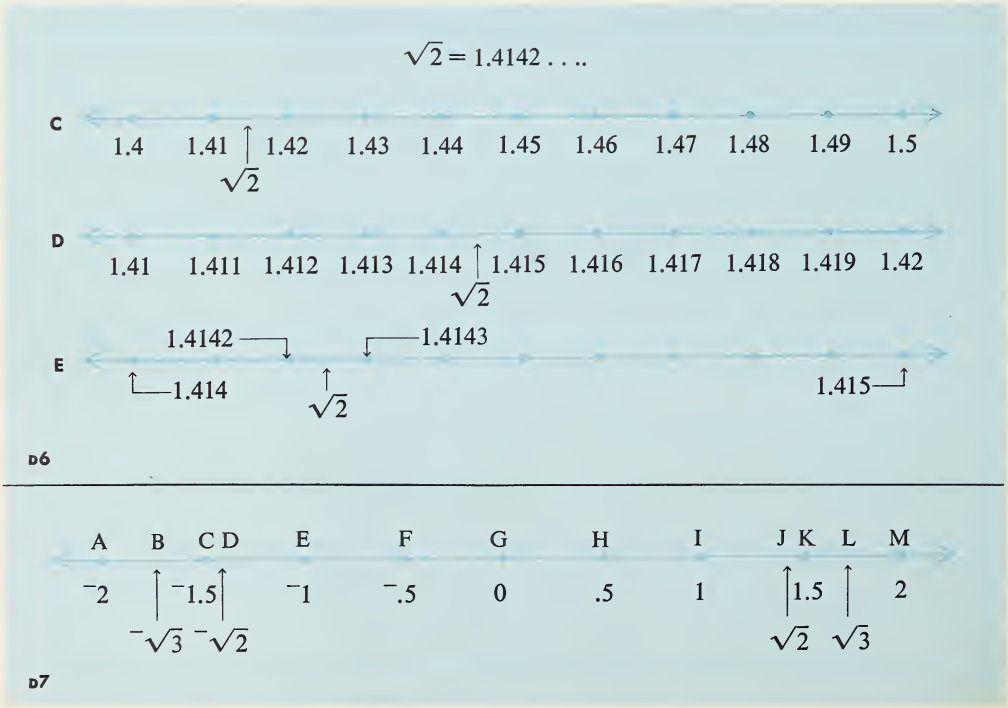
F What subset of ℓ_1 does picture D in D6 represent? Is $1.414 < \sqrt{2} < 1.415$? How does picture D show that point $\sqrt{2}$ is between point 1.414 and point 1.415?

G Look at picture E. Is $1.4142 < \sqrt{2} < 1.4143$? Describe the position of the dot for point $\sqrt{2}$ in picture E.

You could continue the method just explained indefinitely. At each step, you would come closer to locating point $\sqrt{2}$. You can find only an approximate location for point $\sqrt{2}$ by this method. Later in your study of mathematics, you will learn how to use other methods to locate point $\sqrt{2}$ exactly.

H Remember that $\sqrt{3} = 1.7325\ldots$. How do you know that the point in ℓ_1 that is associated with $\sqrt{3}$ is located between point 1.7 and point 1.8? Between point 1.73 and point 1.74? Between point 1.732 and point 1.733?

Each point in ℓ_1 can be associated with a real number. Every point that is not associated with a rational number is associated with an irrational number. A number line whose coördinates are all the real numbers is a *real-number line*. The set of real numbers can be put in one-to-one correspondence with the set of all points in a line. This means that a real-number line is actually a line of the kind that you have worked with in geometry in *Book 1* and in this book. No matter where you happen



to locate a point in a line, there is a real number that can be associated with that point. Because there are no "gaps" in a real-number line, the set of real numbers is sometimes called the *continuum* (kən tin′ū əm) of real numbers.

I A real-number line is represented in D7. The symbol for the coördinate of point D is read "the negative square root of two." How should you read the symbol $^{-}\sqrt{3}$?

J Is $^{-}\sqrt{2} = {}^{-}1.4142\ldots$? Is $^{-}\sqrt{2}$ an irrational number?

K Which of the points represented by dots in D7 are associated with rational numbers? Which are associated with irrational numbers?

In this lesson you learned that the set of real numbers contains all the numbers that can be expressed by repeating infinite decimals and all the numbers that can be expressed by nonrepeating infinite decimals. You also learned that the set of real numbers and the set of all points in a line can be put in one-to-one correspondence.

On your own

Make a Venn diagram like the one in D3 on page 262. Make your diagram large enough to give you room for several numerals in each part of the diagram that represents a subset. Then, for the exercises below, write each numeral in the appropriate part of the diagram.

1 $\frac{1}{4}$	4 $^{-}3\frac{2}{3}$	7 $10\frac{1}{2}$	10 $^{-}\frac{6}{5}$
2 $^{-}3$	5 $\sqrt{4}$	8 $^{-}\sqrt{10}$	11 100
3 $\sqrt{5}$	6 0	9 $^{-}\sqrt{25}$	12 $\sqrt{3}$

Each irrational number named in exercises 13 through 21 is between two successive integers. Name the integers for each exercise. Use the table of squares and square roots to help you.

13 $\sqrt{11}$	16 $\sqrt{410}$	19 $\sqrt{4100}$
14 $^{-}\sqrt{35}$	17 $^{-}\sqrt{22}$	20 $^{-}\sqrt{650}$
15 $\sqrt{113}$	18 $^{-}\sqrt{309}$	21 $\sqrt{12305}$

Algebraic properties of the real numbers

In lesson 151, you learned that a real number is a number that can be expressed by an infinite decimal. In this lesson you will study the properties of the operations of addition and multiplication of real numbers. You will also study the properties of equality.

When you studied the set of natural numbers, the set of rational numbers of arithmetic, and the set of rational numbers, you learned that the operations of addition and multiplication had certain properties, such as the closure properties and the commutative properties. In *Book 1* and in this book we used a certain procedure in presenting properties. First, we defined the kind of number that was being studied. Next, we developed definitions of the sum and the product of two such numbers. Finally, we used these definitions to help prove various properties of the operations of addition and multiplication.

In the case of the real numbers, our procedure must be somewhat different. You know the definition of a real number in terms of its decimal representation. You do not, however, know the definitions of the sum and the product of numbers expressed by infinite decimals. Thus, you cannot use the definitions of sum and product of real numbers to prove the properties of the operations of addition and multiplication of real numbers. This means that you will need to *accept* the properties without proof.

A Look at D1. Name the property expressed in D1 that tells you that $3 + \sqrt{5}$ is a real number. Is $6.05 + {}^{-}.027 \in D$? Is $.\overline{4} + 2.6457\ldots$ a real number?

B Name the property that tells you that the product of 3 and $\sqrt{5}$ is a real number. You may express this product by the symbol $3\sqrt{5}$. Is $6.05 \times .027 \in D$? Is $.\overline{4} \times 2.6457\ldots$ a real number?

To find the sum or the product of two real numbers that are also rational numbers, you use the same processes that you use to find the sum or the product of any two rational numbers. You can also use these processes to find *approximations* of sums or products that involve irrational numbers. For example, $3 + \sqrt{5}$ involves an irrational number. The rational number 2.236 is an approximation, to the nearer thousandth, of $\sqrt{5}$. A *rational approximation* of $3 + \sqrt{5}$ is $3 + 2.236$, or 5.236, to the nearer thousandth.

C Suppose that you want to find a rational approximation of $3\sqrt{5}$. Why should you multiply 2.236 by 3? Is 6.708 a rational approximation of $3\sqrt{5}$?

D What is a rational approximation of the sum of $.\overline{4}$ and $2.6457\ldots$? Of the product of these two numbers? Use approximations, to the nearer thousandth, of both $.\overline{4}$ and $2.6457\ldots$.

E Look again at D1. What property expressed in D1 tells you that $3 + \sqrt{59} = \sqrt{59} + 3$?

F What property expressed in D1 tells you that $.7 \times \sqrt{26} = \sqrt{26} \times .7$?

G To which operation of real numbers does the property $(a + b) + c = a + (b + c)$ apply? To which operation does the property $(ab)c = a(bc)$ apply?

H Name the identity element for addition of real numbers. Name the identity element for multiplication of real numbers.

I Two real numbers whose sum is 0 are additive inverses of each other. Which property ex-

Universe for a, b, and c = D.		
Operations ⟶	Addition	Multiplication
Closure properties ⟶	For each a and b, $a + b \in D$.	For each a and b, $ab \in D$.
Commutative properties ⟶	For each a and b, $a + b = b + a$.	For each a and b, $ab = ba$.
Associative properties ⟶	For each a, b, and c, $(a + b) + c = a + (b + c)$.	For each a, b, and c, $(ab)c = a(bc)$.
Identity-element properties ⟶	For each a, $0 + a = a$.	For each a, $1 \times a = a$.
Inverse properties ⟶	For each a, there is an element ${}^{-}a$ so that $a + {}^{-}a = 0$.	For each $a \neq 0$, there is an element $1/a$ so that $a(1/a) = 1$.
Distributive property ⟶	For each a, b, and c, $a(b + c) = ab + ac$.	

D1

pressed in D1 tells you that each real number has an additive inverse?

J You can use the additive-inverse property of rational numbers to help you find the additive inverse of a real number that is also a rational number. What is the additive inverse of the real number 5? Of the real number $^-.32$?

The following examples of irrational numbers that are additive inverses of each other should help you find the additive inverses of other irrational numbers. The additive inverse of $\sqrt{11}$ is $^-(\sqrt{11})$ or $^-\sqrt{11}$. The additive inverse of $^-\sqrt{59}$ is $\sqrt{59}$.

K Name the additive inverse of $\sqrt{43}$. Of $^-\sqrt{6}$. Of $^-\sqrt{112}$.

Just as for rational numbers, the difference of any two real numbers is the number that is the sum of the minuend and the additive inverse of the subtrahend. $7 - \frac{5}{3} = 7 + {}^-\frac{5}{3}$. $\sqrt{13} - \sqrt{6} = \sqrt{13} + {}^-\sqrt{6}$.

L How do you know that the set of real numbers is closed under subtraction?

M Look again at D1. Two real numbers whose product is 1 are multiplicative inverses, or reciprocals, of each other. Which property tells you that each non-zero real number has a reciprocal?

N $\sqrt{16} \times 1/\sqrt{16} = 1$. What is the reciprocal of $\sqrt{16}$? Of $1/\sqrt{16}$? What is the reciprocal of 4.5? Of 1/4.5?

O Name the reciprocal of $\sqrt{145}$. Of $^-\sqrt{32}$.

The quotient of any two non-zero real numbers is the same as the product of the dividend and the reciprocal of the divisor.

$13.2 \div .4 = 13.2(1/.4)$.

$\sqrt{23} \div \sqrt{2} = \sqrt{23}(1/\sqrt{2})$.

P How do you know that the set of non-zero real numbers is closed under division?

Q Which property expressed in D1 relates multiplication and addition? Is $\sqrt{47}(95 + \sqrt{10}) = (\sqrt{47} \times 95) + (\sqrt{47} \times \sqrt{10})$?

R Does the set of real numbers, together with the operations of addition and multiplication, form a number system? Explain your answer.

So far in this lesson, you have studied certain properties of addition and multiplication of real numbers. Now you will study a property of the non-negative real numbers that is not a property of the non-negative rational numbers.

You have been working with the operations of addition and multiplication. Finding the square root of a number is also an operation.

A Think of the non-negative rational numbers 0, 16, 49, and 144. Is each of these numbers a perfect square? Name the square root of each number. Is the square root of each number a non-negative rational number?

B Think of the rational numbers 2, 3, and 5. Is each of these numbers a non-negative rational number? Is $\sqrt{2}$ a rational number? Is $\sqrt{3}$ a rational number? Is $\sqrt{5}$ a rational number?

C Explain how you know that the set of non-negative rational numbers is not closed under the operation of finding the square root of a number.

D Think of the non-negative real numbers 0, 2, 3, 4, 5, 16, and 21. Is $\sqrt{0}$ a non-negative real number? Is $\sqrt{2}$ a non-negative real number? Is the square root of each of the other numbers named in this exercise a non-negative real number?

The square root of each non-negative real number is a non-negative real number. Thus, the set of non-negative real numbers is closed under the operation of finding the square root. This closure property is expressed below.

The universe for a *is the set of non-negative real numbers. For each* a, $\sqrt{a}$ *is a non-negative real number.*

Next you will study the properties of equality for real numbers.

E What equations are expressed in D2? Are any inequalities expressed? Does each equation tell you that a real number is equal to itself?

The fact that a real number is always equal to itself is the *reflexive* (ri flek′siv) *property of equality for real numbers*. The property is expressed below.

The universe for a *is D. For each* a, a = a.

F Why is there no reflexive property of "less than," of "greater than," or of "not equal to" for real numbers?

G Is $3+5=4+4$? Is $4+4=3+5$? Is $\sqrt{16}=4$? Is $4=\sqrt{16}$? For each a and b, if $a=b$, can you conclude that $b=a$? The universe for a and b is D.

In exercise G, you used examples of the *symmetric* (si met′rik) *property of equality for real numbers*. The property is expressed below.

The universe for a *and* b *is D. For each* a *and* b, *if* a = b, *then* b = a.

H Why is there no symmetric property of "less than" or of "greater than" for real numbers?

You have already studied a transitive property of equality that tells you that, if a first number is the same as a second number and the second number is the same as a third number, then the first number is the same as the third number. The *transitive property of equality for real numbers* is expressed below.

The universe for a, b, *and* c *is D. For each* a, b, *and* c, *if* a = b *and* b = c, *then* a = c.

I Look at D3. Are statements G and H true?

J What is the sum of 2.5 and 3.25? What is the sum of $\frac{5}{2}$ and $\frac{13}{4}$? Is the sum of 2.5 and 3.25 the same as the sum of $\frac{5}{2}$ and $\frac{13}{4}$? Is statement I true?

The sum of 2.5 and 3.25 is the same real number as the sum of $\frac{5}{2}$ and $\frac{13}{4}$ because of the *well-defined property of addition of real numbers*. The property is expressed below.

D2

A $32 < 32$.
B $^-65 = {}^-65$.
C $107 = 107$.
D $^-354 > {}^-354$.
E $96 \neq 96$.
F $789 = 789$.

D3

G $2.5 = \frac{5}{2}$.
H $3.25 = \frac{13}{4}$.
I $2.5 + 3.25 = \frac{5}{2} + \frac{13}{4}$.
J $2.5 \times 3.25 = \frac{5}{2} \times \frac{13}{4}$.

The universe for a, b, c, *and* d *is D. For each* a, b, c, *and* d, *if* a = b *and* c = d, *then* a + c = b + d.

K What is the product of 2.5 and 3.25? Of $\frac{5}{2}$ and $\frac{13}{4}$? Is the product of 2.5 and 3.25 the same as the product of $\frac{5}{2}$ and $\frac{13}{4}$? Is statement J true?

The product of 2.5 and 3.25 is the same real number as the product of $\frac{5}{2}$ and $\frac{13}{4}$ because of the *well-defined property of multiplication of real numbers*. The property is expressed below.

The universe for a, b, c, *and* d *is D. For each* a, b, c, *and* d, *if* a = b *and* c = d, *then* ac = bd.

L How do you know that 7.9 + 8.5 is the same real number as $\frac{79}{10} + \frac{17}{2}$? How do you know that 7.9×8.5 is the same real number as $\frac{79}{10} \times \frac{17}{2}$?

For some time now, you have been working with sets, operations, and properties of operations. These properties are *algebraic* (al′jə-brā′ik) *properties*.

In this lesson you studied various properties of addition and multiplication of the real numbers. You also learned that the square root of any non-negative real number is a non-negative real number. Then you studied the reflexive, symmetric, and transitive properties of equal-

ity for the real numbers and the well-defined properties of addition and multiplication.

On your own

Tell whether each of the sentences below expresses a true statement or a false statement. For each true statement, name the property on which your decision depends.

1 $\sqrt{31} + \frac{1}{\sqrt{31}} = 1$.

2 $(\sqrt{\frac{2}{3}} + \frac{4}{9}) + \frac{8}{5} = (\frac{4}{9} + \sqrt{\frac{2}{3}}) + \frac{8}{5}$.

3 $\sqrt{16} + {}^{-}\sqrt{16} = 1$.

4 $\sqrt{\frac{11}{10}}(\frac{17}{4} \times \sqrt{\frac{15}{6}}) = \sqrt{\frac{11}{10}}(\sqrt{\frac{15}{6}} \times \frac{17}{4})$.

5 $1 \times \sqrt{89} = \sqrt{89}$.

6 $1 + \sqrt{6.2} = \sqrt{6.2}$.

7 $2(\sqrt{3} + 8) = 2\sqrt{3} + 16$.

8 $0 \times \sqrt{99} = \sqrt{99}$.

9 $0 + \sqrt{15} = \sqrt{15}$.

10 ${}^{-}\sqrt{123} + \sqrt{123} = 0$.

For each of exercises 11, 12, and 13, name the property, or properties, that tells you that the number expressed is a real number.

11 $15 + \sqrt{86}$ 12 $\sqrt{54} \times \sqrt{17}$ 13 $95 \div \sqrt{3}$

14 Which property of equality for real numbers tells you that $\sqrt{149} = \sqrt{149}$?

15 $\sqrt{1521} = 39$. Is $39 = \sqrt{1521}$? Which property of equality for real numbers did you use to answer this question?

16 $50 = \sqrt{2500}$. $2\sqrt{625} = \sqrt{2500}$. Which two properties of equality for real numbers can you use to explain why $50 = 2\sqrt{625}$?

17 Is $\sqrt{81} = 9$? Is $\sqrt{64} = 8$? Which properties tell you that $\sqrt{81} + \sqrt{64} = 17$? Which properties tell you that the product of $\sqrt{81}$ and $\sqrt{64}$ is 72?

For each of exercises 18 through 21, use the table of squares and square roots to find the two square roots named in the exercise. Then use properties to find the sum and the product of the numbers named in the exercise.

18 $\sqrt{10000}$, $\sqrt{141}$ 20 $\sqrt{6889}$, $\sqrt{900}$

19 $\sqrt{9801}$, $\sqrt{576}$ 21 $\sqrt{5625}$, $\sqrt{1764}$

Order and completeness properties

You have studied properties of equality for real numbers. In this lesson you will study properties of "less than," together with several other properties of real numbers.

A Look at the picture of a real-number line in D1. Is point .5 to the left of point 1, or is it to the right of point 1? Is .5 less than 1, or is it greater than 1?

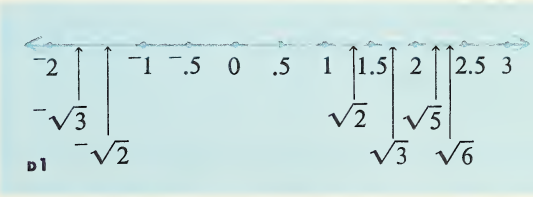
In a real-number line, if the point associated with a first number is to the left of the point associated with a second number, then the first number is less than the second number.

B Look again at D1. Is ${}^{-}2 < 0$? Is $\sqrt{5} < 2$? Is $\sqrt{2} < {}^{-}1$? Is $\sqrt{3} < 2$? Is $\sqrt{2} < 1$? Explain your answers.

You can also use infinite decimals to help you decide if one real number is less than another real number.

C $\sqrt{3} = 1.732\ldots$. Is $1\frac{3}{4} = 1.75\overline{0}$?

To decide whether the decimal for $\sqrt{3}$ or the decimal for $1\frac{3}{4}$ expresses the lesser number, you need to compare only the first three digits from the left in the two numerals. You can see that the first digit in each numeral expresses 1 one, and that the second digit expresses 7



D1

tenths; but the third digit in the numeral $1.732\ldots$ expresses 3 hundredths, while the third digit in the numeral $1.750\overline{0}$ expresses 5 hundredths.

D Which decimal, $1.732\ldots$ or $1.750\overline{0}$, expresses the lesser number? Is $\sqrt{3} < 1\frac{3}{4}$?

E Use the table of square roots to find a decimal approximation of $\sqrt{5}$. Is $\sqrt{5} < 2\frac{1}{4}$? Is $\sqrt{5} < 2\frac{1}{5}$? Explain your answers.

F Is $^{-}.\overline{3} < {}^{-}.32$? Explain your answer.

G Use the table of square roots to find a decimal approximation of $\sqrt{6}$. Is $^{-}\sqrt{6} < {}^{-}2.5$? Is $^{-}\sqrt{6} < {}^{-}2.448$? Explain your answers.

H How do you know, without using decimals, that $^{-}1 < \sqrt{2}$? Is $^{-}\sqrt{8} < \frac{1}{4}$? Is $^{-}\sqrt{92} < 3$?

Now you will see how the idea of "greater than" is related to the idea of "less than."

I You know that $^{-}\frac{1}{4} < \frac{1}{4}$. The inequality $\frac{1}{4} > {}^{-}\frac{1}{4}$ gives you the same information as $^{-}\frac{1}{4} < \frac{1}{4}$. Does $6 > 0$ give you the same information as $0 < 6$?

J What inequality that involves the idea of "greater than" gives you the same information as $\sqrt{14} < \sqrt{15}$? As $^{-}\frac{2}{3} < {}^{-}\frac{1}{3}$? As $^{-}.\overline{6} < {}^{-}.6$?

Now you see that, for each a and b, $b > a$ means $a < b$. The universe for a and b is D.

K Does $b > a$ mean $a < b$ if the universe for a and b is R? If the universe for a and b is I? Explain your answers.

Use the table of square roots to decide whether each of the following sentences expresses a true statement or a false statement.

L $^{-}\frac{2}{3} < {}^{-}\frac{3}{5}$. **O** $\sqrt{50} > 7.0\overline{7}$. **R** $6.\overline{9} > 6.99$.

M $\sqrt{4} > 2$. **P** $^{-}.11 < .1\overline{0}$. **S** $\sqrt{57} < 7.\overline{5}$.

N $0 < {}^{-}5$. **Q** $^{-}\sqrt{5} > {}^{-}\sqrt{3}$. **T** $6.\overline{3} < {}^{-}6.\overline{4}$.

In the following exercises, you will study three properties of "less than" for real numbers.

A Read sentences A and B in D2. Is statement A true? What real number has been added to both 2 and 3 to obtain statement B? Is statement B true?

A $2 < 3$.

B $2 + 5 < 3 + 5$.

C $\frac{1}{3} < \frac{1}{2}$.

D $\frac{1}{3} + {}^{-}\frac{2}{3} < \frac{1}{2} + {}^{-}\frac{2}{3}$.

D2

E $^{-}4 < \frac{3}{2}$.

F $^{-}4(\frac{2}{3}) < \frac{3}{2}(\frac{2}{3})$.

G $1.65 < 2.4$.

H $1.65(.3) < 2.4(.3)$.

D3

I $2 < 4$.

J $2({}^{-}\frac{1}{2}) > 4({}^{-}\frac{1}{2})$.

K $^{-}3\frac{1}{2} < {}^{-}1$.

L $^{-}3\frac{1}{2}({}^{-}10) > {}^{-}1({}^{-}10)$.

D4

B Do sentences C and D in D2 express true statements? What real number has been added to both $\frac{1}{3}$ and $\frac{1}{2}$ to obtain statement D?

The two pairs of statements expressed in D2 are examples of the *sum property of "less than" for real numbers*. For any three real numbers, if the first number is less than the second number, then the sum of the first and third numbers is less than the sum of the second and third numbers. The property is expressed below.

The universe for each variable is D. For each a, b, *and* c, *if* $a < b$, *then* $a + c < b + c$.

C You know that $\sqrt{2} = 1.414\ldots$ and that $\sqrt{3} = 1.732\ldots$. Is $\sqrt{2} < \sqrt{3}$? How do you know that $\sqrt{2} + 5.\overline{4} < \sqrt{3} + 5.\overline{4}$?

D The universe for a, b, and c is D. You know that $b > a$ means $a < b$. Does $b + c > a + c$

mean $a + c < b + c$? Use the idea of "greater than" to express in a different way the sum property of "less than" for real numbers.

E Now look at D3. Does sentence E express a true statement? Read sentence F. Both $^{-}4$ and $\frac{3}{2}$ have been multiplied by what third real number to obtain statement F? Is the third number positive? Is the third number greater than 0? Is statement F true?

F Is statement G expressed in D3 true? By what third real number have both 1.65 and 2.4 been multiplied to obtain statement H? Is the third number positive? Is this number greater than 0? Is statement H true?

The two pairs of statements expressed in D3 are examples of the *positive-multiplier property of "less than" for real numbers*. For any three real numbers, if the first number is less than the second number and if the third number is greater than 0, then the product of the first and third numbers is less than the product of the second and third numbers. The property is expressed below.

The universe for each variable is D. For each a, b, *and* c, *if* a $<$ b *and* c $>$ *0, then* ac $<$ bc.

G Use the table of square roots to find a decimal approximation of $\sqrt{19}$. Is $\sqrt{19} < 4.4$? Is $.2 > 0$? Is $\sqrt{19}(.2) < 4.4(.2)$? How do you know?

H The universe for a, b, and c is D. You know that $b > a$ means $a < b$. Does $bc > ac$ mean $ac < bc$? Use these ideas to express in a different way the positive-multiplier property.

Now you will discover what happens when you multiply both a first and a second number by a third number that is negative.

I Does sentence I in D4 express a true statement? By what third number have both 2 and 4 been multiplied to obtain statement J? Is the third number negative? Is $^{-}\frac{1}{2} < 0$? Is statement J true?

Universe for a, b, and c = D.

M For each a, b, and c, if $a < b$, then $a + c < b + c$.

N For each a, b, and c, if $a < b$ and $c > 0$, then $ac < bc$.

O For each a, b, and c, if $a < b$ and $c < 0$, then $ac > bc$.

D5

J Read sentences K and L in D4. Is statement K true? By what third number have both $^{-}3\frac{1}{2}$ and $^{-}1$ been multiplied to obtain statement L? Is $^{-}10 < 0$? Is statement L true?

Each of the two pairs of statements expressed in D4 is an example of the *negative-multiplier property of "less than" for real numbers*. For any three real numbers, if the first number is less than the second number and if the third number is less than 0, then the product of the first and third numbers is greater than the product of the second and third numbers. The property is expressed below.

The universe for each variable is D. For each a, b, *and* c, *if* a $<$ b *and* c $<$ *0, then* ac $>$ bc.

K Explain how the positive-multiplier property is different from the negative-multiplier property.

L Is $1.3 < 4.2$? Is $^{-}\sqrt{5} < 0$? How do you know that $1.3(^{-}\sqrt{5}) > 4.2(^{-}\sqrt{5})$?

M The universe for a, b, and c is D. You know that $b > a$ means $a < b$. Does $bc < ac$ mean $ac > bc$? Use these ideas to express in a different way the negative-multiplier property.

N Look at D5. Name the three properties expressed in the display.

Two statements are expressed in each of exercises O through V. The first statement is true. Without computing, decide if the second statement is also true. If so, tell what property of "less than" you used to make your decision.

O $\frac{1}{5} > \bar{}\frac{1}{5}$.
$\frac{1}{5} + 3 > \bar{}\frac{1}{5} + 3$.

P $\bar{}2 < 3$.
$\bar{}2(\frac{1}{4}) > 3(\frac{1}{4})$.

Q $2.5 > \bar{}9$.
$2.5(2) > \bar{}9(2)$.

R $\bar{}8 < \bar{}2$.
$\bar{}8(4) < \bar{}2(4)$.

S $\bar{}5 > \bar{}7$.
$\bar{}5(\bar{}\sqrt{3}) > \bar{}7(\bar{}\sqrt{3})$.

T $4 < \sqrt{20}$.
$4(\bar{}3) < \sqrt{20}(\bar{}3)$.

U $3.5 < 3.51$.
$3.51(.9) < 3.5(.9)$.

V $\sqrt{6} < \sqrt{8}$.
$\sqrt{8} + 3 < \sqrt{6} + 3$.

Now you will study two other properties of the set of real numbers.

A Is it possible to find at least one rational number between any two given rational numbers? Does R have the density property?

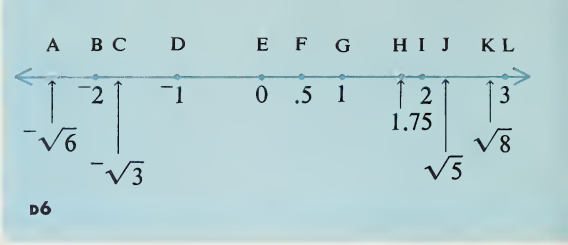
> REMINDER
> The set of rational numbers has the density property because there is at least one rational number between any two given rational numbers.
> *See lesson 147, page 243.*

B You know that R is a subset of D. Do you think that it is possible to find at least one real number between any two given real numbers? Do you think that the set of real numbers has the density property?

The set of real numbers is dense because it is possible to find at least one real number between any two given real numbers. This property is the *density property of the set of real numbers*. The property is expressed below.

The universe for each variable is D. For each x *and* y, *if* $x < y$, *then there is a real number* d *such that* $x < d < y$.

C Name a real number between $\bar{}2$ and 0. Name a real number between $\sqrt{8}$ and 5. Name a real number between $\bar{}.\bar{3}$ and $\bar{}.3331$. Name a real number between $\sqrt{5}$ and $\sqrt{10}$.



D6

D A real-number line is represented in D6. What is the coördinate of point A? Is the coordinate of point A a member of D? A member of R? A member of $\overline{R}$?

E The coördinate of each point that is represented by a dot in D6 is a real number. For each coördinate, tell whether it is a rational number or an irrational number.

In lesson 151 you studied the correspondence between the real numbers and the points in a line. This property is the *completeness property of the set of real numbers*. The property is expressed below.

There is a one-to-one correspondence between the set of real numbers and the set of points in a line.

F Does R have the completeness property? Does I have the completeness property? Explain your answers.

In this lesson you learned that the real numbers can be ordered according to the idea of "less than." You studied the following properties of "less than": the sum property, the positive-multiplier property, and the negative-multiplier property. You also learned that the set of real numbers is dense and complete.

On your own

For each of exercises 1 through 6, list in order the real numbers named. Begin with the name of the least number.

1 $\bar{}\frac{1}{2}, \bar{}2, 0, 3, \bar{}2\frac{1}{2}, 1\frac{3}{4}, \bar{}2\frac{1}{4}, \bar{}3\frac{1}{5}$

2 $.02, \bar{}.3, 2.2, \bar{}.03, \bar{}1.8, 1.6$

3 $\sqrt{7}, \bar{}3, \bar{}\sqrt{15}, 2, \sqrt{11}, \bar{}8$

4 .09, .1, $^-.\overline{6}$, $^-.5$, .01, $^-.675$

5 $^-\frac{1}{6}$, $^-\frac{1}{5}$, $^-\frac{1}{8}$, $^-\frac{1}{3}$, $^-\frac{1}{2}$, $^-\frac{1}{10}$

6 $\sqrt{12}$, 3.46, 3.47, 3.465, 3.463

Two statements about real numbers are expressed in each of exercises 7 through 12. In each case, the first statement is true. Without computing, decide if the second statement is also true. If it is, tell what property of "less than" you used to make your decision.

7 $3 > 2.8$.
$3 + {}^-8 > 2.8 + {}^-8$.

8 $^-5 < 0$.
$0\sqrt{11} > {}^-5\sqrt{11}$.

9 $^-\frac{1}{2} < {}^-\frac{1}{3}$.
$^-\frac{1}{2}(^-\frac{9}{10}) < {}^-\frac{1}{3}(^-\frac{9}{10})$.

10 $.\overline{9} > .\overline{3}$.
$.\overline{9}(^-1) < .\overline{3}(^-1)$.

11 $.7 < 2$.
$.7(2) > 2(2)$.

12 $\sqrt{5} < 3$.
$3 + 3 < \sqrt{5} + 3$.

13 Which members of $\{^-1\frac{3}{10}, 1\frac{1}{2}, 1\frac{3}{10}, {}^-2, 1\frac{1}{4}\}$ are between $1\frac{1}{5}$ and $1\frac{2}{5}$?

14 Which members of $\{^-4.432, {}^-4.51, 0, 4.45, {}^-4.499\}$ are between $^-4.4$ and $^-4.5$?

15 Which elements of $\{3.4, \sqrt{10}, {}^-3.2, 3.2, 2.\overline{9}, 3.3\}$ are between $\sqrt{9}$ and $\sqrt{11}$?

16 Which members of $\{0, .1, {}^-.012, .\overline{02}, .001\}$ are between $.\overline{01}$ and .02?

17 Which elements of $\{^-2.2, {}^-2.05, {}^-2.235, {}^-2.1, {}^-2.\overline{2}\}$ are between $^-\sqrt{5}$ and $^-2$?

KEEPING SKILFUL

For each condition expressed in exercises 1 through 6, tabulate the solution set if possible. Otherwise, give a standard description of the solution set. U = R.

1 $10\frac{1}{2} - y = {}^-30\frac{1}{4}$.

2 $^-13\frac{1}{4} + n = {}^-62\frac{1}{3}$.

3 $x - 67.4 < 100.8$.

4 $\sim(^-12.3 + m > 21)$.

5 $5w - 6.5 > 4.25 \wedge 2w + {}^-8.3 < 2.9$.

6 $2\frac{1}{2} + z < \frac{5}{8} \vee 2z + {}^-\frac{3}{4} < 9\frac{3}{4}$.

Find the positive square root of each number named to the nearer thousandth.

7 150	9 20	11 525	13 1364
8 883	10 722	12 931	14 1484

APPLYING MATHEMATICS

For each problem, first write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem.

For problems 1 through 6, the universe for each variable is R.

1 Roberta buys her lunch in the school cafeteria. She spent \$.55 for her lunch on Monday, \$.40 on Tuesday, \$.60 on Wednesday, \$.30 on Thursday, and \$.50 on Friday. How much did Roberta spend for lunch on these five days?

2 Last month, George earned more than \$12.50. He earned less than John and Tony earned together. John earned \$5.60 last month, and Tony earned \$9.30. How much did George earn last month?

3 At 6:00 A.M. the temperature in Maple City was below zero. By noon the temperature had risen 10 degrees and was more than 5 degrees above zero. What was the temperature in Maple City at 6:00 A.M.?

4 What numbers can be added to 10 so that each sum is either $^-2\frac{3}{4}$ or $5\frac{7}{8}$?

5 If a certain number is added to the product of $3\frac{3}{4}$ and $\frac{4}{9}$, the sum is $2\frac{1}{2}$. What is the number?

6 What numbers can be subtracted from 12.25 so that each difference is not greater than 2.5?

For problems 7 through 17, the universe for each variable is R_p.

7 Oakville is $40\frac{1}{2}$ mi. west of Prospect City. Lakeview is $20\frac{7}{8}$ mi. east of Prospect City. The distance between Oakville and Prospect City is how many miles more than the distance between Lakeview and Prospect City?

8 Mr. Young had less than 32 bu. of tomatoes. After he had sold $20\frac{1}{2}$ bu. of them, he still had more than $7\frac{1}{2}$ bu. left. How many bushels

of tomatoes did he have before he sold the $20\frac{1}{2}$ bu.?

9 Hilda has $4\frac{1}{2}$ yd. of linen. To make a suit, she needs $3\frac{3}{4}$ yd. of the linen. To make a dress, she needs $2\frac{7}{8}$ yd. of the linen. If Hilda makes either a suit or a dress, how many yards of linen will she have left?

10 Mr. Holland paid \$3.69 for 10 gal. of gasoline. At this rate, how much would he pay for 15 gal. of gasoline?

11 During practice for an archery contest, $66\frac{2}{3}\%$ of Bob's shots hit the bull's-eye. 52 of his shots hit the bull's-eye. How many shots did Bob make?

12 Mr. Myers can plant $2\frac{1}{3}$ acres of corn per hour with a two-row corn planter. He can plant $4\frac{3}{4}$ acres of corn per hour with a four-row corn planter. If Mr. Myers uses either a two-row planter or a four-row planter, how many acres of corn can he plant in $8\frac{1}{2}$ hours?

13 The area of the earth's surface is about 2×10^8 square miles. About 71% of the earth's surface is covered with water. About how many square miles of the earth's surface are covered with water?

14 The speed of sound in wood is about 1.0×10^4 feet per second. The speed of sound in air is about 1.1×10^3 feet per second. The speed of sound in wood is about how many times the speed of sound in air? (Tenths)

15 The length of a rectangular parking lot is 670 feet. The width of the lot is $195\frac{3}{4}$ feet. What is the area of the parking lot in square feet?

16 The length of an altitude of rhombus ABCD is 20.5 inches. The length of the base is 35.4 inches. What is the area of rhombus ABCD in square inches?

17 The length of an altitude of triangle JKL is 2.7 ft. The length of the base is 6.2 ft. What is the area of the triangle in square feet?

SPECIAL CHALLENGE

In this Special Challenge you will work only with integers that are expressed by base-ten numerals.

A What do we mean when we say that 40 is not divisible by 7? That 27 is divisible by 3? That 0 is divisible by each positive integer?

B When you divide a given integer by 7, how can you tell from the remainder you obtain whether or not the given integer is divisible by 7?

C What is the greatest remainder you can obtain when you divide positive integers by 3? By 5? By 13?

D Look at D1. The chart expresses the remainders obtained when certain powers of 10 are divided by 3, 7, 9, and 11. The numerals for these divisors are shown in blue. What can you say about the remainders that are obtained when the powers of 10 expressed in D1 are divided by 3? By 7? By 9? By 11?

E Make a chart listing the names of the powers of 10 that are expressed in D1. In your chart, name the remainders you obtain when you divide these powers of 10 by 13 and by 17.

F Now you will determine whether 275,462 is divisible by 3, by 7, by 9, or by 11. What is the expanded form of the decimal 275,462?

	3	7	9	11
10^6	1	1	1	1
10^5	1	5	1	10
10^4	1	4	1	1
10^3	1	6	1	10
10^2	1	2	1	1
10^1	1	3	1	10
10^0	1	1	1	1

D1

275,462

	3	7	9	11
$2(10^5)$	2	3	2	9
$7(10^4)$	1	0	7	7
$5(10^3)$	2	2	5	6
$4(10^2)$	1	1	4	4
$6(10^1)$	0	4	6	5
$2(10^0)$	2	2	2	2
	~~8~~	~~12~~	~~26~~	~~33~~
	2	5	8	0

D2

G What remainder do you obtain when you divide 10^3 by 7? What is the product of this remainder and 5?

H What remainder do you obtain when you divide 30 by 7?

I Now look at D2. The chart names the remainders obtained when the number expressed by each digit in the decimal 275,462 is divided by 3, 7, 9, and 11. In D2, what remainder is expressed for $5(10^3) \div 7$? How does this compare with your answer for exercise H?

J How can you use the chart in D1 to help find the remainder for $2(10^5) \div 3$? For $6(10^1) \div 7$? For $7(10^4) \div 9$? For $6(10^1) \div 11$?

K In D2 the numeral 8 expresses the "sum of the remainders" 2, 1, 2, 1, 0, and 2. What do the numerals 12, 26, and 33 express? Why were these numerals crossed out, and the numerals 2, 5, 8, and 0 written below them?

L 8 is not divisible by 3. So 275,462 is not divisible by 3. Is 275,462 divisible by 9?

M How do you know that 275,462 is not divisible by 7? That 275,462 is divisible by 11?

N Use the chart that you made for exercise E to make another chart similar to the one in D2. In your chart, name the remainder you obtain when the number expressed by each digit in the numeral 275,462 is divided by 13. By 17. What remainder do you obtain when you divide 275,462 by 13? By 17?

O Is 275,462 divisible by either 13 or 17? How do you know?

P If the "sum of the digits" of an integer is divisible by 3, then the integer is divisible by 3. Is 47,652 divisible by 3? Use the "sum of the digits" of 32,985 to explain how you can determine if an integer is divisible by 9.

Q To determine if 34,914 is divisible by 4, you can divide 14 by 4 and observe the remainder. Is 34,914 divisible by 4? What is a simple method of determining if an integer is divisible by 4? Explain your answer.

R How do you know without actually carrying out the division process that 4791 is not divisible by 2? That 4791 is divisible by 3?

S How do you know without actually carrying out the division process that 29,486 is divisible by 2? That 29,486 is not divisible by 3?

T What is a simple method of determining if an integer expressed by a decimal is divisible by 6?

U To determine if 391,384 is divisible by 8, you can divide 384 by 8 and observe the remainder. Is 391,384 divisible by 8?

V What is a simple method of determining if a given integer is divisible by 5? By 10?

W If each of two given integers is divisible by 7, what do you know about the sum of the given integers? What do you know about the product of the given integers?

X What remainders do you obtain when you divide 312,664 by each of the integers from 2 through 11?

Y What remainders do you obtain when you divide 118,818 by each of the integers from 2 through 11?

Z What is the complete factorization of 55,495,440?

CHECKING UP

If you have trouble with these tests, you can find help in lessons 149 through 153.

Test 140

Use the table of squares and square roots to help you find the square or the square root expressed in each of exercises 1 through 9.

1 $\sqrt{17}$ 4 $\sqrt{3721}$ 7 158^2

2 23^2 5 $\sqrt{1849}$ 8 $\sqrt{43}$

3 209^2 6 $(^{-}71)^2$ 9 $\sqrt{30{,}625}$

For each of exercises 10 through 13, compute to find the square root of the number named to the nearer hundredth.

10 240 11 310 12 475 13 768

Test 141

In each of exercises 14, 15, and 16, an example of a property of equality for real numbers is expressed. Name the property.

14 If $69 = \sqrt{4761}$, then $\sqrt{4761} = 69$.

15 If $24 = (\sqrt{24})^2$, and $(\sqrt{24})^2 = \sqrt{576}$, then $24 = \sqrt{576}$.

16 $^{-}\sqrt{801} = {}^{-}\sqrt{801}$.

Each number named in exercises 17 through 20 belongs to at least one subset of D. Name the subsets of D of which the number is an element. Use the following subsets: $\overline{R}$, R, R_p, R_n, I, I_p, and I_n.

17 $^{-}\frac{5}{4}$ 18 35 19 $^{-}\sqrt{36}$ 20 $^{+}.\overline{79}$

In each of exercises 21 through 28, an example of a property of D is expressed. Name the property.

21 If $\sqrt{16} = 4$ and $5 = \sqrt{25}$, then $\sqrt{16} + 5 = 4 + \sqrt{25}$.

22 $\sqrt{8873} + .07\overline{6} \in$ D.

23 $.00\overline{4}(1/.00\overline{4}) = 1$.

24 $1 \times \sqrt{915} = \sqrt{915}$.

25 If $\sqrt{67} < 9\frac{1}{2}$, then $\sqrt{67} + {}^{-}\frac{5}{8} < 9\frac{1}{2} + {}^{-}\frac{5}{8}$.

26 $13(\sqrt{67} + 31) = (13\sqrt{67}) + (13 \times 31)$.

27 If $^{-}7.5 < {}^{-}3.8$, then $^{-}7.5\sqrt{6} < {}^{-}3.8\sqrt{6}$.

28 $\sqrt{690} + 4.5 = 4.5 + \sqrt{690}$.

Solution sets of conditions for equality

In lesson 146 you learned to obtain, or derive, equivalent conditions by using certain definitions and properties. In this lesson you will use the same definitions and properties, together with the distributive property, to derive equivalent conditions.

A What condition is expressed by sentence A in D1? What is the universe for x?

To find the solution set of condition A, remember that you want to derive a condition that is equivalent to condition A, but that enables you to think of x rather than of $x - 4\frac{1}{2}$.

B Compare sentences A and B. For each x, is $x - 4\frac{1}{2}$ the same as $x + {}^{-}4\frac{1}{2}$? How do you know?

C Use the transitive property of equality to explain why condition B is equivalent to condition A.

D To obtain a condition that enables you to think of x rather than of $x + {}^{-}4\frac{1}{2}$, you can first find a condition that involves $x + 0$. Look at sentence C in D1. What number can you add to $x + {}^{-}4\frac{1}{2}$ to obtain $x + 0$?

From condition B, you know that you want to find an x so that $x + {}^{-}4\frac{1}{2}$ is equal to $^{-}\frac{1}{8}$. Therefore, in $(x + {}^{-}4\frac{1}{2}) + ({}^{+}4\frac{1}{2})$, you can replace $(x + {}^{-}4\frac{1}{2})$ by $^{-}\frac{1}{8}$ without affecting the sum. The sum $^{-}\frac{1}{8} + ({}^{+}4\frac{1}{2})$ is the same as the sum $(x + {}^{-}4\frac{1}{2}) + ({}^{+}4\frac{1}{2})$. The x that satisfies condition B also satisfies condition C.

E Look again at sentence C. To obtain a condition that involves $x + 0$, you must regroup

U = D.

A $x - 4\frac{1}{2} = {}^{-}\frac{1}{8}$.

B $x + {}^{-}4\frac{1}{2} = {}^{-}\frac{1}{8}$.

C $(x + {}^{-}4\frac{1}{2}) + ({}^{+}4\frac{1}{2}) = {}^{-}\frac{1}{8} + ({}^{+}4\frac{1}{2})$.

D $x + ({}^{-}4\frac{1}{2} + {}^{+}4\frac{1}{2}) = {}^{-}\frac{1}{8} + {}^{+}4\frac{1}{2}$.

E $x + 0 = {}^{-}\frac{1}{8} + {}^{+}4\frac{1}{2}$.

F $x = 4\frac{3}{8}$.

D1

U = D.

G $x - 4\frac{1}{2} = {}^{-}\frac{1}{8}$.

H $x + 0 = {}^{-}\frac{1}{8} + 4\frac{1}{2}$.

I $x = 4\frac{3}{8}$.

D2

U = D.

J $2x + 3x = 10$.

K $(2 + 3)x = 10$.

L $5x = 10$.

M $x = 2$.

D3

the numbers in $(x + {}^{-}4\frac{1}{2}) + ({}^{+}4\frac{1}{2})$. Which property of addition tells you that, for each x, $(x + {}^{-}4\frac{1}{2}) + {}^{+}4\frac{1}{2} = x + ({}^{-}4\frac{1}{2} + {}^{+}4\frac{1}{2})$? Is condition D equivalent to each of conditions A, B, and C?

F Compare sentences D and E. How do you know that $x + ({}^{-}4\frac{1}{2} + {}^{+}4\frac{1}{2}) = x + 0$, for each x? Use the transitive property of equality to explain why conditions E and D are equivalent.

G Is condition E also equivalent to each of conditions A, B, and C?

H For each x, is $x + 0 = x$? Upon which property of real numbers does your answer depend?

I What is a standard name of ${}^{-}\frac{1}{8} + {}^{+}4\frac{1}{2}$? Is condition F equivalent to condition E and to each of the other conditions expressed in D1?

J What number satisfies condition F? Tabulate $\{x | x - 4\frac{1}{2} = {}^{-}\frac{1}{8}\}$.

As you develop skill in finding the solution sets of conditions like $x - 4\frac{1}{2} = {}^{-}\frac{1}{8}$, you may not need to write a sentence for each of the equivalent conditions in a chain. D2 shows the sentences you might write to find the solution set of condition A. You should always be able to tell what properties you used to obtain each condition from the preceding condition.

Next you will use the distributive property to help derive equivalent conditions. First you will see that the distributive property may be expressed in more than one form.

K As you know $a(b + c) = ab + ac$, for each a, b, and c, when the universe for each variable is D. What property tells you that $ab + ac = ba + ca$? What property tells you that $a(b + c) = ba + ca$?

L What property tells you that $(b + c)a = a(b + c)$, for each a, b, and c? What property tells you that $(b + c)a = ba + ca$?

From now on in this book, we will use both forms of the distributive property of real numbers. That is, we will use both $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

M Now look at D3. What condition is expressed by sentence J? To find the solution set of $2x + 3x = 10$, it is convenient first to find the sum of $2x$ and $3x$. From the distributive property of real numbers you know that, for each x, the sum of $2x$ and $3x$ is equal to $(2 + 3)x$.

N From condition J you know that you want to find an x such that $2x + 3x$ is equal to 10. Use the transitive property of equality to explain why condition K is equivalent to condition J.

O Is $2+3=5$? Which property of real numbers tells you that $(2+3)x=5x$, for each x? Is condition L equivalent to condition K? Is condition L also equivalent to condition J?

P Explain how to derive condition M from condition L. What number satisfies $x=2$?

Q Tabulate $\{x|2x+3x=10\}$.

R One way to make sure that you have made no mistakes in deriving a chain of equivalent conditions is to see whether or not each member of the solution set of the last condition satisfies the original condition. Look again at D3. What is the original condition? What is $\{x|x=2\}$? Does 2 satisfy $2x+3x=10$? Explain your answer.

S Another way to decide if you have derived a chain of equivalent conditions is to see if you can derive the original condition from the last condition in the chain. Look again at D3. How can you obtain condition L from condition M? How can you obtain condition K from condition L? How can you obtain condition J from condition K? Can you derive $2x+3x=10$ from $x=2$?

U = D.

J $2x+3x=10$.

K $(2+3)x=10$.

L $5x=10$.

M $x=2$.

D3

U = D.

A $10a=7(a+2)$.

B $10a=7a+14$.

C $^{-}(7a)+10a=^{-}(7a)+(7a+14)$.

D $^{-}(7a)+10a=[^{-}(7a)+7a]+14$.

E $^{-}(7a)+10a=0+14$.

F $^{-}(7a)+10a=14$.

G $^{-}(7a)+(7a+3a)=14$.

H $[^{-}(7a)+7a]+3a=14$.

I $0+3a=14$.

J $3a=14$.

K $a=\frac{14}{3}$.

D4

In the following exercises, you will use the distributive property to help you obtain the solution set of another condition.

A What condition is expressed by sentence A in D4?

B For each a, is $7(a+2)=7a+14$? Upon which property of real numbers does your answer depend? How do you know that condition B is equivalent to condition A?

To find the solution set of condition B, you want to obtain a condition that enables you to think of a rather than of $10a$ and $7a$.

C You can first find some number that you can add to $7a+14$ to obtain $0+14$. $^{-}(7a)$ is the additive inverse of $7a$. Is $^{-}(7a)+7a=0$, for each a?

D Compare sentences B and C in D4. Which property of addition can you use to obtain condition C from condition B?

E Compare sentences C and D. Which property of addition tells you that, for each a, $^{-}(7a)+(7a+14)=[^{-}(7a)+7a]+14$? Explain why conditions D and C are equivalent.

F Compare sentences D and E. Which property of real numbers tells you that, for each a, $[^{-}(7a)+7a]+14=0+14$? Use the transitive property of equality to help you explain why condition E is equivalent to condition D.

G How do you know that $0+14=14$? Is condition F equivalent to condition E?

H Compare sentences F and G. To find the sum of $^{-}(7a)$ and $10a$, you can first express

$10a$ as the sum of $7a$ and $3a$. Use the distributive property to help you explain why $10a = 7a + 3a$. Are conditions G and F equivalent?

I Compare sentences G and H. How do you know that condition H is equivalent to condition G?

J How can you derive condition I from condition H? How can you derive condition J from condition I?

K Explain how to derive condition K from condition J. What number satisfies $a = \frac{14}{3}$?

L Tabulate $\{a \mid 10a = 7(a + 2)\}$.

M What statement do you obtain when you replace a by $\frac{14}{3}$ in $10a = 7(a + 2)$? Is the statement you obtained a true statement?

So far in this lesson, you have found the solution sets of conditions for equality. Next you will find the solution set of a condition for equivalence.

Look at D5. Notice that the universe for m is D. In *Book 1*, and so far in this book, the universe for the variable in a condition for equivalence has not contained 0 or any negative numbers. However, from now on, we will accept 0 and negative numbers as components of rate pairs.

A Is condition L, expressed in D5, a condition for equivalence? Name the components of each rate pair.

In the following exercises, you are going to derive a condition for equality that has the same solution set as condition L and that tells you that m is equal to a certain number.

B You can use the definition of equivalent ordered pairs to help you find the real number that satisfies a condition for equivalence. How can you use this definition to obtain condition M from condition L?

C Compare sentences M and N. You know that multiplication distributes over addition of real numbers. Explain why multiplication distributes over subtraction of real numbers. For each m, is $.8(m - .5) = .8m - .8(.5)$?

D For each m, is $.8m - .8(.5) = .8m - .4$? Is $.8(m - .5) = .8m - .4$? Is condition N equivalent to condition M? Upon which property of equality does your answer depend?

E Compare sentences N and O. For each m, is $.8m - .4 = .8m + {}^{-}.4$? How do you know? Is condition O equivalent to condition N?

F Look at sentence P. What number has been added to $.8m + {}^{-}.4$ and also to 7.2? Which property of real numbers tells you that condition P is equivalent to condition O?

G Compare sentences P and Q. Which property of real numbers tells you that condition Q is equivalent to condition P?

H How can you derive condition R from condition Q? Are conditions R and Q equivalent?

I Why are conditions S and R equivalent?

J Name the reciprocal of .8. Explain how you can obtain condition T from condition S.

K What number satisfies condition T?

L Tabulate $\{m \mid 1/(m - .5) \sim .8/7.2\}$.

M Do you obtain a true statement when you replace m by 9.5 in $1/(m - .5) \sim .8/7.2$?

U = D.

L $1/(m - .5) \sim .8/7.2$.

M $7.2 = .8(m - .5)$.

N $7.2 = .8m - .4$.

O $7.2 = .8m + {}^{-}.4$.

P $7.2 + ({}^{+}.4) = (.8m + {}^{-}.4) + ({}^{+}.4)$.

Q $7.2 + ({}^{+}.4) = .8m + ({}^{-}.4 + {}^{+}.4)$.

R $7.2 + ({}^{+}.4) = .8m + 0$.

S $7.6 = .8m$.

T $9.5 = m$.

D5

In this lesson you reviewed how to derive a chain of equivalent conditions and how to use these conditions to find the solution set of a given condition. You learned how to use the distributive property to derive equivalent conditions, and you also learned how to check your work in deriving equivalent conditions.

On your own

For each of exercises 1 through 10, tell whether the sentence expresses a true statement or a false statement. U = D.

1 $\{c \mid \frac{9}{14} = c + \frac{5}{7}\} = \{\frac{1}{14}\}$.

2 $\{x \mid \overline{}\frac{4}{3} + x = \frac{1}{6}\} = \{\overline{}\frac{1}{12}\}$.

3 $\{n \mid 13.2 = 5.7 - n\} = \{7.5\}$.

4 $\{w \mid w - \frac{13}{8} = \overline{}\frac{3}{4}\} = \{\frac{7}{8}\}$.

5 $\{x \mid 8/(x - .3) \sim 36/9\} = \{2.3\}$.

6 $\{x \mid 4x + 39x = \overline{}172\} = \{\overline{}4\}$.

7 $\{x \mid 15(x + \overline{}.2) = 21x\} = \{.5\}$.

8 $\{z \mid 150/175 \sim (z + 2)/77\} = \{66\}$.

9 $\{x \mid .9/2 \sim .3/(4 + x)\} = \{3.\overline{3}\}$.

10 $\{x \mid 1.9x = 2.5(x + .2)\} = \{.83\}$.

For each of exercises 11 through 14, first write sentences to express a chain of equivalent conditions. The last condition should tell you that x is equal to a certain number. Then tell what definition or property you used to obtain each condition. Finally, tabulate the solution set of the original condition. U = D.

11 $\frac{1}{5}(x - \frac{2}{3}) = \frac{6}{5}$.

12 $13x + 7x = \overline{}180$.

13 $(5 - x)/2 \sim 6/1$.

14 $5(x - 3) = 2x$.

Tabulate the solution set of each condition expressed in exercises 15 through 26. U = D.

15 $\overline{}29 = b + 25$.

16 $\overline{}2.45 = m - 6.09$.

17 $\frac{3}{4}x - \frac{5}{8}x = \overline{}\frac{13}{8}$.

18 $\frac{2}{3} = \frac{1}{9}(8 + c)$.

19 $\frac{4}{9}x - \frac{8}{3}x = \overline{}\frac{10}{9}$.

20 $\frac{7}{8} - a = 1\frac{1}{2}$.

21 $\overline{}21\frac{2}{3} = \overline{}32 + w$.

22 $15\frac{1}{3} = 12\frac{1}{6}x + 10\frac{5}{6}x$.

23 $4/(y + 2) \sim 5/3$.

24 $16x + 5 = 21x$.

25 $1.3/2.6 \sim 1/(x - .1)$.

26 $\frac{7}{10}k = \frac{2}{5}k - \frac{11}{15}$.

Solution sets of conditions for inequality

You have learned how to use certain properties and definitions to help find the solution sets of conditions for equality. In this lesson you will see that you can use properties of "less than" to find the solution sets of conditions for inequality.

Look at D1. In the work that follows, you will derive a condition that is equivalent to condition A, but that enables you to think of y rather than of $4y$.

A Compare sentences A and B in D1. To obtain condition B from condition A, what number has been multiplied by both $4y$ and 7? Explain why we have chosen to use the reciprocal of 4.

B Is $\frac{1}{4}$ greater than 0? For each y, if $4y < 7$, is $\frac{1}{4}(4y) < \frac{1}{4}(7)$? Upon what property of "less than" does your answer depend?

C Is condition B equivalent to condition A?

D What property of multiplication can you use to obtain condition C from condition B? Is condition C equivalent to condition B?

U = D.

A $4y < 7$.

B $\frac{1}{4}(4y) < \frac{1}{4}(7)$.

C $(\frac{1}{4} \cdot 4)y < \frac{1}{4}(7)$.

D $1y < \frac{1}{4}(7)$.

E $y < \frac{7}{4}$.

D1

U = D.

F $y < \frac{7}{4}$.
G $4 \cdot y < 4 \cdot \frac{7}{4}$.
H $4y < 7$.

D2

U = D.

I $^{-}3x < 4$.
J $^{-}\frac{1}{3}(^{-}3x) > ^{-}\frac{1}{3}(4)$.
K $(^{-}\frac{1}{3} \cdot ^{-}3)x > ^{-}\frac{1}{3}(4)$.
L $1x > ^{-}\frac{1}{3}(4)$.
M $x > ^{-}\frac{4}{3}$.

D3

U = D.

N $3 - m > ^{-}2$.
O $3 + ^{-}m > ^{-}2$.
P $^{-}3 + (3 + ^{-}m) > ^{-}3 + (^{-}2)$.
Q $(^{-}3 + 3) + ^{-}m > ^{-}3 + (^{-}2)$.
R $0 + ^{-}m > ^{-}3 + (^{-}2)$.
S $^{-}m > ^{-}5$.
T $5 > m$.

D4

E How can you obtain condition D from condition C? How can you obtain condition E from condition D? Is condition D equivalent to condition C? Is condition E equivalent to condition D?

F Is condition E equivalent to condition A?

G Use condition E to help you give a standard description of the solution set of condition A.

H Why can you not tabulate $\{y \mid y < \frac{7}{4}\}$?

In lesson 154 you studied two methods of checking to see if you made any mistakes in deriving a chain of equivalent conditions for equality. One method is to see whether or not each member of the solution set of the last condition in the chain satisfies the original condition. The other method is to try to derive the original condition from the last condition.

I Why is it impossible to try each member of $\{y \mid y < \frac{7}{4}\}$ to see if it satisfies $4y < 7$?

J Now look at D2. Use sentences F, G, and H to help you explain how you can obtain $4y < 7$ from $y < \frac{7}{4}$.

Look at D3. Notice that the condition expressed by sentence I involves the product of $^{-}3$ and x. To find the solution set of condition I, you will obtain an equivalent condition that enables you to think of x rather than of $^{-}3x$.

A Compare sentences I and J. What is the reciprocal of $^{-}3$? In condition J, what number has been multiplied by both $^{-}3x$ and $^{-}4$?

B Is $^{-}\frac{1}{3}$ less than 0? For each x, if $^{-}3x < 4$, is $^{-}\frac{1}{3}(^{-}3x) > ^{-}\frac{1}{3}(4)$? Upon what property of "less than" does your answer depend?

C Explain why condition K is equivalent to condition J.

D How do you know that condition L is equivalent to condition K? That condition M is equivalent to condition L?

E Use condition M to help you give a standard description of the solution set of $^{-}3x < 4$.

Now look at D4. The sentences in D4 will help you find the solution set of condition N.

F Compare sentences N and O. What property of real numbers tells you that $3 - m = 3 + ^{-}m$? Is condition O equivalent to condition N?

G Compare sentences O and P. You are going to obtain a condition that enables you to think of ^{-}m rather than of $3 + ^{-}m$. Notice that, in condition P, both $3 + ^{-}m$ and $^{-}2$ have been added to $^{-}3$. Explain why the additive inverse of 3 is used in condition P.

U = D.

N $3 - m > {}^{-}2$.

O $3 + {}^{-}m > {}^{-}2$.

P ${}^{-}3 + (3 + {}^{-}m) > {}^{-}3 + ({}^{-}2)$.

Q $({}^{-}3 + 3) + {}^{-}m > {}^{-}3 + ({}^{-}2)$.

R $0 + {}^{-}m > {}^{-}3 + ({}^{-}2)$.

S ${}^{-}m > {}^{-}5$.

T $5 > m$.

D4

U = D.

U ${}^{-}6a + 2a < 8 \wedge a - \frac{7}{8} < \frac{1}{8}$.

D5

U = D.

$\{a | {}^{-}2 < a < 1\}$

D6

H What property tells you that, for each ? m, if $3 + {}^{-}m > {}^{-}2$, then $({}^{-}3 + (3 + {}^{-}m) > {}^{-}3 + ({}^{-}2)$?

I Is condition P equivalent to condition O?

J Is condition Q equivalent to condition P? ? Is condition R equivalent to condition Q? Is condition S equivalent to condition R?

K Next you will obtain a condition that enables you to think of m rather than of ${}^{-}m$. Look again at sentence S. How can you obtain $m + {}^{-}m > m + {}^{-}5$ from condition S? How can you obtain $0 > m + {}^{-}5$ from $m + {}^{-}m > m + {}^{-}5$?

L What steps must you take to obtain condition T from $0 > m + {}^{-}5$?

M Give a standard description of the solution set of $3 - m > {}^{-}2$.

So far in this lesson you have worked with simple conditions for inequality. Next you will find solution sets of compound conditions that involve conditions for inequality.

A Look at D5. What condition does sentence U express? What connective is used? What is the universe for a?

B How do you know that the solution set of condition U is the intersection of the solution sets of the simple conditions?

C Think about ${}^{-}6a + 2a < 8$. How can you obtain ${}^{-}4a < 8$ from this condition?

D How can you obtain $a > {}^{-}2$ from ${}^{-}4a < 8$? Is $a > {}^{-}2$ equivalent to ${}^{-}6a + 2a < 8$?

E Give a standard description of the solution set of ${}^{-}6a + 2a < 8$.

F Now think about $a - \frac{7}{8} < \frac{1}{8}$. How can you obtain $a < 1$ from $a - \frac{7}{8} < \frac{1}{8}$?

G Give a standard description of the solution set of $a - \frac{7}{8} < \frac{1}{8}$.

H Is $\{a | {}^{-}2 < a < 1\}$ the solution set of condition U?

I Does $\{a | {}^{-}2 < a < 1\}$ contain all the real numbers between ${}^{-}2$ and 1? Does this set contain ${}^{-}2$? Does it contain 1?

J $\{a | {}^{-}2 < a < 1\}$ is an open interval in D. What are the limits of this interval? Are the limits contained in the interval?

K Look at the graph in D6. Are the dots for point ${}^{-}2$ and point 1 in the graph of the interval $\{a | {}^{-}2 < a < 1\}$? Do the dots for the points between point ${}^{-}2$ and point 1 form the graph of $\{a | {}^{-}2 < a < 1\}$? How are these dots indicated? Is $\{a | {}^{-}2 < a < 1\}$ an infinite set?

L Now look at D7. What condition does sentence V express? What is the universe for r?

M What connective is used in condition V? Is the solution set of the compound condition the union, or is it the intersection, of the solution sets of the simple conditions?

U = D.

V $\bar{3}/4 + r < 2 \vee r + 10/4 = 21/4$.

D7

U = D.

$\{r \mid r \leqq 11/4\}$

D8

U = D.

W $\sim(.5x + .2 > 1.2)$.

D9

N Explain the steps that you can use to obtain $r < 11/4$ from $\bar{3}/4 + r < 2$. Is $r < 11/4$ equivalent to $\bar{3}/4 + r < 2$?

O Give a standard description of the solution set of $\bar{3}/4 + r < 2$.

P Tabulate $\{r \mid r + 10/4 = 21/4\}$.

Q How do you know that the solution set of condition V is $\{r \mid r < 11/4 \vee r \mid r = 11/4\}$?

R Use the symbol $\leqq$ in a standard description of the solution set of $\bar{3}/4 + r < 2 \vee r + 10/4 = 21/4$.

S Is $11/4$ a member of $\{r \mid r \leqq 11/4\}$? Is 0 a member? Is $\bar{}\sqrt{2}$ a member? Is every negative real number a member of $\{r \mid r < 11/4\}$?

T Look at the graph in D8. How can you tell that the graph contains the dot for point $11/4$? Is the dot for point 0 contained in the graph? What dots form the graph of $\{r \mid r \leqq 11/4\}$? How are these dots indicated?

U Look at D9. Is $\{\overline{x \mid .5x + .2 > 1.2}\}$ the solution set of condition W?

V How can you obtain the equivalent condition $.5x > 1$ from $.5x + .2 > 1.2$? How can you obtain $x > 2$ from $.5x > 1$?

W Is $\{x \mid x \leqq 2\} = \{x \mid \sim(.5x + .2 > 1.2)\}$?

X Make a graph of $\{x \mid \sim(.5x + .2 > 1.2)\}$.

In this lesson you learned how to use properties of "less than" and properties of addition and multiplication of real numbers to find solution sets of simple and compound conditions for inequality.

On your own

For each of exercises 1 through 6, first express a chain of equivalent conditions. The last condition should tell you that x is greater than, or less than, a certain real number. For each condition, tell what definition or property you used to obtain it. Finally, give a standard description of the solution set of the original condition. U = D.

1 $x + \bar{4} < 1\frac{1}{2}$.

2 $\bar{1}2x > 3$.

3 $\frac{1}{10}x < 7$.

4 $x - .3 < 2.8$.

5 $\frac{5}{4}x + \frac{1}{2}x > \bar{1}4$.

6 $15 - x > 12$.

For each of exercises 7 through 14, tell whether the sentence expresses a true statement or a false statement. U = D.

7 $\{y \mid y + 3 < 7\} = \{y \mid y < 4\}$.

8 $\{x \mid .5x < .15\} = \{x \mid x > .3\}$.

9 $\{k \mid \bar{2}/3k > 1/9\} = \{k \mid k < \bar{1}/6\}$.

10 $\{x \mid .8x + .6x > 2.8\} = \{x \mid x < 2\}$.

11 $\{p \mid 9 - p < 16\} = \{p \mid p < 7\}$.

12 $\{x \mid x > \bar{2}/3 \wedge x - 1/4 < 7/8\} = \{x \mid \bar{2}/3 < x < 9/8\}$.

13 $\{r \mid r > \bar{7} \vee 4/3 - r = 8\frac{1}{3}\} = \{r \mid r \geqq \bar{7}\}$.

14 $\{t \mid \sim(3t + 3 < 5)\} = \{t \mid t \geqq 2/3\}$.

For each of the conditions expressed below, give a standard description of the solution set. Then make a graph of the solution set. U = D.

15 $\frac{4}{5}x > 8$.

16 $2.1 - x > 2.6$.

17 $\bar{1}/6x < 1/12$.

18 $4r + 2r > \bar{2}$.

19 $\bar{2}/3y > \bar{6} \wedge y + 6 > 4$.

20 $4(x + 3) = 12 \vee \bar{2}/7 - x < \bar{2}/7$.

21 $\sim(\bar{1}/2k > \bar{2}2/3)$.

Coördinate axes and the real plane

In unit 3 of *Book 1* and in lesson 111 of this book, you learned how points in a plane can be associated with ordered pairs of natural numbers. In this lesson, you will learn how points in a plane can be associated with ordered pairs of real numbers.

A What is a name of the Cartesian set that contains all the ordered pairs whose first and second components are natural numbers?

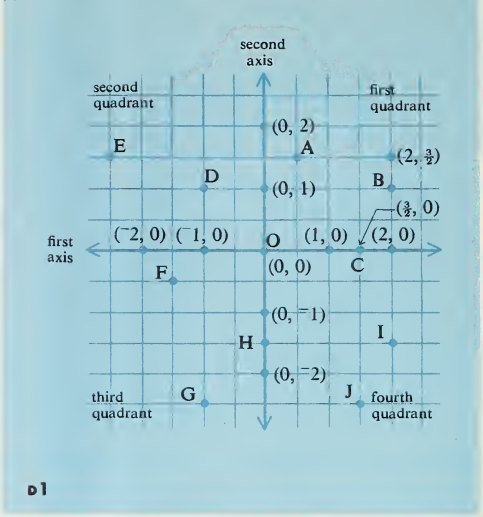
B $D \times D$ is the Cartesian set that contains all the ordered pairs whose first and second components are real numbers. How do you know that $(\sqrt{5}, {}^{-}8)$ is a member of $D \times D$? Is $(1, \frac{5}{8})$ a member of $D \times D$? Is $({}^{-}2, {}^{-}2) \in D \times D$? Is $(0, 0) \in D \times D$?

C How do you know that $D \times D$ is an infinite set?

Look at D1. All the points represented in the display are coplanar. Two perpendicular lines, called *coördinate axes*, can be used to locate points in a plane. The horizontal axis pictured in D1 is the *first* coördinate axis, or first axis. The vertical axis is the *second* coördinate axis, or second axis.

Each point represented by a dot in D1 is associated with a member of $D \times D$. The first and second components of the ordered pair that is associated with a point are the first and second coördinates of the point.

The set of dots whose points are associated with members of $D \times D$ forms a graph of $D \times D$. The point of intersection of the two



D1

axes is associated with (0, 0). Point (0, 0) is the *origin* (ôr′ə jin). Notice that, in the graph, the dot for the origin is labelled with the letter O.

D What is the first coördinate of point (0, 0)? What is the second coördinate of the origin?

E In D1, the pictures of the lines form a *grid*. The pictures of lines printed in blue help you to locate points with reference to the coördinate axes. Find the dot for point C. In which axis is point C located? Notice that point C is located $\frac{3}{2}$ units to the right of the second axis. Name the coördinates of point C.

F In which axis is point (0, 1) located? How many units above the first axis is point (0, 1) located?

Each point in the coördinate axes is associated with an ordered pair of real numbers, at least one of whose components is 0. If the second coördinate of a point is 0, the point is in the first axis. If the first coördinate is 0, the point is in the second axis.

G In which axis is point (0, 2)? In which axis is point (2, 0)? Find the dots in D1 that represent these points.

H Look again at D1. Is point A located $\frac{1}{2}$ unit to the right of the second axis? How many units above the first axis is point A located? Point A is associated with $(\frac{1}{2}, \frac{3}{2})$. The first component of $(\frac{1}{2}, \frac{3}{2})$ tells you where point A is located with respect to the second axis. The second component of $(\frac{1}{2}, \frac{3}{2})$ tells you where point A is located with respect to the first axis.

I Are the coördinates of point B the components of (2, 1), or are they the components of (4, 2)? How do you know?

J Now find the dot for point $(^{-}1, 0)$. In which axis is point $(^{-}1, 0)$? Is point $(^{-}1, 0)$ located to the left of the second axis?

K In which axis is point $(^{-}2, 0)$? Where is point $(^{-}2, 0)$ located with respect to the second axis?

L Find the dot for point $(0, ^{-}2)$. In which axis is point $(0, ^{-}2)$? Is point $(0, ^{-}2)$ located above the first axis or below the first axis?

M Point D, represented in D1, is the same as point $(^{-}1, 1)$. Describe the location of point D.

N Describe the location of point E. With what ordered pair of real numbers is point E associated?

O Find the dot for point F. With what ordered pair is point F associated? Are both coördinates of point F negative?

P Describe the location of point G. With what ordered pair is point G associated?

Q Why is point H associated with $(0, ^{-}\frac{3}{2})$?

R Why is point I associated with $(2, ^{-}\frac{3}{2})$?

S Describe the location of point J. With what ordered pair is point J associated?

T In which direction from the second axis is a point located if the first coördinate of the point is positive? If the first coördinate of the point is negative?

U In which direction from the first axis is a point located if the second coördinate of the point is positive? If the second coördinate of the point is negative?

Each point in a plane can be associated with an ordered pair of real numbers, and each ordered pair of real numbers can be associated with a point in the plane. This means that you can set up a one-to-one correspondence between $D \times D$ and the set of points in a plane. For this reason, the plane that is determined by the coördinate axes is called the *real plane*.

V Suppose the first coördinate of a point is 15, and the second coördinate is $^{-}6$. Where in the real plane is point $(15, ^{-}6)$ located? Describe the location of point $(^{-}5\frac{3}{4}, 18)$. Describe the location of point $(^{-}\sqrt{17}, ^{-}\sqrt{17})$.

The coördinate axes separate the plane represented in D1 into four regions that are called *quadrants* (kwod′rənts). Each of the four quadrants is the intersection of two half-planes bounded by the coördinate axes. Because the axes are the boundaries of the quadrants, the points *in* the axes are not contained in any of the quadrants.

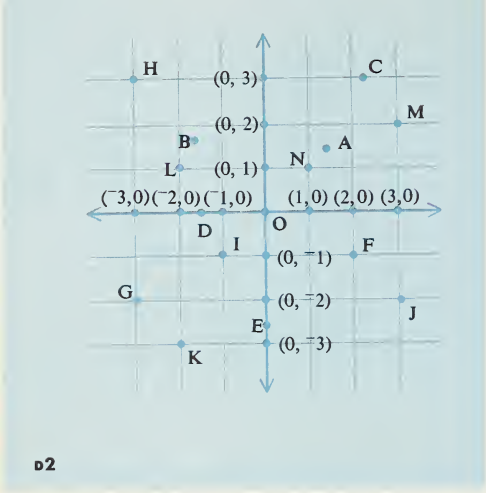
W Look again at D1. Notice the names of the quadrants. For each of points A through J, in which quadrant, if any, is the point contained?

X In which quadrant are both coördinates of a point positive? In which quadrant are both coördinates of a point negative?

Y In which quadrant is a point contained if the first coördinate of the point is positive and the second coördinate is negative? In which quadrant is a point contained if the first coördinate is negative and the second coördinate is positive?

Z In which quadrant is each of the following points contained: $(^{-}8, 2)$, $(.\overline{3}, .6)$, $(^{-}\sqrt{2}, ^{-}11)$, $(4, ^{-}4)$?

Now you know how to associate ordered pairs of real numbers with points in a plane. You



D2

also know how to tell in which quadrant of the real plane a point is contained.

On your own

Use the graph in D2 in connection with exercises 1 through 12.

For each of exercises 1 through 5, describe the location of the point named.

1 Point A is associated with $(\sqrt{2}, \sqrt{2})$.

2 Point B is associated with $(\overline{\ }\sqrt{3}, \frac{5}{3})$.

3 Point C is associated with $(2\frac{1}{4}, 3)$.

4 Point D is the same as point $(\overline{\ }1\frac{1}{2}, 0)$.

5 Point E is the same as point $(0, \overline{\ }2.7)$.

6 Name the ordered pair that is associated with each of points F through N.

7 Which points represented by dots are in the second quadrant?

8 Which points represented by dots are in the first quadrant?

9 Which points are in the fourth quadrant?

10 Which points are in the third quadrant?

11 Which points represented by dots are in the line that is parallel to the first axis and 2 units below the first axis?

12 Which points represented by dots are in the line that is parallel to the second axis and 3 units to the left of the second axis?

13 Think of the line included in the real plane that is parallel to the second axis and also $\sqrt{2}$ units to the right of the second axis. What is the first coördinate of each point in this line?

14 Describe the location of the set of points each of whose second coördinates is $\sqrt{3}$.

Now make a picture of the coördinate axes. On your picture of the axes, use dots to represent the points you need to help you locate the points named in the exercises below. Then make a dot for each point named. Beside the dot write the name of the ordered pair associated with the point.

15 $(\overline{\ }3, 2)$

16 $(1, \overline{\ }1)$

17 $(\frac{1}{2}, \frac{1}{2})$

18 $(\overline{\ }4, \overline{\ }2)$

19 $(2, 4)$

20 $(0, \overline{\ }\frac{3}{2})$

21 $(\overline{\ }2, \frac{1}{2})$

22 $(\frac{3}{2}, \overline{\ }3)$

23 $(\overline{\ }\frac{5}{2}, 0)$

KEEPING SKILFUL

For each exercise below, use a mixed numeral or a numeral for a basic fraction to express the sum, difference, product, or quotient.

1 $1\frac{3}{16} + 2\frac{5}{8}$

2 $\frac{3}{16} \times \overline{\ }\frac{4}{5} \times \frac{8}{9}$

3 $\frac{3}{4} + \overline{\ }\frac{1}{2} + \overline{\ }\frac{1}{3}$

4 $\frac{3}{8} - \frac{1}{3}$

5 $65 \times \overline{\ }5 \times \overline{\ }1$

6 $\overline{\ }6\frac{3}{4} + \overline{\ }8\frac{2}{5}$

7 $\overline{\ }8\frac{1}{2} - 6$

8 $\overline{\ }1\frac{3}{8} \div \frac{1}{2}$

9 $\overline{\ }25 - \overline{\ }30\frac{1}{2}$

10 $2\frac{1}{3} \times \overline{\ }12\frac{3}{4}$

11 $13\frac{3}{4} \div 22\frac{1}{2}$

12 $\overline{\ }\frac{5}{6} + \overline{\ }12\frac{1}{2} + \overline{\ }\frac{3}{4}$

13 $\frac{5}{16} - \overline{\ }\frac{1}{4}$

14 $\overline{\ }24 \div \overline{\ }\frac{1}{16}$

For each exercise below, use a decimal to express the sum, difference, product, or quotient.

15 $.506 - \overline{\ }.452$

16 $10.3 + 85.1$

17 $.6 \times .304$

18 $164 \div \overline{\ }.5$

19 $1.121 + \overline{\ }.154 + \overline{\ }.7$

20 $\overline{\ }75 \times \overline{\ }6.8 \times \overline{\ }2$

21 $\overline{\ }12 - \overline{\ }16.45$

22 $.362 \div .9$

23 $\overline{\ }90 + 85.001$

24 $1.1 \times 1.01 \times \overline{\ }10$

25 $\overline{\ }.754 - .6$

26 $\overline{\ }5.14 \div 6$

27 $\overline{\ }62.15 + \overline{\ }3.86$

28 $64 - 89$

29 $\overline{\ }.004 \times 100.6$

30 $\overline{\ }65.34 \div \overline{\ }1.1$

Conditions in two and three variables

In this lesson you will study conditions in two and three variables. You will also learn how to make graphs of solution sets of conditions in two variables.

A Study the chart in D1. If you replace a by $^{-}\frac{1}{4}$ in $b = 2a + 1$, what condition do you obtain? What replacement for b satisfies this condition? What numeral is placed directly below the numeral $^{-}\frac{1}{4}$ and opposite the letter b in the chart? Is $(^{-}\frac{1}{4}, \frac{1}{2})$ a member of $\{(a, b) | b = 2a + 1\}$?

B Copy the chart in D1. Then find the replacements for b that you need to complete the chart.

C Tabulate the set of ordered pairs that you can obtain from your chart. Is each of these ordered pairs a member of the solution set of $b = 2a + 1$? Is each pair a member of D × D?

D Name two more ordered pairs that satisfy $b = 2a + 1$. Is the solution set of the condition $b = 2a + 1$ an infinite set? How do you know?

E Look at D2. Notice how the dots that represent points in the axes are labelled. The dot for point (3, 0) is labelled with the numeral 3. In which axis is point (3, 0)? How is the dot for point (0, — 2) labelled? In which axis is point (0, — 2)? Which two dots are labelled with the numeral $^{-}3$?

F Each of points A through G represented in D2 is associated with an ordered pair that you named for exercise C. Name the coördinates of each of these points. Is each of points A through G contained in ℓ_1?

The picture of ℓ_1 in D2 is a graph of the solution set of $b = 2a + 1$. Each member of $\{(a, b) | b = 2a + 1\}$ is associated with a point in ℓ_1, and each point in ℓ_1 is associated with a member of $\{(a, b) | b = 2a + 1\}$. Later in your study of mathematics, you will learn why the set of points that is associated with the solution set of $b = 2a + 1$ is a line.

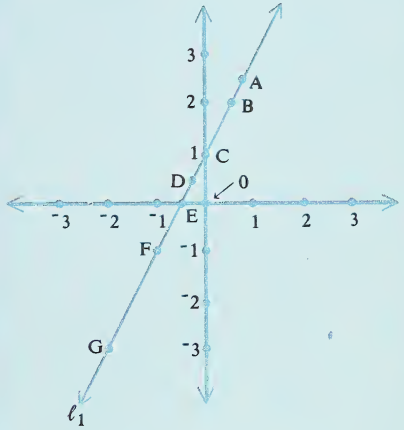
G Is $(25, 51) \in \{(a, b) | b = 2a + 1\}$? Is $(100, 201) \in \{(a, b) | b = 2a + 1\}$? Why are the words "incomplete graph" used in D2?

Universe for $(a, b) = D \times D$.
$b = 2a + 1$.

a	$^{-}\frac{1}{4}$	$\frac{1}{2}$	3	0	$^{-}1$	$^{-}2$	$^{-}\frac{1}{2}$	$\frac{3}{4}$	$^{-}3$
b	$\frac{1}{2}$	2	7	1					

D1

Universe for $(a, b) = D \times D$.



$\{(a, b) | b = 2a + 1\}$
Incomplete graph

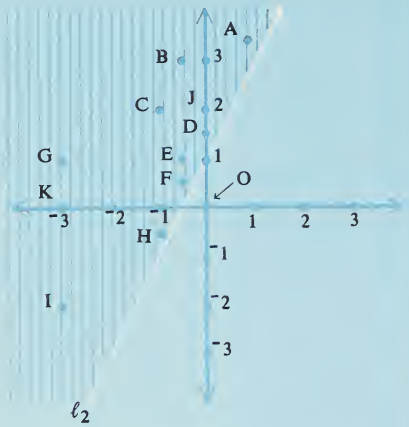
D2

Universe for $(a, b) = D \times D$.
$b > 2a + 1$.

a	0	1	$-\frac{1}{2}$	-1	-3
b	$1\frac{1}{2}$	$3\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	-4
	2	4	1	0	-2
	3	$4\frac{1}{2}$	3	2	0
	15	7	8	5	1

D3

Universe for $(a, b) = D \times D$.



$\{(a, b) | b > 2a + 1\}$
Incomplete graph

D4

H Now study the chart in D3. What replacements for a are named in the chart? What is the set of all possible replacements for a? Is this set finite, or is it infinite?

I What number do you obtain from $2a + 1$ when you replace a by 0? By 1? By $-\frac{1}{2}$? By -1? By -3? For each of these replacements, tell what must be true of each b that satisfies $b > 2a + 1$.

J Tabulate the set of ordered pairs that you can obtain from the chart in D3. Does each of these ordered pairs satisfy $b > 2a + 1$?

K Explain how you know that, for each replacement of a, there is an infinite set of replacements for b that satisfy $b > 2a + 1$. Is $\{(a, b) | b > 2a + 1\}$ an infinite set?

L Now look at D4. Find the dots for points A through K. The coördinates of these points were obtained from the chart in D3. Name the coördinates of each point. Is each ordered pair that you named a member of the solution set of $b > 2a + 1$?

Notice the picture of ℓ_2 in D4. This picture is the graph of $\{(a, b) | b = 2a + 1\}$. This graph is shown in white because it is not part of the graph of $\{(a, b) | b > 2a + 1\}$.

The graph of $\{(a, b) | b > 2a + 1\}$ is indicated by the vertical shading in D4. Each dot in the shaded portion represents a point associated with a member of $\{(a, b) | b > 2a + 1\}$.

M Is the set of points associated with $\{(a, b) | b > 2a + 1\}$ a half-plane? Explain your answer.

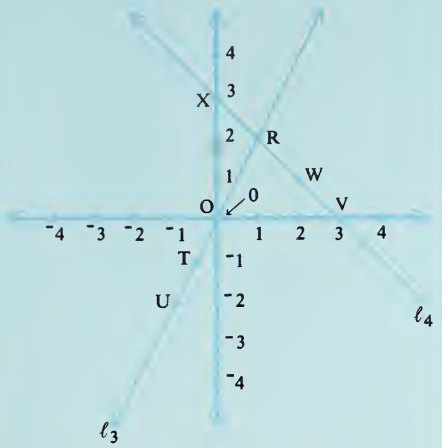
N Now think about $b < 2a + 1$. Where would the graph of $\{(a, b) | b < 2a + 1\}$ be located with respect to the graph of $\{(a, b) | b = 2a + 1\}$? The universe for (a, b) is $D \times D$.

O Explain how you know that the graph of $\{(a, b) | b \geqq 2a + 1\}$ includes pictures of both a line and a half-plane. The universe for (a, b) is $D \times D$.

Next you will study compound conditions in two variables that include the connective "and." You will also learn how to make graphs of the solution sets of such conditions.

A Look at D5. How do you know that each dot in the graph of the solution set of the compound condition is also in the graphs of the solution sets of both simple conditions?

Universe for $(x, y) = D \times D$.



$\{(x, y) | y = 2x \wedge x + y = 3\}$

D5

Universe for $(x, y) = D \times D$.

A $y = 2x \wedge x + y = 3$.

B $y = 2x \wedge x + 2x = 3$.

C $y = 2x \wedge 3x = 3$.

D $y = 2x \wedge x = 1$.

E $y = 2 \wedge x = 1$.

D6

B Study D5. Name the coördinates of points R, O, T, and U. Which line that is represented in D5 contains each of these points? Is each of points R, O, T, and U associated with a member of $\{(x, y) | y = 2x\}$?

C Next you will make a graph of the solution set of $x + y = 3$. The graph of this solution set is a picture of a line. Suppose that you are given two dots in the graph of $\{(x, y) | x + y = 3\}$. How can you use these dots to help you draw the graph? Explain your answer.

D If you replace x by 0 in $x + y = 3$, what y satisfies $0 + y = 3$? Name the member of $\{(x, y) | x + y = 3\}$ that has a first component of 0.

E Now replace y by 0 in $x + y = 3$. Name the member of $\{(x, y) | x + y = 3\}$ that has a second component of 0.

F Look again at D5. What are the coördinates of point V? Of point X? What line do these two points determine? Is the picture of ℓ_4 the graph of $\{(x, y) | x + y = 3\}$?

G You should find a third solution of $x + y = 3$ to be sure that the picture of ℓ_4 is the graph of $\{(x, y) | x + y = 3\}$. Replace x by 2 in $x + y = 3$. What y satisfies $2 + y = 3$? Does the picture of ℓ_4 contain the dot for point (2, 1)? How is the dot labelled?

H Name the coördinates of the intersection of ℓ_3 and ℓ_4. Which dots in D5 are in the graph of the solution set of $y = 2x \wedge x + y = 3$? Tabulate $\{(x, y) | y = 2x \wedge x + y = 3\}$.

In the following exercises, you will study another method of finding the solution set of $y = 2x \wedge x + y = 3$.

Look at sentences A and B in D6. From the first simple condition in condition A, you know that y is the same as $2x$. What property tells you that you can replace y in $x + y$ by $2x$ without affecting the sum? Is condition B equivalent to condition A?

Notice that each simple condition in condition A contains two variables, but that one of the simple conditions in condition B contains only one variable.

J Now compare condition B and condition C. How can you obtain $3x = 3$ from $x + 2x = 3$? Is condition C equivalent to condition B?

K Compare conditions C and D. How can you obtain condition D from condition C? Is condition D equivalent to condition C?

Universe for $(x, y) = D \times D$.

A $y = 2x \wedge x + y = 3$.

B $y = 2x \wedge x + 2x = 3$.

C $y = 2x \wedge 3x = 3$.

D $y = 2x \wedge x = 1$.

E $y = 2 \wedge x = 1$.

D6

Universe for the variables in $(x, y, z) = D$.

$(x + y + 2z = 12) \wedge (x + y = 6) \wedge (x - 4 = 2)$.

D7

L How can you obtain condition E from condition D? Tabulate the solution set of condition E. How do you know that the set you tabulated is also the solution set of condition A?

M The chain of conditions expressed in D6 enables you to find the solution set of condition A. Use the method just explained to find the solution set of $x = y + 2 \wedge 2y = x + 1$. The universe for (x, y) is $D \times D$.

Now you will find the solution set of a compound condition in three variables.

A What compound condition is expressed in D7? The condition is made up of how many simple conditions? What connective is used?

B Look again at D7. The universe for the ordered triple of variables is the set of all ordered triples whose first, second, and third components are real numbers. Write a standard description of this universe.

C Is the solution set of the compound condition expressed in D7 the intersection, or is it the union, of the solution sets of the three simple conditions?

D To find the solution set of the compound condition, you will first work with the simple condition that involves only one variable. Which condition is this?

E What is the first component of each ordered triple that satisfies $x - 4 = 2$?

F Which simple condition expressed in D7 involves only two variables?

G If you replace x in $x + y = 6$ by 6, what condition do you obtain? What real number satisfies $6 + y = 6$?

So far, you know that any ordered triple that satisfies $(x + y + 2z = 12) \wedge (x + y = 6) \wedge (x - 4 = 2)$ must have 6 as its first component and 0 as its second component.

H If you replace x by 6 and y by 0 in the condition $x + y + 2z = 12$, what condition do you obtain? What z satisfies this condition?

I Tabulate the solution set of $(x + y + 2z = 12) \wedge (x + y = 6) \wedge (x - 4 = 2)$.

In this lesson you studied simple and compound conditions in two and three variables. You also learned how to make graphs of the solution sets of conditions in two variables.

On your own

Give a standard description of the solution set of each condition expressed in exercises 1 through 8. Then make a graph of each solution set. The universe for $(x, y) = D \times D$.

1 $4y = 3x$.

2 $y = {}^-x$.

3 ${}^-2x = 4y - 6$.

4 $x = \frac{1}{2}y \wedge y \leqq 0$.

5 ${}^-x = \frac{3}{4} \wedge y \geqq 0$.

6 $y = 0 \wedge x < {}^-2$.

7 $3y = 3x + 3 \vee 3y > 3x + 3$.

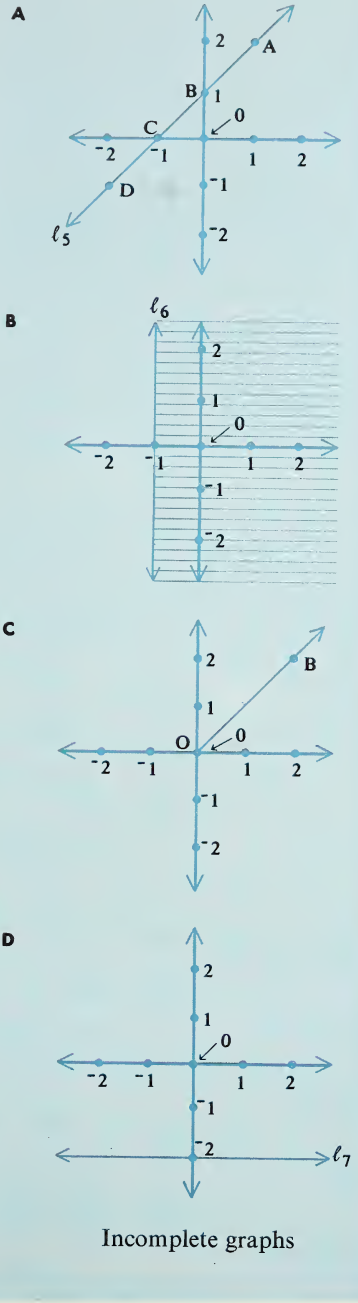
8 $y = x \wedge {}^-1 < x < 1$.

Tabulate the solution set of each condition expressed in exercises 9 through 12. The universe for the variables in $(r, s, t) = D$.

9 $(2r + s + 3t = 15) \wedge (r = 4 - t) \wedge (t + 2 = 5\frac{1}{2})$.

10 $({}^-18 = 3s) \wedge (r + 2s = t) \wedge (r - 5 = 7)$.

$U = D \times D$.



D8

11 $(3r - s = 2t) \wedge (\frac{1}{2}r = 3s) \wedge (14 = s + 11)$.

12 $(r + s + t = 0) \wedge (s = t) \wedge (6 - r = {}^{-}10)$.

Use the graphs in D8 for the exercises below. The universe for (x, y) is $D \times D$.

13 Study graph A in D8. The picture of ℓ_5 is an incomplete graph of the solution set of a condition. Name the coördinates of the points in ℓ_5 that are represented by labeled dots.

14 In each of the ordered pairs you named for exercise 13, the second coördinate is how much greater than the first coördinate? Is the picture of ℓ_5 an incomplete graph of the solution set of $y = x + 1$?

15 Look at graph B. The picture of ℓ_6 and all the dots to the right of ℓ_6 form an incomplete graph of the solution set of a condition. Write a sentence that expresses a condition whose solution set is represented by graph B.

16 Look at graph C. The picture of $\overrightarrow{OB}$ is an incomplete graph of the solution set of a condition. Write a sentence that expresses a condition whose solution set is represented by graph C.

17 The picture of ℓ_7 in graph D is an incomplete graph of the solution set of a condition. Write a sentence that expresses a condition whose solution set is represented by graph D.

KEEPING SKILFUL

For each condition expressed below, tabulate the solution set if possible. Otherwise, give a standard description of the solution set. $U = D$.

1 $2\frac{1}{2} < x + 1\frac{7}{10} \wedge 2\frac{4}{5} + x < 4\frac{1}{2}$.

2 ${}^{-}3y = 21.9 \vee 2y + 3y > {}^{-}36.5$.

3 ${}^{-}12r = 3.6 \wedge {}^{-}10.2 + s = r$.

4 $m = 4n \wedge m + {}^{-}34 = n - 25$.

5 $(a + b + c = 50) \wedge (2\frac{1}{4}b - {}^{-}2 = 30\frac{1}{2}) \wedge (b - a = 7\frac{5}{8})$.

6 $(x - 2\frac{1}{2} = y) \wedge (y - z = {}^{-}15) \wedge (\frac{4}{5}z = 20)$.

Problems involving conditions in one variable

You have learned how to use properties of the real numbers to find solution sets of simple and compound conditions in one variable. In this lesson you will learn to solve problems that involve such conditions. The universe for each variable is D.

Read the problem in D1. First you will develop the condition for the problem.

A You know that Mt. Robson is 12,972 ft. above sea level. Does the positive number expressed at the right represent the elevation of Mt. Robson in feet? 12972

B Do you know the elevation of Ripple Rock? Use y as a variable for the elevation of Ripple Rock in feet. y

C The sentence in blue below expresses the condition for the problem. To what does $12972 - y$ refer?

$$12972 - y = 13019.$$

D How can you obtain $^{-}y = 47$ from the condition $12972 - y = 13019$? Are $^{-}y = 47$ and $12972 - y = 13019$ equivalent conditions?

E From $^{-}y = 47$, you know that the additive inverse of the replacement for y is equal to 47. What number has 47 for its additive inverse?

F Tabulate $\{y \mid 12972 - y = 13019\}$.

You can use $^{-}47$ to get the answer to the problem in D1.

The elevation of Ripple Rock is $^{-}47$ ft., or 47 ft. below sea level.

Mt. Robson is 12,972 ft. above sea level. The difference of the elevation of Mt. Robson and the elevation of Ripple Rock, B.C., is 13,019 ft. What is the elevation of Ripple Rock?

D1

Mr. Rogers bought $12\frac{3}{4}$ ft. of copper wire. After he had used some of the wire, he still had more than $7\frac{1}{2}$ ft. left. How much wire did Mr. Rogers use?

D2

Read the problem in D2. You will develop a compound condition for this problem.

G Explain why the sentence below at the right expresses a simple condition for the problem. For what is c a variable? $12\frac{3}{4} - c > 7\frac{1}{2}$.

H Is $12\frac{3}{4} - c > 7\frac{1}{2}$ equivalent to $c < 5\frac{1}{4}$?

I Is $2\frac{1}{2}$ a solution of $c < 5\frac{1}{4}$? Is 0 a solution? Is $^{-}3\frac{1}{4}$ a solution? Is every real number that is less than $5\frac{1}{4}$ a solution of $c < 5\frac{1}{4}$?

Now you will see if you can use $\{c \mid c < 5\frac{1}{4}\}$ to get the answer to the problem in D2.

J You know that $2\frac{1}{2}$, 0, and $^{-}3\frac{1}{2}$ are members of $\{c \mid c < 5\frac{1}{4}\}$. Can you use $2\frac{1}{2}$ to get a sensible answer to the problem? Can you use 0? Can you use $^{-}3\frac{1}{2}$?

K Can you use any real number that is less than or equal to 0 to get a sensible answer to the problem?

You have discovered that you can use only elements of $\{c \mid c < 5\frac{1}{4}\}$ greater than 0 to get sensible answers to the problem in D2. So you can use the condition expressed at the right as another simple condition for the problem. $c > 0$.

The sentence below in blue expresses a compound condition for the problem.

$$12\tfrac{3}{4} - c > 7\tfrac{1}{2} \wedge c > 0.$$

L You know that $\{c|c < 5\frac{1}{4}\}$ is the solution set of $12\frac{3}{4} - c > 7\frac{1}{2}$. Does each member of $\{c|c < 5\frac{1}{4}\}$ satisfy $c > 0$? Explain your answer.

M Is $\{c|0 < c\}$ the same set as $\{c|c > 0\}$? How do you know? Is $\{c|0 < c < 5\frac{1}{4}\}$ the solution set of $12\frac{3}{4} - c > 7\frac{1}{2} \wedge c > 0$? Explain your answer.

You can use $\{c|0 < c < 5\frac{1}{4}\}$ to get the answer to the problem in D2. You can give the answer in either of the following ways:

Mr. Rogers used more than 0 ft. of wire and less than $5\frac{1}{4}$ ft. of wire.

Mr. Rogers used between 0 ft. and $5\frac{1}{4}$ ft. of wire.

N You can verify the answer to the problem by asking yourself the following questions: If Mr. Rogers had used between 0 ft. and $5\frac{1}{4}$ ft. of wire, would he have more than $7\frac{1}{2}$ ft. of wire left? If he had used between 0 ft. and $5\frac{1}{4}$ ft. of wire, would he have used more than 0 ft.?

Think again about what we did in exercises G through M. First we developed a simple condition for the problem. After we had obtained the solution set of this condition, we discovered that some members of the solution set could not be used to get sensible answers. For this reason, we developed another simple condition for the problem. Finally, we used the solution set of the compound condition to get the answer to the problem.

It is not always necessary to develop a condition such as $c > 0$. For example, read the following problem.

Don had 75 stamps in his collection. After Paul had given him some more stamps, Don had 100 stamps in his collection. How many stamps did Paul give to Don?

After Mr. Brennan had bought 114.5 acres of land, he still owned less than 350 acres. How many acres of land did he own before he bought the 114.5 acres?

D3

You know that $75 + x = 100$ is the condition for the problem about the stamps. You also know that $\{25\}$ is the solution set of the condition. Because you can use 25 to get a sensible answer to the problem, it is not necessary to use $x > 0$ as another simple condition for the problem.

Now read the problem in D3.

O Explain why the condition expressed at the right is a simple condition for the problem. For what is n a variable?

$$n + 114.5 < 350.$$

P Is $n + 114.5 < 350$ equivalent to $n < 235.5$?

Q Is 95 a solution of $n + 114.5 < 350$? Is 0 a solution? Is $^-20$ a solution? Is every real number that is less than 235.5 a solution of $n + 114.5 < 350$? Explain your answers.

Read the problem in D3 again. Now you will see if you can use $\{n|n < 235.5\}$ to get the answer to the problem.

R You know that 95, 0, and $^-20$ are solutions of $n + 114.5 < 350$. Can you use 95 to get a sensible answer to the problem? Can you use 0? Can you use $^-20$? Can you use any real number less than 0 to get a sensible answer?

Because the real numbers that are greater than or equal to 0 are the only members of $\{n|n < 235.5\}$ that provide sensible answers to the problem in D3, you can develop another condition for the problem. This condition is expressed at the right.

$$n \geq 0.$$

The sentence in blue below expresses a compound condition for the problem in D3.

$$n + 114.5 < 350 \wedge n \geq 0.$$

S You know that $\{n|n < 235.5\}$ is the solution set of $n + 114.5 < 350$. Does each member of $\{n|n < 235.5\}$ satisfy $n \geq 0$? Explain your answer.

T Is $\{n|0 \leq n\}$ the same set as $\{n|n \geq 0\}$? Is $\{n|0 \leq n < 235.5\}$ the solution set of $n + 114.5 < 350 \wedge n \geq 0$? Explain your answers.

Now you can use $\{n|0 \leq n < 235.5\}$ to get the answer to the problem in D3.

Mr. Brennan owned from 0 to 235.5 acres of land before he bought the 114.5 acres.

Next you will solve a problem that involves a compound condition made up of three simple conditions. Read the problem in D4.

A Mrs. Turner bought a package of 24 drapery hooks and then had more than 50. Use x as a variable for the number of hooks that she had before she bought 24 more. The sentence at the right expresses a simple condition for the problem. To what does $x + 24$ refer?

$$x + 24 > 50.$$

B How do you know that Mrs. Turner had fewer than 31 drapery hooks to begin with?

C To what does the variable in the condition expressed at the right refer?

$$x < 31.$$

The sentence below in blue expresses a compound condition for the problem in D4.

$$x + 24 > 50 \wedge x < 31.$$

D How do you know that every real number greater than 26 is a solution of $x + 24 > 50$?

E Each solution of $x < 31$ is less than what number?

After Mr. Brennan had bought 114.5 acres of land, he still owned less than 350 acres. How many acres of land did he own before he bought the 114.5 acres?

D3

Mrs. Turner needed 31 drapery hooks. She did not have enough drapery hooks on hand, so she bought a package of 24. She then had more than 50 hooks. How many drapery hooks did she have before she bought the 24?

D4

The sentence in blue below expresses a compound condition for the problem in D5.

$$x + 24 > 50 \wedge x < 31.$$

The sum of a number and $^{-}6\frac{2}{3}$ is the same as the product of that number and 5. What is the number?

D5

F Are $x + 24 > 50 \wedge x < 31$ and $26 < x < 31$ equivalent conditions? Is 27 a solution of $26 < x < 31$? Is $28\frac{1}{2}$ a solution? Is $30\frac{1}{4}$ a solution?

Read the problem in D4 again. Now you will decide if you can use $\{x|26 < x < 31\}$ to get the answer to the problem.

G You know that 27, $28\frac{1}{2}$, and $30\frac{1}{4}$ are all solutions of $26 < x < 31$. Can you use 27 to get a sensible answer to the problem in D4? Can you use $28\frac{1}{2}$? Can you use $30\frac{1}{4}$?

H Can you use any real number that is not an integer to get a sensible answer to the problem?

You have discovered that you can use only certain members of $\{x|26 < x < 31\}$ to get sensible answers to the problem. So you can develop another simple condition.

I You can use the condition expressed below at the right as another simple condition for the problem. The condition requires that x be a member of the set of integers. Is the set of integers a subset of the set of real numbers?

$x \in I$.

The sentence in blue below expresses a compound condition for the problem in D4.

$$(x + 24 > 50) \wedge (x < 31) \wedge (x \in I).$$

J You know that $\{x \mid 26 < x < 31\}$ is the solution set of $x + 24 > 50 \wedge x < 31$. Does each member of $\{x \mid 26 < x < 31\}$ satisfy $x \in I$? Explain your answer.

K Tabulate $\{x \mid 26 < x < 31 \wedge x \in I\}$.

You can use each member of the set that you tabulated for exercise K to get an answer to the problem in D4.

Mrs. Turner had between 26 and 31 drapery hooks before she bought 24 more.

In the work that follows, you will study a problem that concerns numbers only. Read the problem in D5.

L Use m as a variable for the number you are to find. Is $5m = m + {}^{-}6\frac{2}{3}$ the condition for the problem?

M Are $5m = m + {}^{-}6\frac{2}{3}$ and $4m = {}^{-}6\frac{2}{3}$ equivalent conditions? How do you know?

N What number satisfies $4m = {}^{-}6\frac{2}{3}$?

O Tabulate $\{m \mid 5m = m + {}^{-}6\frac{2}{3}\}$

P Is the product of 5 and ${}^{-}1\frac{2}{3}$ equal to the sum of ${}^{-}1\frac{2}{3}$ and ${}^{-}6\frac{2}{3}$?

In this lesson you used simple and compound conditions in one variable to solve problems involving real numbers.

On your own

For each problem, write a sentence that expresses a condition. If possible, tabulate the solution set. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The universe for each variable is D.

1 Nancy bought 35 post cards. She used more than 14 of them, but still had more than 17 left. How many post cards did she use?

2 The speed of sound in air is about 1088 ft. per second. At this speed, how many feet does sound travel in 60 seconds?

3 The batteries that Frank buys for his transistor radio cost \$.20 apiece. He has \$1.25 to spend. How many batteries can he buy?

THINK How can you use $6\frac{1}{4}$ to obtain a sensible answer for the problem?

4 A support for a bridge has $12\frac{1}{2}$ ft. of its height embedded in the ground and $9\frac{1}{4}$ ft. of its height in the water. The total height of the support is less than 35 ft. How many feet of the height of the support are above water?

5 On July 15, Mrs. Allen deposited \$96.75 in her savings account. After she had deposited this money, she still had less than \$185 in her account. How much money did she have in her account before she deposited \$96.75?

6 Barbara spent more than \$4.75 for art supplies. She gave the clerk \$10 and received more than \$2.50 in change. How much did Barbara spend for art supplies?

7 Mr. Henderson bought $5\frac{1}{2}$ gal. of paint. After he had used some of the paint to paint his garage, he still had more than $1\frac{5}{8}$ gal. left. How much paint did Mr. Henderson use?

8 The Emerson Shipping Company plans to buy 10,000 ft. of hemp rope. They can buy rope that costs \$.02 per foot or rope that costs \$1.95 per 100 ft. How much will it cost the company to buy 10,000 ft. of hemp rope?

9 What numbers greater than ${}^{-}6\frac{3}{4}$ can be added to $4\frac{1}{8}$ so that each sum is less than $2\frac{1}{2}$?

10 The sum of $3\frac{1}{2}$ and what numbers can be multiplied by 4 so that each product is equal to the quotient of ${}^{-}8\frac{1}{3}$ and $2\frac{1}{2}$?

Problems involving conditions in two and three variables

In this lesson you will use the methods developed in lesson 158 to solve problems that involve compound conditions in two and three variables. The universe for each variable is D.

Read the problem in D1.

A You know that the length of the field is 5 times the width of the field. The rate pair expressed below at the right represents this comparison. To what does the first component refer? To what does the second component refer? $5/1$

B You know one rate pair that represents the comparison. You want to find another rate pair equivalent to 5/1. Use a as a variable for the length of the field in feet. Use b as a variable for the width of the field in feet. Why do you use a/b, and not b/a? a/b

C The sentence at the right below expresses one of the simple conditions for the problem. Why is the condition a condition for equivalence? $5/1 \sim a/b.$

D You know that the perimeter of the field is 900 ft. Does the sentence below in blue express another simple condition for the problem?

$$a + b + a + b = 900.$$

The sentence below in blue expresses a compound condition for the problem.

$$5/1 \sim a/b \wedge a + b + a + b = 900.$$

The length of a rectangular field is 5 times the width of the field. The perimeter of the field is 900 ft. What are the length and the width of the field in feet?

D1

A $a + b + a + b = 900.$
B $5b + b + 5b + b = 900.$
C $(5 + 1)b + (5 + 1)b = 900.$
D $6b + 6b = 900.$
E $(6 + 6)b = 900.$
F $12b = 900.$

D2

Now you will find each (a, b) that satisfies the compound condition for the problem.

E How was the condition expressed below at the right obtained from $5/1 \sim a/b$? How do you know that each ordered pair that is a solution of the compound condition must have a first component that is 5 times the second component? $a = 5b.$

F Look at D2. You know that condition A is one of the simple conditions for the problem in D1. If you replace a by $5b$ in condition A, do you obtain condition B?

G Each of the other conditions expressed in D2 is equivalent to condition B. Explain how each condition was obtained.

H How was the condition expressed at the right obtained from $12b = 900$? $b = 75.$

You know that 75 is the replacement for b that satisfies the compound condition for the problem. The next step is to find a replacement for a that satisfies the compound condition.

I Replace b by 75 in $a = 5b$. What number satisfies the condition that you obtain?

J Tabulate the solution set of the compound condition for the problem.

K What are the length and the width of the field in feet?

L Verify your answer for the problem. If the field is 375 ft. long and 75 ft. wide, is the length 5 times the width? Is the perimeter 900 ft.?

So far in this lesson, you have solved a problem that involves a compound condition in two variables. Now you will solve a problem that involves a compound condition in three variables.

Read the problem in D3.

A You are asked to find the number of hours it took the two trains to reach points that were 475 mi. apart. Did the two trains travel the same distance to reach these points? Explain your answer.

B Do you know how many miles the passenger train travelled? Use x as a variable for the number of miles the passenger train had travelled by the time the two trains were 475 mi. apart. x

C Do you know how many miles the freight train travelled? Use y as a variable for the number of miles the freight train had travelled by the time the two trains were 475 mi. apart. y

D The sentence at the right expresses a simple condition for the problem. To what does $x + y$ refer? $x + y = 475.$

E What does the problem ask you to find? You know that the two trains travelled the same number of hours. Use z as a variable for the number of hours each train had travelled by the time they were 475 mi. apart. z

A passenger train left Carlton station and travelled east at a rate of 60 mi. per hour. At the same time a freight train left Carlton station and travelled west at a rate of 35 mi. per hour. In how many hours were the two trains 475 mi. apart?

D3

To find the number you can use to get the answer to the problem, you need to develop conditions that involve the third variable, z. You can use the rates at which the two trains traveled to develop other conditions.

F You know that the passenger train travelled at a rate of 60 mi. per hour. The rate pair expressed at the right represents this rate. To what does each component refer? $60/1$

G Why do you use x/z, instead of z/x, to represent a rate pair equivalent to 60/1? x/z

H The sentence at the right below expresses another simple condition for the problem. Why is the condition a condition for equivalence? $60/1 \sim x/z.$

I The rate pair expressed below at the right represents the rate at which the freight train travelled. To what does each component of 35/1 refer? $35/1$

J You want to find a rate pair equivalent to 35/1. Why do you use y/z to represent this rate? y/z

K The sentence at the right below expresses another simple condition for the problem. Why is the condition a condition for equivalence? $35/1 \sim y/z.$

The sentence below in blue expresses a compound condition for the problem in D3.

$$(x + y = 475) \wedge (60/1 \sim x/z) \wedge (35/1 \sim y/z).$$

Now you can find the solution set of the compound condition. Remember that each (x, y, z) that is a solution of the compound condition must satisfy each of the simple conditions.

L How was the condition expressed at the right obtained from $60/1 \sim x/z$? $x = 60z.$

M How was the condition expressed at the right obtained from $35/1 \sim y/z$? $y = 35z.$

N Replace x in $x + y = 475$ by $60z$. Replace y by $35z$. What condition do you obtain?

O Look at D4. Is each condition expressed in D4 equivalent to $60z + 35z = 475$? Explain how each condition was obtained.

P How was the condition expressed at the right obtained from $95z = 475$? What number satisfies $z = 5$? $z = 5.$

You know that 5 is the third component of each solution of the compound condition. Now you can give the answer to the problem.

After 5 hours, the two trains were 475 mi. apart.

Since the problem asks you to find only the number of hours that it took the two trains to reach points 475 mi. apart, you can give the answer without finding the replacements for x and y that satisfy the compound condition. To verify the answer to the problem, however, you need to find the replacements for x and y.

Q Replace z by 5 in $x = 60z$. Replace z by 5 in $y = 35z$. What number satisfies $x = 60(5)$? What number satisfies $y = 35(5)$?

R Verify the answer to the problem in D3.

A passenger train left Carlton station and travelled east at a rate of 60 mi. per hour. At the same time a freight train left Carlton station and travelled west at a rate of 35 mi. per hour. In how many hours were the two trains 475 mi. apart?

D3

G $(60 + 35)z = 475.$

H $95z = 475.$

D4

In this lesson you used compound conditions in two and three variables to solve problems involving real numbers.

On your own

For each problem, first write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The universe for each variable is D.

1 The distance from Winnipeg to Regina by commercial air line is $\frac{8}{7}$ of the distance from Winnipeg to Edmonton. The distance from Winnipeg to Edmonton is 420 mi. more than the distance from Winnipeg to Regina. What is the distance from Winnipeg to Regina by commercial air line, and what is the distance from Winnipeg to Edmonton?

2 One Thursday, the temperature in Saskatoon was 12 degrees below zero. The temperature on Friday was 5 degrees higher than the temperature on Thursday. The temperature on Saturday was 4 degrees higher than the temperature on Friday. What was the temperature in Saskatoon on Friday and Saturday?

3 Hamilton is 305 ft. above sea level. If the elevation of Hamilton were 90 ft. less, it would be the same as the elevation of Ottawa. The elevation of Ottawa is about $\frac{1}{16}$ of the elevation of Calgary. What is the elevation of Ottawa in feet, and what is the elevation of Calgary in feet?

4 Ken had a piece of pine plywood. He bought a piece of oak plywood 4 times as long as the piece of pine and a piece of birch plywood 5 times as long as the piece of pine. The total length of the three pieces was 36 ft. What was the length of each piece of plywood?

5 Mrs. Watson received a 20% discount on a refrigerator. She paid \$335 for the refrigerator. What was the regular price of the refrigerator, and what was the amount of the discount?

6 Together, Jane and Martha spent less than \$3.50 for school supplies. Jane spent \$1.05 for notebooks, \$.25 for pencils, and \$.95 for paints. How much did each girl spend?

THINK How do you know that each girl spent more than 0 cents for school supplies?

7 In 1900 the population of Hartville was 35,105. The increase in population from 1900 to 1910 was 431. The decrease in population from 1910 to 1920 was 230. What was the population of Hartville in 1920?

8 One afternoon, Jim left school and cycled at a rate of 5 mi. per hour. At the same time, Sam left school and cycled in the opposite direction at a rate of $7\frac{1}{2}$ mi. per hour. After 3 hours, the boys were how many miles apart?

9 Together, Clyde and Henry weigh 265 lb. Clyde weighs $7\frac{1}{2}$ lb. more than Henry. How much does each boy weigh?

THINK Are $x - y = 7\frac{1}{2}$ and $x = 7\frac{1}{2} + y$ equivalent conditions?

10 The sum of two numbers is more than $^{-}6$. The first number is equal to the product of 2 and $3\frac{1}{2}$. What are the numbers?

11 The difference of two numbers is less than 3. When $3\frac{1}{2}$ is added to the second number, the sum is 6. What are the numbers?

12 The sum of three numbers is $^{-}10\frac{1}{2}$. The second number is 4 times the first number. The third number is 2 times the first number. What are the three numbers?

KEEPING SKILFUL

Use compasses, a straightedge, and protractor to help you with the exercises below.

1 Construct circle A whose radii are congruent to a given segment.

2 Use a straightedge to draw a picture of diameter BD of $\odot A$.

3 Use a straightedge and a protractor to draw a picture of central angle BAC of $\odot A$. $\angle BAC^\circ = 60$.

4 What is minor $\overset{\frown}{BC}^\circ$? What is major $\overset{\frown}{BC}^\circ$?

5 Name a semicircle of $\odot A$.

6 What is the measure in degrees of the semicircle that you named for exercise 5?

7 Name an arc of $\odot A$ that is adjacent to minor $\overset{\frown}{BC}$.

8 What is the measure in degrees of the arc that you named for exercise 7?

9 Draw a picture of any acute triangle MNO. Construct the perpendicular bisector of $\overline{MN}$ to determine P, the midpoint of side MN. Then construct median OP of $\triangle MNO$.

10 Draw a picture of any obtuse triangle RST. Then construct the bisector of the obtuse angle.

11 Draw a picture of any scalene triangle UVW. Then use the s-a-s property of congruence for triangles to construct a triangle that is congruent to $\triangle UVW$.

12 Draw a picture of any acute triangle DEF. Then use the a-s-a property of congruence for triangles to construct a triangle that is congruent to $\triangle DEF$.

APPLYING MATHEMATICS

For each problem, first write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The universe for each variable is D.

1 In 12 years, Don will be more than 30 years old. Eleven years ago, he was less than 15 years old. How old is Don now?

2 The distance from Vancouver to Fort William by air is 3 times the distance from Fort William to Sarnia by air. The distance from Vancouver to Fort William is 1512 mi. What is the distance from Fort William to Sarnia?

3 On October 1, Mr. Berry had \$500. On October 31, he was \$200 in debt. What is the difference of the amounts of money he had on October 1 and on October 31?

THINK Can you use $^-200$ to represent the amount of money Mr. Berry had on October 31?

4 The area of Mr. Johnson's farm is 205 acres. Mr. Novak wants to buy 40 acres of Mr. Johnson's land. Mr. Evans wants to buy $25\frac{1}{2}$ acres of it. If Mr. Johnson sells to one of these men, how many acres will he have left in his farm?

5 The regular price of a certain camera was \$35.95. During a sale, the price of the camera was \$28.76. What was the rate of discount?

6 One year the lowest temperature recorded in Canada was 81.4 degrees below zero. The highest temperature recorded that year was 191.6 degrees higher than the lowest temperature. What was the highest temperature recorded in Canada that year?

7 Together, Al and Stewart spent less than 12 hr. working at Palmer's Drugstore. Stewart worked $6\frac{1}{2}$ hr. How many hours did Al work?

8 Mr. Wills bought 2 fluorescent lights priced at \$10.95 each; 8 wall switches priced at 2 for \$1.15; and an overload switch priced at \$6.95. How much did he spend in all for these items?

9 During a 3-hour period, rock temperatures in the mountains have been known to rise from 15° below zero centigrade to 27.9° above zero centigrade. What is the average rise per hour in rock temperatures during such a period?

10 The population of Morton City is 9 times the population of Huntsville. The population of Warner is twice the population of Huntsville. The total population of Morton City and Warner is 278,762. What is the population of each of the three cities?

11 Altogether, Ray bought fewer than 12 five-cent and four-cent stamps. If he had bought 6 more five-cent stamps, he would have bought 10. How many stamps of each kind did he buy?

12 The length of side AC of $\triangle ABC$ is $7\frac{1}{2}$ in. The length of side CB is 10 in. The length of side BA is less than the sum of the lengths of sides AC and CB. The perimeter of $\triangle ABC$ is at least $25\frac{1}{2}$ in. What is the length of side BA in inches?

13 The length of a base of a parallelogram is 11.25 ft. This base is 2.5 times as long as the altitude. What is the area in square feet?

14 Certain rate pairs are equivalent to $3\frac{1}{2}/5\frac{1}{2}$. The first component of each of these rate pairs is either 12 or 15. What are the rate pairs?

15 The sum of two numbers is at most 2.5. The first number is equal to the quotient of $^-.68$ and $^-.4$. What are the numbers?

16 What numbers are not greater than the difference of $^-25.5$ and $^-16.4$?

17 A first number is equal to the product of 5 and a second number. The sum of the first and second numbers is $13\frac{1}{2}$. The difference of the second number and a third number is $1\frac{1}{2}$. What are the three numbers?

CHECKING UP

The small numerals within parentheses tell what pages to turn to for help if you need it.

Test 142

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 The square of 16 is 4. (253)

2 The square of a negative rational number is a positive rational number. (254)

3 The decimal for $\sqrt{2}$ is a repeating infinite decimal. (259)

4 The set of points in a line can be put in one-to-one correspondence with D. (272)

5 For each x and a, if $x^2 = a$, then x is a square root of a. U = R. (255)

6 The square of a positive integer is also a positive integer. (254)

7 $^-8 < {}^-\sqrt{75} < {}^-9$ is a true statement. (269)

8 $4x + 5x = {}^-3.7$ and $(4 + 5)x = {}^-3.7$ are equivalent conditions. U = D. (277)

9 A graph of $\{c \mid \sim ({}^-c + \frac{1}{2} = {}^-\frac{1}{2})\}$ contains the dot for point 1. U = D. (289)

10 A graph of the solution set of the condition $n + 2 = {}^-4 \wedge m = 3$ is a picture of a line. The universe for (m, n) is D × D. (289)

Test 143

Tell what words or symbols best complete each of the following exercises.

11 The principal square root of 100 is ~~~. (256)

12 The set of real numbers is the union of the set of rational numbers and the set of ~~~ numbers. (263)

13 A standard name of $\sqrt{6}(1/\sqrt{6})$ is the numeral ~~~. (267)

14 The second and third quadrants are to the ~~~ of the second axis. (285)

15 Each of the numbers 8 and ~~~ is a square root of 64. (256)

16 If $\sqrt{625} = 25$, then $\sqrt{625} + .452 = 25 + .452$ is an example of the ~~~ property of addition of real numbers. (268)

17 There is a real number between any two given real numbers because of the ~~~ property of D. (272)

18 A rational approximation of $4 + \sqrt{24}$ is ~~~. (Hundredths) (266)

19 In the real plane, point $({}^-7, .8\overline{3})$ is located in the ~~~ quadrant. (285)

20 To the nearer one, the positive square root of 405 is ~~~. (259)

Test 144

For each of exercises 21 through 27, tabulate the solution set of the condition expressed, if possible. Otherwise, write a standard description of the solution set. The universe for each variable is D.

21 $19 + x = 8$. (276)

22 $.9(z - .5) = .63$. (278)

23 $\frac{1}{3}s/\frac{3}{4} \sim {}^-\frac{5}{2}/\frac{9}{2}$. (279)

24 $^-7m > 32$. (281)

25 $k - 1.3 < {}^-6$. (281)

26 $\sim ({}^-3.6 - x > {}^-4.8)$. (283)

27 $(3a + 2b - c = 15) \wedge (\frac{1}{2}b = 3\frac{1}{2}) \wedge (b - c = 3\frac{1}{2})$. (290)

Two equivalent conditions are expressed in each of exercises 28 through 32. Name the property or definition that was used to obtain the second condition from the first condition. U = D.

28 $1 \cdot d = \frac{13}{17}$. $\quad d = \frac{13}{17}$. (266)

29 $14 = 4m + 3m$. $\quad 14 = (4 + 3)m$. (266)

30 $0 + t = {}^-9$. $\quad t = {}^-9$. (266)

31 $^-4x < 29$. $\quad {}^-\frac{1}{4}({}^-4x) > {}^-\frac{1}{4}(29)$. (271)

32 $.4/.3b \sim 8.7/1.5$. $\quad .4(1.5) = 8.7(.3b)$. (279)

Make a graph of the solution set of each condition expressed below. The universe for (r, s) is D × D.

33 $2r = s$. (287)

34 $r = s + \frac{3}{4}$. (287)

35 $r = 2 \wedge s \leqq 2\frac{1}{2}$. (289)

36 $s \geqq {}^-1\frac{2}{3} \wedge r \geqq {}^-3$. (289)

CHECKING UP

The small numerals within parentheses tell what pages to turn to for help if you need it.

Test 145

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 An open interval in R_a contains its limits. (83)

2 A hexagon has six sides. (122)

3 A circle and a tangent to that circle have two points of tangency. (115)

4 Zero is the only rational number that does not have a reciprocal. (208)

5 An angle has two bisectors. (127)

6 Division of rational numbers is commutative. (211)

7 The set of positive rational numbers, together with addition and multiplication, forms a number system. (198)

8 $(^{-}4)^5$ is a positive integral power. (224)

9 $\{x \mid x < {}^{-}.326\}$ contains $^{-}\sqrt{9}$. U = D. (270)

10 $(\frac{3}{7})^2 \times (\frac{3}{7})^3 = (\frac{3}{7})^5$. (225)

Test 146

Tell what words or symbols best complete the exercises below.

11 A symbol for circle H is ——. (114)

12 $\overset{\frown}{MN}$ and $\overset{\frown}{NP}$ are adjacent arcs in circle S. If $\overset{\frown}{MN}° = 40$ and $\overset{\frown}{NP}° = 38$, then minor $\overset{\frown}{MP}° =$ ——. (120)

13 A standard name of 8.26×10^{-2} is the numeral ——. (233)

14 The endpoints of a directed segment are the starting point and the —— point. (153)

15 The three properties of congruence for triangles that you have studied are the s-a-s, the a-s-a, and the —— properties. (137)

16 The reciprocal of $^{-}\frac{1}{7}$ is ——. (209)

17 $\frac{5}{4} + {}^{-}\frac{4}{6} = \frac{5}{4} + {}^{-}\frac{2}{3}$ because of the —— property of addition of rational numbers. (178)

18 $^{-}\frac{7}{10}(^{-}\frac{1}{2}) \in R$ because of the —— property of multiplication of rational numbers. (184)

19 A common fraction numeral for $^{-}.032$ is ——. (213)

20 Equivalent conditions are conditions that have the same ——. (238)

Test 147

For each problem, first write a sentence that expresses a condition. Tabulate the solution set of the condition, if possible. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The universe for each variable is D.

21 A ticket for a bleacher seat in a ball-park costs 75¢. A ticket for a box seat costs $3\frac{1}{3}$ times as much as a ticket for a bleacher seat. How much does a ticket for a box seat cost? (102)

22 In a certain 1-hr. television show, 14 min. are used for commercials. What per cent of the hour is used for commercials? (103)

23 Five different readings of the depth of a submarine were: 48 ft., 65 ft., 110 ft., 216 ft., and 205 ft. What was the average reading? (Tenths) (104)

24 Mr. Parker paid 50% of the cost of a new front walk. The walk is 45 ft. long and cost $1.60 per foot. How much did he pay? (105)

25 Together, Don and Oscar have less than 120 yd. of fishing line. Don has 25 yd. of 9-lb. line and 40 yd. of 15-lb. line. How many yards of fishing line does each boy have? (296)

26 What numbers can be multiplied by 5.7 so that each product is at least $^{-}1.9$? (295)

27 The difference of a first and a second number is $2\frac{3}{4}$. The product of $6\frac{2}{3}$ and the second number is $^{-}\frac{60}{9}$. What are the numbers? (110)

28 When the first of three numbers is subtracted from $^{-}8.2$, the difference is $^{-}4$. The sum of the first and second numbers is $^{-}5$. The third number is the product of the first and second numbers. Name the three numbers. (297)

Unit 12

Measurement

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Standard units of length

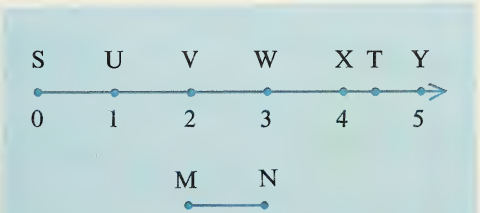
In unit 8 you learned what is meant by the measure of a segment. You also learned how to use standard units in measurement. In this lesson you will review some important ideas concerning measure. Then you will learn more about the metric system of measures.

You know that the measure of a segment is a number associated with the segment, and that you can obtain the measure of a segment by finding the absolute difference of the coördinates of the endpoints. First you will see how you can use a unit segment to find the measures of other segments.

A Look at D1. You can find the measure of $\overline{ST}$ by comparing it with the unit segment MN. What is $m(\overline{MN})$? Remember that you read the symbol $m(\overline{MN})$ as "the measure of segment MN."

B First, the number 0 was assigned to point S. Then point U was located so that $\overline{SU} \cong \overline{MN}$. What is the coördinate of point U?

C How were points V, W, X, and Y located? What is the measure of each of the segments UV, VW, WX, and XY?



D1

D What is the measure of $\overline{SX}$? Of $\overline{SY}$? How do you know that $m(\overline{ST}) > m(\overline{SX})$? That $m(\overline{ST}) < m(\overline{SY})$?

E Suppose that you subdivide $\overline{XY}$ into ten congruent segments. Is .1 the measure of each of these segments? Suppose that T is the right endpoint of the fourth of these segments. What is the measure of $\overline{XT}$? Of $\overline{ST}$?

F You know how to find the measure of $\overline{ST}$ by comparing it with the unit segment MN. Will you obtain the same measure for $\overline{ST}$ if you compare it with a unit segment that is not congruent to $\overline{MN}$?

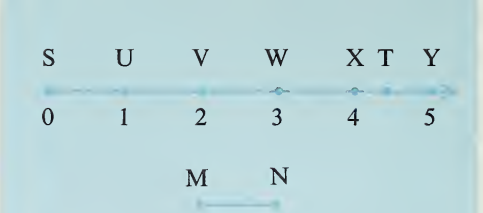
You know that certain physical objects have a characteristic called length. If you choose a unit segment with which to compare the segment suggested by the length of the object, you can find a number that is the measure of the length of the object. For convenience in this book, we will use the term "length" to mean both the characteristic of length and the number that is the measure of length.

G Suppose that each of two students in your class measures the length of the same desk. Suppose also that these students choose, as unit segments, the segments represented by two pieces of chalk that are not alike. Why may these students obtain different numbers for the length of the desk?

H Does the measure of the length of a physical object depend upon the unit segment with which the object is compared?

You can see that the measure of the length of a physical object depends upon the unit segment that is used. This is why people have agreed upon certain standard units.

The familiar standard units, such as the inch, the foot, the yard, and the mile, belong to the *English system of measures*. This system is in common use in most English-speaking countries. Many other countries, however, use the *metric system of measures*. The metric system is also used by scientists in every country.



D1

10^{-3} m.	1 millimetre (mm.) (mil′ə mē′tər)
10^{-2} m.	1 centimetre (cm.) (sen′tə mēt′ər)
10^{-1} m.	1 decimetre (dm.) (des′ə mēt′ər)
10^{0} m.	1 metre (m.) (mē′tər)
10^{1} m.	1 decametre (dkm.) (dek′ə mēt′ər)
10^{2} m.	1 hectometre (hm.) (hek′tə mēt′ər)
10^{3} m.	1 kilometre (km.) (kil′ə mēt′ər or kə lom′ə tər)

D2

The English system developed gradually over a period of centuries. The metric system, on the other hand, was invented in the eighteenth century by a group of French scientists who believed that a convenient system of measures could be constructed according to the principles of the decimal system of numeration.

The *metre* is the standard unit of length upon which the metric system is based. Originally, a metre was defined as one ten-millionth of the distance from the North Pole to the equator. In 1960, the metre was redefined in

terms of the wave length of radiation from the chemical element Krypton 86. A length of one metre is about $3\frac{1}{3}$ inches longer than a length of one yard. Later in this lesson, you will learn more about comparisons between metric units and English units. First, however, you will see how the metric system is organized.

A Study the table in D2. Each length named in the first column is expressed in terms of the metre (m.). Each of these lengths is the same as the metric unit of length named opposite it in the second column. What unit of length is the same as 10^0 m.? Is $10^0 = 1$?

REMINDER

The zero power of any non-zero rational number is 1.

A negative integral power is a power that is the reciprocal of a given positive integral power.

See lesson 144, pages 228 and 229.

B What unit of length is the same as 10^{-1} m.? Is $10^{-1} = \frac{1}{10}$? What unit of length is the same as $\frac{1}{10}$ metre?

C What unit of length is the same as 10^{-2}m.? What unit is the same as $\frac{1}{100}$ metre?

D What unit of length is the same as $\frac{1}{1000}$ m.?

E Which is longer, a decimetre or a centimetre? A centimetre or a millimetre? Are the decimetre, the centimetre, and the millimetre longer or shorter than the metre?

F What unit of length is the same as 10 metres? The same as 100 metres? The same as 1000 metres? Explain your answers.

G What standard metric units of length are longer than the metre?

H Which is longer, a kilometre or a hectometre? A decametre or a hectometre? A millimetre or a kilometre?

One of the events in the Olympic Games is the 50,000-metre walk. What is the length of the walk in kilometres?

D3

I How do you know that the metric system of measures is a "decimal system"?

Because 10^{-1} m., or .1 metre, is the same as 1 decimetre, we sometimes say there is .1 metre *in* 1 decimetre. You can use the rate pair .1/1 to represent the comparison of the number of metres in 1 decimetre with 1 decimetre. Each component of .1/1 refers to the same length. However, since the units of measure are different, the components of the rate pair are different.

J What rate pair can you use to represent the comparison of the number of metres in 1 centimetre with 1 centimetre?

K What rate pair represents the comparison of the number of metres in 1 decametre with 1 decametre? The comparison of the number of metres in 1 kilometre with 1 kilometre? The comparison of the number of metres in 1 millimetre with 1 millimetre?

Now read the problem in D3. The universe for each variable is D.

L Why can you use 1000/1 in a condition for the problem?

M You know that the length of the walk is 50,000 metres. You are to find the length in kilometres. Why will you use $50000/x$, and not $x/50000$, in a condition for the problem? To what does x refer?

The sentence below in blue expresses a condition for the problem.

$$1000/1 \sim 50000/x.$$

N Tabulate $\{x \mid 1000/1 \sim 50000/x\}$. Give the answer to the problem in D3.

For some problems that concern standard metric units, you may find it convenient to use a compound condition.

The work that follows will help you develop a compound condition for the problem in D4. The universe for each variable is D.

The table you have been using does not tell you directly how many millimetres are in 1 centimetre. The table does, however, tell you how many metres are in 1 centimetre and also how many metres are in 1 millimetre. First you will develop a condition to help you find the number of metres in 2.1 cm.

O Why can you use .01/1 in one of the simple conditions for the problem? Why will you use $x/2.1$, and not $2.1/x$, in this condition? For what is x a variable?

P Is $.01/1 \sim x/2.1$ one of the simple conditions for the problem? You can use this condition to find the number of metres in 2.1 cm.

Next you can develop a condition to find the number of millimetres in x metres.

Q Does .001/1 represent the comparison of the number of metres in 1 millimetre with 1 millimetre?

You want to find a rate pair that is equivalent to .001/1 and whose second component is the number of millimetres in x metres. Use x/y to represent this rate pair.

R To what does x refer? To what does y refer?

S What is the second simple condition for the problem?

The sentence below in blue expresses a compound condition for the problem in D4.

$$.01/1 \sim x/2.1 \wedge .001/1 \sim x/y.$$

T Tabulate the solution set of the compound condition.

U What is the width of the wrench opening in millimetres?

The spark plug wrench that Mr. Lewis uses on his car has an opening that is 2.1 cm. wide. What is the width of the opening in millimetres?

D4

LENGTH

.039 in.	1 millimetre
.394 in.	1 centimetre
1.094 yd.	1 metre
.621 mi.	1 kilometre

D5

Standard atmospheric pressure is the pressure at sea level that will support a column of mercury 76 cm. high. What is the approximate height of the column of mercury in inches? (Thousandths)

D6

Notice what you have done. You know that the width of the wrench opening in centimetres is 2.1 You used a compound condition involving rate pairs to find that the opening in millimetres is 21. Later on, after you have studied formulas in unit 13, you will learn a shorter way of converting from one standard unit to another.

Sometimes you will need to change a measure given in metric units to a measure in English units. The exercises that follow will show you how to convert from one system of measures to another.

The international inch, adopted in 1959, is the same length as 2.54 centimetres. This com-

parison makes it possible to obtain the comparisons expressed in the table in D5. Each length named in the first column is approximately equal to the length named opposite it in the second column.

A Which is longer, the centimetre or the inch? The metre or the yard? The kilometre or the mile?

Now read the problem in D6. The universe for each variable is D.

B Why can you use .394/1 in a condition for the problem?

C Why will you use $x/76$, and not $76/x$, in the condition? To what does x refer?

$.394/1 \sim x/76$ is a condition for the problem in D6.

D Tabulate $\{x|.394/1 \sim x/76\}$. What is the approximate height of the column of mercury in inches?

Read the problem in D7. The universe for each variable is D.

You do not know the comparison of metres with inches, but you do know how many yards are in 1 metre and also how many inches are in 1 yard.

E First you will develop a condition to help you find the length in yards that is the same as 1.3 metres. Why will you use 1.094/1 in this condition? Which rate pair, $k/1.3$ or $1.3/k$, is equivalent to 1.094/1? Write a sentence that expresses this condition.

F Next you will develop a condition to help you find the number of inches in k yards. Why will you use 36/1 in this condition?

G Let h represent the number of inches in k yards. Will you use h/k or k/h in the second condition for the problem? Explain your answer.

The sentence in blue below expresses a compound condition for the problem.

$$1.094/1 \sim k/1.3 \wedge 36/1 \sim h/k.$$

While Sarah was in France, she bought some silk cloth. The cloth is 1.3 m. wide. What is the approximate width of the cloth in inches?

D7

$\overline{JK} \cong \overline{KL}$. $\overline{LM} \cong \overline{JL}$. $\overline{MN} \cong \overline{KL}$.

D8

H Tabulate the solution set of the compound condition. Then give the answer to the problem in D7.

In this lesson you reviewed the measure of a segment. You also studied metric units of length, and you learned how to convert a measure of length expressed in the metric system to a measure expressed in the English system.

On your own

Use the segments represented in D8 and the sentences in the display to help you answer the questions in exercises 1 through 4.

1 Suppose that you are using a unit segment AB to find the measure of $\overline{JP}$. If $\overline{JK} \cong \overline{AB}$, what coördinates should you assign to points K, L, M, and N?

2 Using the coördinates you named for exercise 1, find $m(\overline{JK})$. What is $m(\overline{JL})$? What is $m(\overline{LM})$?

3 Using the same coördinates, find $m(\overline{JM})$. What is $m(\overline{JN})$? How do you know that $m(\overline{JP}) > 4$? That $m(\overline{JP}) < 5$?

4 Suppose that $\overline{MN}$ is subdivided into two congruent segments and that P is the right endpoint of the first of these segments. What is $m(\overline{MP})$? What is $m(\overline{JP})$?

In each of exercises 5 through 10, a length is expressed in one unit of measure, and a second unit is named in parentheses. Express the given length in terms of the unit indicated in parentheses. Use the table if necessary.

5 1 m. (kilometres)
6 1 m. (millimetres)
7 1 mm. (centimetres)
8 5 hm. (metres)
9 1242 mi. (kilometres)
10 42 hm. (decametres)

For each of problems 11 through 17, first write a sentence that expresses a condition. Then give the answer to the problem. The universe for each variable is D.

11 One model of a European car has an overall length of 488 cm. What is the length of the car in metres?

12 In the 1960 Olympic Games, an entry from Poland won the 3000-metre steeplechase. What is the length of the steeplechase in kilometres?

13 Carol bought $\frac{3}{4}$ m. of lace in Brussels. How many centimetres of lace did she buy?

14 The bore of an engine is the measure of the diameter of a piston of the engine. The engine of one European car has a bore of 78 mm. What is the approximate bore of the engine in inches?

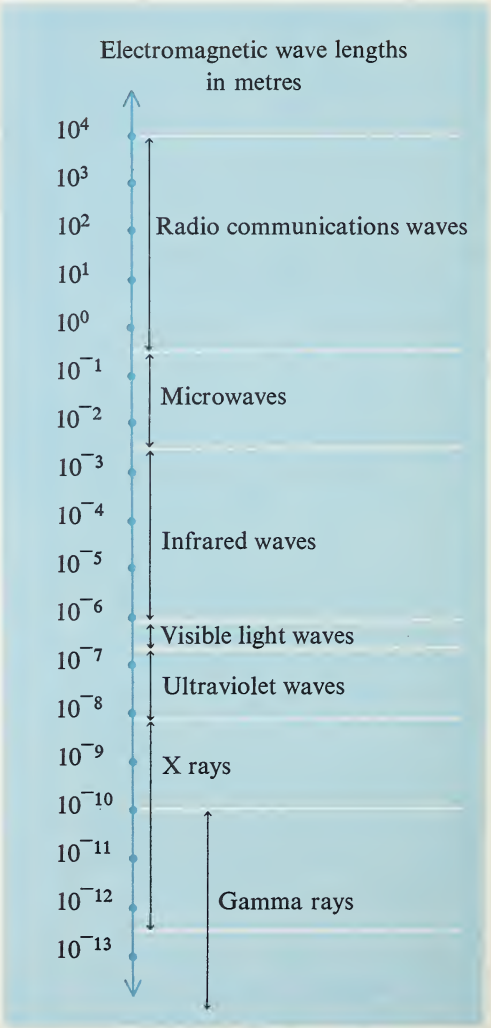
15 The filament of a flash bulb is made by shredding a sheet of metal that has been rolled to a thickness of 8×10^{-4} cm. What is the approximate thickness of the sheet in inches?

16 The moon's apogee is its distance from the earth when it is at that point in its orbit that is farthest from the earth. This distance is about 252,710 mi. What is the approximate apogee of the moon in kilometres? (Tens)

17 In some countries, a person's height is measured in centimetres. How many centimetres tall is a person who is 6 feet tall? (Ones)

APPLYING MATHEMATICS

The diagram in the display shows the classifications of certain types of electromagnetic waves according to their wave lengths in metres. Use the diagram in connection with exercises 1 through 8. In each exercise, the length of an electromagnetic wave is given. Tell how



you would classify each wave. For example, since 2×10^2 metres, or 200 metres, is between 10^1 and 10^2 metres, for exercise 1 you should write "radio communications wave."

1 2×10^2 m.
2 9×10^{-4} m.
3 9.9×10^{-2} m.
4 25×10^{-11} m.
5 1 m.
6 .000065 m.
7 .000000085 m.
8 .00000000000010 m.

For each problem below, first write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. The universe for each variable is D.

9 The length of a particular microwave is 10^{-2} m. The length of a particular X ray is 10^{-9} m. The length of the microwave is how many times the length of the X ray?

10 The length of an ultraviolet wave is 10^{-7} m. The length of an infrared wave is 10^{-5} m. The length of the ultraviolet wave is what fraction of the length of the infrared wave?

11 A particular radio communications wave is 10^2 m. long. What is its length in decametres?

12 The length of an X ray is 10^{-10} m. What is its length in millimetres?

13 An ultraviolet wave is 10^{-6} cm. in length. What is its length in metres?

14 The length of an infrared wave is 10^{-5} m. What is its length in kilometres?

15 The length of a gamma ray is 10^{-12} cm. What is its length in metres?

16 A particular radio communications wave is 6.5×10^2 m. long. What is its length in kilometres?

17 The length of a microwave is 2.2×10^{-2} m. What is its length in hectometres?

18 The length of a radio communications wave is 85 cm. What is its length in millimetres?

19 A particular ultraviolet wave is .000075 mm long. What is its length in centimetres?

APPLYING MATHEMATICS

For each of the following problems, first write a sentence that expresses a condition. Then find the number that satisfies the condition and give the answer to the problem. The universe for each variable is D.

1 The contents of a can of tomato juice weighed 720.8 grams. About what was the weight of the tomato juice in ounces?

2 Last year, a town in Holland produced 265,000 kilograms of cheese. This amount is about how many pounds?

3 A storage tank holds 4100 liters of oil. How many kilolitres of oil does the tank hold?

4 Mr. Campbell takes .0025 gram of a certain medicine each day. How many milligrams of the medicine does he take each day?

5 Mr. Harvey prepared 2.5 litres of a glucose solution. How many millilitres of the glucose solution did he prepare?

6 A sample of steel weighs 500 milligrams. What is the weight in grams of the sample of steel?

WEIGHT (METRIC)

1000 milligrams (mg.) (mil′ə grams)	1 gram (g.) (gram)
1000 grams	1 kilogram (kg.) (kil′ə gram)
.035 ounce (approx.)	1 gram
2.2 pounds (approx.)	1 kilogram

CAPACITY (METRIC)

1000 millilitres (ml.) (mil′ə lē′tərs)	1 litre (l.) (lē′tər)
1000 litres	1 kilolitre (kl.) (kil′ə lē′tər)
.88 quarts (approx.)	1 litre

7 Betty weighs 95 lb. Betty's weight in kilograms is about how much? (Tenths)

8 A chemist made a solution of hydrochloric acid that contained 79 grams of hydrogen chloride per litre. He made 20 litres of the solution. How many grams of hydrogen chloride did the 20 litres contain?

9 Mrs. Wilmington bought 3.5 kilograms of squash at a street market in Acapulco, Mexico. About how many pounds of squash did she buy?

10 Fred bought a candy bar that weighed 2 oz. About what was the weight of the candy bar in grams? (Tenths)

11 There are 716 calories in 100 grams of butter. How many calories are in 1 kilogram of butter?

12 On certain international flights, passengers are allowed to carry 20 kilograms of baggage without extra charge. Miss Klein's baggage weighs 51 lb. Her baggage weighs about how many pounds more than the baggage allowance?

13 Mr. Heath needs 5.5 litres of oil to fill the crankcase of his car. The oil costs $.45 per quart. About how much will it cost Mr. Heath to fill the crankcase with oil?

14 Mr. Green bought 45 litres of gasoline. About how many gallons of gasoline did he buy? (Tenths)

THINK Use x as a variable for the number of quarts of gasoline that Mr. Green bought. Use y as a variable for the number of gallons he bought. Is $4/1 \sim x/y$ one of the conditions for the problem?

15 A chemist dissolved 750 milligrams of silver in an acid. How many kilograms of silver did he dissolve in the acid?

16 The gasoline tank in Mr. Fisher's car holds 15.9 gal. of gasoline. About how many litres of gasoline does the tank hold?

Greatest possible error in measurement

In your previous study of measurement, you learned how to measure certain objects. You also learned that it is impossible to find the exact, or *mathematical*, measure of a physical object. In this lesson you will learn how mathematicians work with approximations of mathematical measures.

A D1 shows a picture of a card and a rule. The length of the rule between its left edge and the mark labelled "1" is 1 inch. What is the length of the rule between its left edge and the mark labelled "2"?

B Each inch of the rule is marked off into parts of equal length. Into how many parts is each inch marked off? What is the length of the rule between any two consecutive marks?

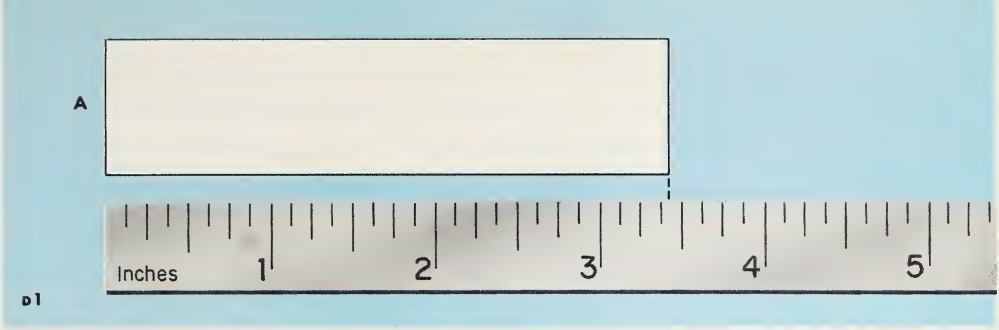
C Is there a mark on the rule for $\frac{1}{4}$ in.? For $\frac{1}{2}$ in.? For $\frac{1}{8}$ in.?

Now you are going to find approximations of the length of the card represented in D1. To begin with, you will use 1 in. as the unit of length. This means that you will round the approximation to the nearer inch.

D Is the right edge of the card closer to the mark for 3 in. or to the mark for 4 in.? Is the length of the card nearer to 3 in. or to 4 in.?

E When you use 1 in. as the unit of length, what approximation do you obtain for the length of the card?

Next you will use $\frac{1}{2}$ in. as the unit of length. This means that you will round the approximation to the nearer $\frac{1}{2}$ in.



D1

F Look again at D1. How many $\frac{1}{2}$-inch units of length are represented by the mark labelled "1"? How many such units of length are represented by the mark labelled "3"?

G Is the length of the card nearer to $\frac{6}{2}$ in. or to $\frac{7}{2}$ in.? Is the length nearer to $3\frac{0}{2}$ in. or to $3\frac{1}{2}$ in.? When you use $\frac{1}{2}$ in. as the unit of length, what approximation do you obtain for the length of the card?

Next you will use $\frac{1}{4}$ in. as the unit of length.

H How many $\frac{1}{4}$-inch units of length are represented by the mark labelled "1"? By the mark labelled "3"?

I Is the length of the card nearer to $3\frac{1}{4}$ in. or to $3\frac{2}{4}$ in.? When you use $\frac{1}{4}$ in. as the unit of length, what approximation do you obtain for the length of the card?

You can see that, each time you use a shorter unit of length, you are making a closer approximation of the length of the card. You come closer to finding the mathematical length when you use $\frac{1}{2}$ in. as the unit of length than when you use 1 in. When you use $\frac{1}{4}$ in. as the unit of length, you come closer still.

In exercises E, G, and I, you should have obtained 3 in., $3\frac{1}{2}$ in., and $3\frac{2}{4}$ in. as approximations of the length of the card. We will call these approximations *physical measures* of the length of the card.

Whether or not a physical measure of an object is a good approximation depends upon the unit of measure used. Therefore, scientists and mathematicians have agreed to express physical measures in such a way that it is obvious what unit of measure was used. For example, a physical measure of 3 in. indicates that 1 in. was used as the unit of length. A physical measure of $3\frac{1}{2}$ in. indicates that $\frac{1}{2}$ in. was used as the unit.

J When you use $\frac{1}{4}$ in. as the unit of length, you should give $3\frac{2}{4}$ in., and not $3\frac{1}{2}$ in., as the physical measure of the card. Explain the reason for this.

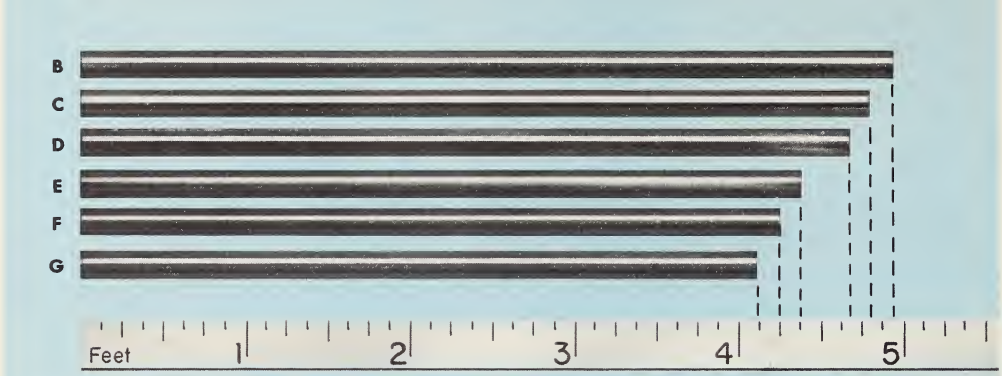
K What unit of length was used to obtain a physical measure of $7\frac{5}{12}$ ft.? Of 9 cm.? Of $14\frac{1}{3}$ yd.?

L How do you know that a physical measure of 14 yd. and a physical measure of $14\frac{0}{3}$ yd. were obtained by using different units of measure?

In a decimal numeral for a physical measure, the digit farthest to the right indicates the unit used to obtain the measure. You know that the digit farthest to the right in the numeral 13.5 is associated with tenths. So when a physical measure is expressed as 13.5 ft., you know that .1 ft. was used as the unit of length.

M What unit of length was used to obtain a physical measure of 7.8 mm.? Of 1.756 km.? Of 17.92 mi.?

N How does a physical measure of 45 m. differ from a physical measure of 45.0 m.?



D2

O D2 shows a picture of six pipes and a rule. Think of the distance between the left edge of the rule and the mark labelled "1" as 1 ft. What is the distance between the left edge of the rule and the mark labelled "4"?

P Is there a mark on the rule for $4\frac{1}{4}$ ft.? For $4\frac{3}{4}$ ft.? For 5 ft.?

Q Two of the pipes represented in D2 appear to have lengths that are greater than $4\frac{1}{4}$ ft. and less than $4\frac{3}{4}$ ft. Name these pipes. What is the physical measure of each of these pipes if you use $\frac{1}{2}$ ft. as the unit of length?

R Which pipe appears to have a length that is less than $4\frac{1}{4}$ ft.? Which pipes appear to have lengths greater than $4\frac{3}{4}$ ft.?

S Which pipe appears to have a length of $4\frac{1}{4}$ ft.?

If you are using $\frac{1}{2}$ ft. as the unit of length, you could give the physical measure of pipe F either as $4\frac{0}{2}$ ft. or as $4\frac{1}{2}$ ft. However, we will agree always to use the physical measure that contains an even number of unit lengths.

T Explain why we use $4\frac{0}{2}$ ft., rather than $4\frac{1}{2}$ ft., as the physical measure of pipe F.

U Suppose that the length of some object appears to be $4\frac{3}{8}$ ft. If you use $\frac{1}{4}$ ft. as the unit of length, which is the physical measure, $4\frac{1}{4}$ ft. or $4\frac{2}{4}$ ft.? Explain your answer.

V Suppose that the length of an object is less than $4\frac{1}{4}$ ft. If you use $\frac{1}{2}$ ft. as the unit of length, should you obtain $4\frac{1}{2}$ ft. as the physical measure? Suppose that the length of an object is greater than $4\frac{3}{4}$ ft. If you use $\frac{1}{2}$ ft. as the unit of length, should you obtain $4\frac{1}{2}$ ft. as the physical measure?

Now you are ready to study the mathematical meaning of a physical measure. A physical measure of $5\frac{0}{2}$ ft. means that $\frac{1}{2}$ ft. was used as the unit of length. A physical measure of $5\frac{0}{2}$ ft. also means that the mathematical length is equal to or greater than $4\frac{3}{4}$ ft. and equal to or less than $5\frac{1}{4}$ ft.

A How do you know that the set of numbers that contains $4\frac{3}{4}$, $5\frac{1}{4}$, and all the numbers between $4\frac{3}{4}$ and $5\frac{1}{4}$ is a closed interval in D? Name the limits of the interval.

REMINDER

The set of all x that satisfy $a \leqq x \leqq b$ is a closed interval. a and b are the limits of the interval. The universe for a, b, and x may be R_a, R, or D.

See lesson 113, page 85.

We say that a physical measure *determines* a closed interval. A physical measure of $5\frac{0}{2}$ ft. determines $\{x|4\frac{3}{4} \leqq x \leqq 5\frac{1}{4}\}$. If we know that the physical measure of an object is $5\frac{0}{2}$ ft., we assume that the mathematical length is a member of $\{x|4\frac{3}{4} \leqq x \leqq 5\frac{1}{4}\}$. U = D.

B What is the absolute difference of $5\frac{0}{2}$ and each limit of $\{x|4\frac{3}{4} \leqq x \leqq 5\frac{1}{4}\}$?

The *greatest possible error* of a physical measure of $5\frac{0}{2}$ ft. is $\frac{1}{4}$ ft. The mathematical length of an object will seldom differ from the physical measure by as much as the greatest possible error.

C Is the greatest possible error of a physical measure of $5\frac{0}{2}$ ft. the same as $\frac{1}{2}$ of the unit of length?

The greatest possible error of a physical measure is $\frac{1}{2}$ of the unit of measure.

For each of the physical measures expressed in exercises D through I, name the unit of length and the greatest possible error.

D $7\frac{2}{3}$ ft. **F** 16.03 mm. **H** $39\frac{5}{8}$ yd.

E 25.4 in. **G** 1.800 m. **I** 190 km.

J How can you determine the unit of measure from the greatest possible error of a physical measure?

A physical measure expressed as 150 ft. indicates that 1 ft. was used as the unit of length. When 10 ft. is used as the unit of length, a physical measure of 150 ft. may be expressed as shown in D3. Study D3.

K What unit of length was used to obtain a physical measure of 14 ft.? A physical measure of 3000 ft. ± 500? A physical measure of 690 ft.?

L Write a symbol for a physical measure of 2600 ft. with a unit of length of 1 ft. Write a symbol for a physical measure of 2600 ft. with a unit of length of 10 ft. Write another symbol for this physical measure with a unit of length of 100 ft.

"A length of 150 ft.

150 ft. ± 5

with a greatest possible error of 5 ft."

D3

greatest possible error of a physical measure. One half of the unit of measure that is used to obtain the physical measure.

M Name the unit of length for each of these physical measures: 160 yd. ± 5, 1200 mi. ± 50, 4000 mi. ± 5.

N You know that a physical measure of $5\frac{0}{2}$ ft. determines $\{x|4\frac{3}{4} \leqq x \leqq 5\frac{1}{4}\}$. Is $5\frac{0}{2} - \frac{1}{4} = 4\frac{3}{4}$? Can you obtain the lesser limit of this interval by subtracting the greatest possible error of the physical measure of $5\frac{0}{2}$ ft. from $5\frac{0}{2}$? How can you obtain the greater limit?

O Name the limits of the interval determined by a physical measure of 5.8 in. Give a standard description of this interval. U = D.

P Give a standard description of the interval determined by a physical measure of 7200 km. ± 50. U = D.

Q In lesson 114 you learned that the solution set of a condition like $x \approx_{.05} 25.4$ is a closed interval when the universe for x is R_a. When the universe for x is R or D, the solution set of $x \approx_{.05} 25.4$ is also a closed interval. Write a standard description of this interval in D.

R Explain how you use the tolerance to determine the interval that is the solution set of $x \approx_{.05} 25.4$. How do you use the greatest possible error to find the limits of the interval determined by a physical measure of 25.4?

S From now on, when the tolerance of a condition such as $x \approx 25.4$ is not indicated, the

tolerance is the same as the greatest possible error of a physical measure of 25.4. How do you know that the tolerance of $x \approx 25.4$ is .05? U = D.

T 9.35 is a physical measure. Write a standard description of the interval that is the solution set of $x \approx 9.35$. U = D.

In this lesson you studied units of length and learned how to express a physical measure by a numeral that indicates the unit of measure used. You also learned that the physical measure of an object determines a closed interval and that the mathematical measure of the object is a member of the interval. Finally, you learned that the greatest possible error of a physical measure is $\frac{1}{2}$ of the unit of measure.

On your own

For each of the physical measures expressed in exercises 1 through 20, first name the unit of length and the greatest possible error. Then name the limits of the interval determined by the physical measure. Finally, give a standard description of the interval. U = D.

1 $8\frac{3}{32}$ in.
2 .04 cm.
3 600 ft. ± 5
4 13.6 mi.
5 $18\frac{2}{3}$ yd.
6 49 km.
7 $54\frac{2}{8}$ in.
8 1900 yd. ± 50
9 $15\frac{7}{12}$ ft.
10 .0040 cm.
11 320 m. ± 5
12 $16\frac{0}{2}$ in.
13 162.9 km.
14 3410 ft.
15 250,000 mi. ± 500
16 $1000\frac{0}{8}$ in.
17 4.0 mm.
18 $60\frac{0}{2}$ yd.
19 .3 mm.
20 16.00 cm.

Write a standard description of the solution set of each condition expressed in exercises 21 through 29. U = D.

21 $x \approx 14.8$.
22 $w \approx 37\frac{0}{2}$.
23 $a \approx 621$.
24 $x \approx 10\frac{4}{64}$.
25 $x \approx 9.05$.
26 $z \approx \frac{2}{4}$.
27 $m \approx 6.0$.
28 $x \approx 33\frac{1}{3}$.
29 $n \approx \frac{7}{1000}$.

12 162 Exploring ideas

Precision and accuracy

You have learned that the greatest possible error of a physical measure is equal to $\frac{1}{2}$ of the unit of length. In this lesson you will study some of the ways in which physical measures can be compared and interpreted.

A Each length expressed in D1 is a physical measure. What lengths are expressed in examples A and B?

B What is the greatest possible error of $6\frac{1}{4}$ m.? Of $5\frac{1}{2}$ m.?

C How does $\frac{1}{4}$ m. compare with $\frac{1}{2}$ m.? How does the greatest possible error of $6\frac{1}{4}$ m. compare with the greatest possible error of $5\frac{1}{2}$ m.?

The physical measure $6\frac{1}{4}$ m. is more *precise* (pri sīs′) than the physical measure $5\frac{1}{2}$ m. because the greatest possible error of $6\frac{1}{4}$ m. is less than the greatest possible error of $5\frac{1}{2}$ m. For any two physical measures, if the greatest possible error of the first physical measure is less than the greatest possible error of the second, then the first physical measure is more precise than the second. You can also say that the first physical measure has greater *precision* (pri sizh′ən) than the second.

D Compare examples C and D in D1. Explain why 1.005 cm. is more precise than 6.04 cm.

E Compare examples E and F. Which of these physical measures has the greater precision? Explain your answer.

F Do the physical measures expressed in examples G and H have the same greatest possible error? Is either of these physical measures more precise than the other?

Comparison of physical measures in terms of precision and in terms of accuracy; significant digits and scientific notation

A $6\frac{1}{4}$ m.
B $5\frac{1}{2}$ m.
C 6.04 cm.
D 1.005 cm.
E 135 yd.
F 6.2 mi.
G $14\frac{5}{12}$ ft.
H $5\frac{3}{12}$ ft.

D1

U = D.
I $.05/2.5 \sim r/1$.
J $2.5r = .05$.
K $r = .02$.

D2

relative error (rel′ə tiv). For any given physical measure, the error per 1 standard unit. If 12 ft. is a given physical measure, then $\frac{1}{2}$ ft. is the greatest possible error. $\frac{1}{24}$, the solution of $\frac{1}{2}/12 \sim x/1$, is the relative error. The universe for x is D.

Two physical measures are named in each exercise. Tell which of the two is more precise.

G $6\frac{5}{10}$ yd., $3\frac{1}{36}$ yd.
H 10.8 km., 10.80 km.
I $6\frac{1}{2}$ rd., $6\frac{4}{8}$ rd.
J $8\frac{3}{4}$ in., 3 ft.
K .01 mm., .01 cm.
L $5\frac{1}{4}$ mi., 6 ft.

So far in this lesson, you have seen how the greatest possible errors can be used to compare the precision of two physical measures. Now you will learn how to make another kind of comparison between two physical measures.

A Suppose that the distance between two towns is given as 2.5 mi. What unit of length was used to obtain this physical measure? What is the greatest possible error of 2.5 mi.?

You can use the greatest possible error to determine the *average* error per mile for the physical measure of 2.5 mi.

B What condition for equivalence is expressed in D2? What is the universe for r? To what does each component of .05/2.5 refer? Is r a variable for the average error per mile?

C How do you know that condition J is equivalent to condition I? That condition K is equivalent to condition J?

D What number satisfies condition K?. Does this number also satisfy condition I?

.02 mi. is the average error per mile for the physical measure 2.5 mi. The number .02 is the *relative error* of 2.5 mi. The relative error for any given physical measure is the average error per mile, per centimetre, per foot, or whatever the case may be.

E What is the greatest possible error of 2.25 cm.? Of 35.0 cm.? Which of these physical measures is more precise?

By comparing the greatest possible errors of 2.25 cm. and 35.0 cm., you can decide that 2.25 cm. is the more precise physical measure. Next, you will compare the relative errors of these two physical measures.

F Can you use $.005/2.25 \sim x/1$ to determine the relative error of 2.25 cm.? How do you obtain $2.25x = .005$ from $.005/2.25 \sim x/1$? U = D.

G You can find the number that satisfies $2.25x = .005$ by finding the quotient of what two numbers? Is $.005 \div 2.25 = .00\overline{2}$? Is .002 the relative error (to the nearer thousandth) of 2.25 cm.?

H What condition for equivalence can you use to determine the relative error of 35.0 cm.? U = D.

I Find the relative error of 35.0 cm. to the nearer thousandth.

J Of the two physical measures, 2.25 cm. and 35.0 cm., which has the lesser relative error?

Because the relative error of 35.0 cm. is less than the relative error of 2.25 cm., we can say that 35.0 cm. is more *accurate* (ak′yù rit) than 2.25 cm. When you compare the precision of two physical measures, you compare their greatest possible errors. When you compare

the accuracy of two physical measures, however, you compare their relative errors. Observe that 2.25 cm. is more *precise* than 35.0 cm., but that 35.0 cm. is more *accurate* than 2.25 cm.

K What is the greatest possible error of 20 mi.? Of 50 mi.? How do you know that these two physical measures have the same precision?

L Use a condition for equivalence to determine the relative error of a physical measure of 20 mi. Use a condition for equivalence to determine the relative error of a physical measure of 50 mi. U = D.

M Which physical measure, 20 mi. or 50 mi., has the lesser relative error? Which is more accurate?

For the two physical measures expressed in each of exercises N through Q, first tell which is more precise. Then determine the relative error of each physical measure and tell which is more accurate. U = D.

N $4\frac{1}{2}$ in., $4\frac{3}{4}$ in.
O 2.08 km., 6.45 km.
P .0002 mm., 65 mm.
Q 16 yd., $6\frac{1}{2}$ yd.

Now you know that physical measures can be compared in terms either of their precision or of their accuracy. In the exercises that follow, you will learn how to interpret the digits in a decimal that expresses a physical measure.

A Look at D3. What is the unit of the physical measure expressed in example L?

Notice that, to express the number of 1-foot units contained in 20 ft., you must write two digits, 2 and 0. We say that the numeral that expresses a physical measure of 20 ft. contains two *significant* (sig nif′ə kənt) *digits*. In a decimal that expresses a measure, only those digits needed to express the number of units in the physical measure are significant digits.

B What is the unit of the physical measure expressed in example M? How many of these units are contained in .05 ft.?

L 20 ft.
M .05 ft.
N .0062 cm.
O .0145 cm.
P .020 in.
Q .20 in.
R .75 mi.
S 7500 mi. ± 5

D3

To express the number of units contained in .05 ft., you need write only the digit 5. In other words, .05 ft. can also be expressed as 5 one-hundredths foot. Thus, the digit 0 in the numeral .05 is *not* a significant digit. The numeral .05 contains only one significant digit.

C What is the unit of the physical measure expressed in example N? How many of these units are contained in the given measure?

D How do you know that, in the numeral .0062, neither of the digits for zero is significant? How many significant digits are there in the numeral .0062?

E What is the unit of the physical measure expressed in example O? Explain why there are 3 significant digits in the numeral .0145.

F Look at example P. What unit of length was used to obtain this physical measure? In the numeral .020, which of the digits are significant?

G Look at example Q. How many significant digits are there in the decimal that expresses .20 in.? Explain your answer.

H What is the unit of the physical measure expressed in example R? How many significant digits are there in the decimal .75?

I Now look at example S. How do you know that the unit of length is 10 mi.? How many 10-mile units are contained in 7500 mi.?

J Explain why, in the numeral 7500, the last digit for zero is not significant.

Now you will see how you can use the idea of significant digits to help you compare the accuracy of two physical measures. U = D.

K Look again at examples L and M. What is the relative error of 20 ft. to the nearer thousandth? Of .05 ft. to the nearer thousandth?

L Which of the two physical measures, 20 ft. or .05 ft., is more accurate? Does the numeral that expresses the more accurate physical measure have more significant digits than the other numeral?

M Now compare the physical measures expressed in examples N and O. First find the relative error of each physical measure to the nearer thousandth.

N Which of the physical measures, .0062 cm. of .0145 cm., is more accurate? Which of the numerals that express these physical measures has the greater number of significant digits?

O Next compare the physical measures expressed in examples R and S. Find the relative error of .75 mi. to the nearer ten-thousandth. Find the relative error of 7500 mi. $\pm$ 5 to the nearer ten-thousandth.

P Which physical measure, 7500 mi. $\pm$ 5 or .75 mi., is more accurate? Which of the numerals that express these physical measures has the greater number of significant digits?

The number of significant digits in the numerals that express two physical measures can be used to help you decide which physical measure is more accurate. Next you will see how scientific notation can be used to indicate the number of significant digits in a numeral.

A Look at D4. The physical measure expressed in example T is the average distance in miles from the earth to the sun. What is the unit of this physical measure? How many of these units are contained in 92,870,000 mi.?

B Explain why only the first four digits in the numeral 92,870,000 are significant.

C Now look at example U. Is the numeral in example U scientific notation for 92,870,000?

T 92,870,000 mi. $\pm$ 5000

U 9.287×10^7 mi.

V .0000140 cm.

W 1.40×10^{-5} cm.

D4

Does the numeral 9.287 contain all the significant digits in the numeral 92,870,000?

D Does the numeral 9.2870×10^7 express 92,870,000 $\pm$ 500 in scientific notation?

E The physical measure expressed in example V is the diameter of a certain organic cell. What is the unit of length in .0000140 cm.?

F Which digits are significant in the numeral that expresses .0000140 cm.?

G Is the numeral in example W scientific notation for the physical measure expressed in example V? Why is this measure expressed as 1.40×10^{-5} cm., and not as 1.4×10^{-5} cm.?

In this lesson you learned how to compare physical measures in terms of precision and accuracy. You also learned what relative error is and how to decide which digits are significant in a numeral that expresses a physical measure.

On your own

For each of exercises 1 through 10, determine the relative error of each physical measure expressed. For exercises 4, 6, 7, and 9, find the relative error to the nearer thousandth. If one of the two physical measures is more precise, tell which is more precise. If one of the two physical measures is more accurate, tell which is the more accurate. U = D.

1 $8\frac{1}{2}$ in., 19 in.
2 6 ft., 5 in.
3 $3\frac{2}{3}$ yd., $1\frac{1}{2}$ yd.
4 10.2 m., 5.45 m.
5 $10\frac{1}{2}$ ft., $12\frac{0}{4}$ ft.
6 .025 cm., .25 cm.
7 .0008 mm., .032 mm.
8 $4\frac{5}{36}$ yd., 35 in.
9 1.65 km., .80 km.
10 500 mi., 5000 mi. $\pm$ 5

For each physical measure expressed in exercises 11 through 18, first name the unit of length. Then tell which digits are significant in the numeral that expresses the physical measure. Finally, use scientific notation to express the physical measure.

11 14.030 cm.
12 160 yd.
13 .00050 mm.
14 .02 km.
15 38,000,000 mi. ± 500
16 300.0 ft.
17 .06 cm.
18 350,000 km. ± 5

KEEPING SKILFUL

For each condition expressed below, tabulate the solution set if possible. Otherwise, give a standard description of the solution set. The universe for each variable is D.

1 $\frac{3}{5}(x+5) = {}^{-}9$.
2 $\sim(\frac{3}{4}+a > 1\frac{2}{5})$.
3 $\frac{7}{12}c - \frac{1}{2}c = 1\frac{2}{3}$.
4 $\sim(.8x + {}^{-}.2 > 3)$.
5 $10.5 + d > 3.25 \wedge d + 1.32 < {}^{-}.93$.
6 $4g + 6 = 7\frac{3}{10} \vee g = \frac{1}{2} \div 5$.
7 $y - {}^{-}.3 < 1.8 \wedge .75y < 6.75$.
8 $r = \frac{1}{2}s \wedge \frac{3}{4}s = {}^{-}10$.
9 $a = 3\frac{1}{2}b \wedge a + b = 9$.
10 $(x + y = z) \wedge (10y = 5) \wedge (x - y = {}^{-}.1)$.

Make a graph of each of the sets described in exercises 11 through 15. The universe for (x, y) is $D \times D$.

11 $\{(x, y) | x < {}^{-}.33 + 1.83\}$
12 $\{(x, y) | y = 3\frac{1}{2}x\}$
13 $\{(x, y) | {}^{-}x > y + 6\}$
14 $\{(x, y) | x = y \wedge y + 4x = 10\}$
15 $\{(x, y) | 2y = 3x \vee 3y = 6x\}$

For each exercise below, express the given measure in the unit indicated in parentheses.

16 6500 ml. (litres)
17 5.5 g. (milligrams)
18 .8 kl. (litres)
19 150 g. (kilograms)
20 72 kl. (millilitres)
21 5600 mg. (kilograms)

Computation involving approximations

In lesson 161 you learned that physical measures are approximations. In this lesson you will learn how to compute with numerals that express approximations.

Suppose that you want to find the sum of several physical measures. Since each physical measure is an approximation, the sum of the physical measures will also be an approximation. Now you will see how you can obtain a good approximation of the sum of numbers that are themselves approximations.

A Suppose that you want to find the sum of these physical measures: 3 kilometres, 7.1 kilometres, and 2.8 kilometres. What is the greatest possible error of each physical measure?

B Look at D1. Is interval A determined by the physical measure 3 km.? Which of the physical measures determines interval B? Interval C?

C What is the lesser limit of each interval expressed in D1? What is the sum of 2.5, 7.05, and 2.75? Why is the sum of the mathematical measures greater than or equal to 12.30 km.?

D Name the greater limit of each interval expressed in D1. What is the sum of these limits? Why is the sum of the mathematical measures less than or equal to 13.50 km.?

E Now look at D2. Use w as a variable for the sum of the mathematical measures. How do you know that $12.30 \leqq w \leqq 13.50$? Is the solution set of $12.30 \leqq w \leqq 13.50$ an interval? How is the graph of the interval indicated?

Universe for x, y, and z = D.

A $\{x \mid 2.5 \leqq x \leqq 3.5\}$

B $\{y \mid 7.05 \leqq y \leqq 7.15\}$

C $\{z \mid 2.75 \leqq z \leqq 2.85\}$

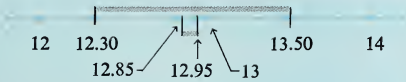
D1

U = D.

$\{w \mid 12.30 \leqq w \leqq 13.50\}$

D2

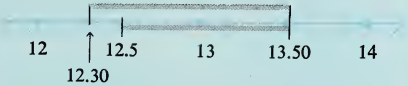
U = D.



$\{w \mid 12.85 \leqq w \leqq 12.95\}$

D3

U = D.



$\{w \mid 12.5 \leqq w \leqq 13.5\}$

D4

You know that the sum of the mathematical measures is a member of $\{w \mid 12.30 \leqq w \leqq 13.50\}$. Now you will see how to obtain an approximation of the sum that will determine an interval that nearly matches this interval.

F You are finding the sum of what physical measures? Is $3 + 7.1 + 2.8 = 12.9$? Look at D3. Explain why an approximation of 12.9 km. determines the interval expressed in D3.

The graphs of both $\{w \mid 12.85 \leqq w \leqq 12.95\}$ and $\{w \mid 12.30 \leqq w \leqq 13.50\}$ are shown in D3. By comparing the two graphs, you can see that the sum of the mathematical measures may not be a member of the interval determined by 12.9 km. Now you will see how you can round 12.9 km. to obtain a different approximation of the sum.

G How do you know that 3 km. is the least precise of the three physical measures? Is 3 km. expressed to the nearer kilometre?

H Now round 12.9 km. so that it is expressed to the same precision as 3 km. You should express 12.9 km. to the nearer kilometre.

I Give a standard description of the interval determined by 13 km.

J Look at D4. Which graph is a graph of $\{w \mid 12.30 \leqq w \leqq 13.50\}$? Which graph is a graph of $\{w \mid 12.5 \leqq w \leqq 13.5\}$?

When you compare the two graphs in D4, you can see that 13 km. determines an interval that very likely contains the sum of the mathematical measures, because this interval very nearly matches $\{w \mid 12.30 \leqq w \leqq 13.50\}$. Therefore, 13 km., which you obtained by rounding 12.9 km. to the same precision as 3 km., is a good approximation of the sum of the mathematical measures.

We will agree to use the following rule for finding the sum of numbers that are approximations:

First find the sum of the given numbers. Then round this sum to the same precision as the least precise of the given approximations.

You can use a similar rule for finding the difference of two numbers that are approximations:

First find the difference of the given numbers. Then round the difference to the same precision as the less precise of the given approximations.

K Find an approximation of the difference of 11.32 yd. and 7.8 yd. Use the rule given for finding such a difference.

We assume that the actual sum or difference is in the interval determined by an approximation of the sum or difference when the approximation is obtained according to the rules just expressed.

Now read the problem in D5. Notice that it concerns physical measures. The universe for each variable is D.

L Use x as a variable for the perimeter of the room. Is $x = 20\frac{1}{12} + 15\frac{3}{4} + 20\frac{1}{12} + 15\frac{3}{4}$ the condition for the problem? How do you know?

M Is $20\frac{1}{12} + 15\frac{3}{4} + 20\frac{1}{12} + 15\frac{3}{4} = 71\frac{8}{12}$?

N Which physical measure, $20\frac{1}{12}$ ft. or $15\frac{3}{4}$ ft., is less precise? How do you know?

O Why should you round $71\frac{8}{12}$ ft. to the nearer $\frac{1}{4}$ ft. to obtain an answer to the problem?

P How do you know that $71\frac{8}{12}$ is between $71\frac{2}{4}$ and $71\frac{3}{4}$?

Q If you express $71\frac{8}{12}$ ft. to the nearer $\frac{1}{4}$ ft., which do you obtain, $71\frac{2}{4}$ ft. or $71\frac{3}{4}$ ft.?

$71\frac{3}{4}$ ft. is a good approximation of the perimeter of the room. Therefore, you assume that the perimeter is in the interval determined by $71\frac{3}{4}$ ft., and you can give an answer to the problem.

The perimeter of the room is $71\frac{3}{4}$ ft.

Next you will learn how to find good approximations of products and quotients of numbers that are themselves approximations.

Read the problem in D6. The universe for each variable is D.

A Use y as a variable for the area of the piece of wood. Is $3.25/1 \sim y/2.5$ a condition for the problem?

B To find the solution of $3.25/1 \sim y/2.5$, you find the product of what two numbers? Are these numbers the physical measures of the length and width of the piece of wood?

C Look at D7. How do you know that interval D is determined by the physical measure of the length of the piece? Is interval E determined by the physical measure of the width?

D Explain why the area of the piece must be greater than or equal to 3.245×2.45. Is 7.95025 the product of 3.245 and 2.45?

E How do you know that the area must be less than or equal to 3.255×2.55? Is $3.255 \times 2.55 = 8.30025$?

F Use m as a variable for the area. How do you know that $7.95025 \leqq m \leqq 8.30025$?

G Look at D8. Is the area a member of the interval expressed in the display? Describe the graph of this interval.

John measured the length of a rectangular room. Roy measured the width. John said the room was $20\frac{1}{12}$ ft. long. Roy said the room was $15\frac{3}{4}$ ft. wide. What is the perimeter of the room?

D5

A rectangular piece of wood is 2.5 ft. wide and 3.25 ft. long. What is the area of the piece of wood in square feet?

D6

Universe for h and k = D.

D $\{h \mid 3.245 \leqq h \leqq 3.255\}$

E $\{k \mid 2.45 \leqq k \leqq 2.55\}$

D7

U = D.

$\{m \mid 7.95025 \leqq m \leqq 8.30025\}$

D8

H Think again about $3.25/1 \sim y/2.5$. To find the solution of the condition, you find the product of 3.25 and 2.5. Is $3.25 \times 2.5 = 8.125$? Does an approximation of 8.125 sq. ft. determine $\{m \mid 8.1245 \leqq m \leqq 8.1255\}$? How do you know?

8.125 sq. ft. is not a good approximation of the area of the piece of wood, because the interval determined by 8.125 does not very closely match the interval that contains the area. Now you will round 8.125 sq. ft. to obtain a good approximation. When you rounded the sum of physical measures, you were concerned with the *precision* of the measures. In rounding the product of physical measures, you will consider the *accuracy* of the measures.

I How do you know that a physical measure of 2.5 ft. is less accurate than a physical measure of 3.25 ft.? Use the idea of significant digits to help you.

J Does the numeral 2.5 have two significant digits? You want the numeral for the product of 3.25 and 2.5 also to have only two significant digits. What number do you obtain when you round 8.125 to the nearer tenth?

K Write a standard description of the interval determined by an approximation of 8.1 sq. ft.

L Why is it better to use 8.1 sq. ft. than to use 8.125 sq. ft. as an approximation of the area?

8.1 sq. ft. is a good approximation of the area. Now you can give an answer to the problem.

The area of the piece of wood is 8.1 sq. ft.

We will agree to use the following rule for finding the product of numbers that are approximations:

First find the product of the given numbers. Then round the product so that it is expressed by a numeral that has exactly as many significant digits as the numeral for the less accurate of the given approximations. If only one of the given numbers is an approximation, the numeral for the product should have exactly as many significant digits as the numeral for the approximation.

You can use a similar rule for finding the quotient of two given numbers that are approximations:

The numeral for the quotient of two numbers, when one or both of the numbers are approximations, should have exactly as many significant digits as the numeral for the less accurate approximation.

M Find a good approximation of the product of 16 and a physical measure of 327 ft. In this case, 16 is not an approximation.

N Find the quotient of these two approximations: 62 and .7.

In this lesson you learned how to find sums, differences, products, and quotients of physical measures.

On your own

Find the sum of the physical measures expressed in each of exercises 1 through 4.

1 .007 in., .05 in., .0109 in., .0083 in.

2 7.2 mi., 8.0 mi., 90 mi., 6 mi., 12.3 mi.

3 7.3 cm., 24 cm., 48 cm., 220 cm.

4 8900 m. $\pm$ 50, 4000 m. $\pm$ 50, 5660 m. $\pm$ 5

In each exercise below, two numbers that are approximations are expressed. For exercises 5 and 6, find the product of the two numbers. For exercises 7 and 8, find the quotient. The first number named is the dividend.

5 .280, 163.5

6 5.3×10^3, 4.8×10^4

7 .089, .6

8 732, 46

The following problems involve physical measures. For each problem, first write a sentence that expresses a condition. Then tabulate the solution set of the condition and give the answer to the problem. The universe for each variable is D.

9 The thickness of one leaf of a book is 6.7×10^{-3} cm. What is the thickness in centimetres of 158 leaves of the book?

10 On three successive days, the precipitation in Chatham was .07 in., 1.3 in., and .52 in. What was the total precipitation in Chatham for the three days?

11 Mount Clemenceau in British Columbia has an elevation of 12,000 ft. ± 50. The elevation of Mount Robson in the same province is 15,300 ft. ± 5. The elevation of Mount Clemenceau is how many feet less than the elevation of Mount Robson?

12 Cynthia listed 25 physical measures for a physics experiment. She found that the sum of the physical measures was 2.98 mm. What was the average measure in millimetres?

KEEPING SKILFUL

For each of the physical measures expressed in exercises 1 through 8, name the unit of length and the greatest possible error.

1 15.0 cm.
2 17.8 ft.
3 19 in.
4 $25\frac{3}{4}$ mi.
5 32 mm.
6 $10\frac{1}{12}$ in.
7 180 ft. ± 5
8 1700 yd. ± 50

For each of exercises 9 through 16, determine the relative error of each length expressed. For exercises 15 and 16, find the relative error to the nearer thousandth. U = D.

9 $3\frac{1}{2}$ in.
10 $16\frac{2}{3}$ yd.
11 25 ft.
12 $2\frac{0}{10}$ in.
13 500 m. ±5
14 2500 mi. ± 50
15 .075 cm.
16 12.9 ft.

Each condition expressed below involves a physical measure. Write a standard description of the solution set of each condition. U = D.

17 $x \approx 16.1$.
18 $y \approx 2\frac{7}{8}$.
19 $a \approx \frac{21}{1000}$.
20 $b \approx .0006$.
21 $m \approx .005$.
22 $y \approx 152$.

SPECIAL CHALLENGE

Many interesting puzzle problems can be solved by using a combination of mathematics and "trial and error."

Look at the pictures in D1. Notice the five disks on rod A. These disks may be moved from one rod to either of the other two rods, but only one disk may be moved each time.

Now read the problem in D2.

A First, suppose that there is only one disk on rod A. How many moves are necessary to transfer one disk to rod B?

B Now suppose that there are two disks on rod A. How many moves are necessary to transfer two disks to rod B? Call the smaller disk "K_1" and the larger "K_2." Why is it necessary first to place K_1 on rod C?

C How many moves are necessary to transfer three disks (K_1, K_2, and K_3) from rod A to rod B? Describe these moves. You may move a disk back to rod A if you need to.

D How many moves are necessary to transfer four disks from rod A to rod B?

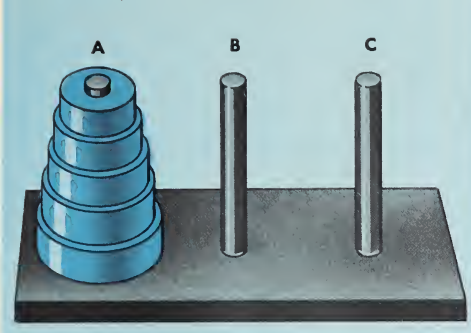
E Give the answer to the problem in D2.

F Use n as a variable for the number of disks on rod A. Use x as a variable for the number of moves necessary to transfer n disks from rod A to rod B. What condition can you use to find the number of moves necessary to transfer a given number of disks from rod A to rod B? The universe for n and x is C.

G How many moves are necessary to transfer 10 disks from rod A to rod B? To transfer 20 disks from rod A to rod B?

H Now read the problem in D3. How do you know that Mrs. Judd does not have 49 buttons in the box? How do you know that she does not have 61 buttons in the box?

I Why can you use $7x$ to represent the number of buttons in the box? The universe for x is C.



D1

How many moves are necessary to transfer the five disks from rod A to rod B, if only one disk is moved at a time and a larger disk is *never* at any time placed on top of a smaller disk?

D2

Mrs. Judd has a box of buttons. When the number of buttons is divided by 2, 3, 4, 5, or 6, there is a remainder of 1, but the number of buttons is divisible by 7. What is the least number of buttons that can be in the box?

D3

J How do you know that x cannot be divisible by 2, 3, 4, 5, or 6? Remember that $7x$ represents the number of buttons.

K Name the first fifteen counting numbers that are not divisible by 2, 3, 4, 5, or 6. Using these numbers as replacements for x in $7x$, find the answer to the problem.

L Now suppose that the number of buttons in another box is divisible by 2, 3, 4, 5, and 6, but leaves a remainder of 1 when divided by 7. What is the least number of buttons that can be in this box?

APPLYING MATHEMATICS

For each problem, first write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The problems involve numbers that are approximations. Remember to use the rules for computation that you studied in lesson 163. Review these rules now if you need to.

1 When weighed in a vacuum, a glass container filled with hydrogen weighs 36.95 grams. The weight of the container alone is 36.83 grams. What is the weight of the hydrogen?

2 During January, Kenton City had 2.16 in. of precipitation. During February, the city had 1.9 in. of precipitation. Altogether, how many inches of precipitation did the city have during January and February?

3 On September 15, Alice weighed $103\frac{1}{4}$ lb. By December 15, she had gained some weight, but she still weighed less than 111 lb. Alice had gained how many pounds between September 15 and December 15?

4 The area of Ontario is 5.95×10^5 sq. mi. The area of Saskatchewan is 2.52×10^5 sq. mi. The area of Ontario is how many times the area of Saskatchewan?

5 Mrs. Perry bought some material. She used $10\frac{1}{2}$ yd. to make draperies and had $\frac{7}{8}$ yd. left. How many yards of material did she buy?

6 Dan bought a piece of lumber $9\frac{1}{2}$ ft. long. He used some of it to build a shelf. Then he had a piece left that was $2\frac{1}{4}$ ft. long. How many feet of lumber did he use for the shelf?

7 Regina had .37 in. of rain on Sunday and .8 in. of rain on Monday. Regina had more rain on these two days than Saskatoon had. How many inches of rain did Saskatoon have on Sunday and Monday together?

8 Last year, Mr. Hawkins planted 45 acres of his farm in corn. The area of his farm is 225 acres. The number of acres that he planted in corn was what fraction of the number of acres in Mr. Hawkins' farm?

9 Mr. Mills drove 680.2 mi. in all on Friday and Saturday. He drove more than 375 mi. on Friday. How many miles did Mr. Mills drive on Saturday?

10 Sam has a piece of slate in the shape of a parallelogram. The length of an altitude is .62 m. The length of the base is 1.24 m. What is the area of this piece in square metres?

11 A rectangular window is $8\frac{1}{2}$ ft. long and $4\frac{3}{4}$ ft. wide. What is the perimeter of the window in feet?

12 Harry has a rectangular piece of sheet aluminum that is 40.2 cm. long and 30.6 cm. wide. What is its area in square centimetres?

13 Mr. Hall plans to fence a field that is in the shape of a parallelogram. The lengths of two sides of the field are 300.6 rods and 285.2 rods. How many rods of fencing does he need to enclose the field?

14 Sally has a triangular silk scarf. The length of an altitude of the triangle is 16.5 in. The length of the base is 32 in. What is the area of the scarf in square inches?

THINK In $32/1 \sim 2x/16.5$, is the number 2 an approximation?

15 The perimeter of a triangular patio is $97\frac{1}{2}$ ft. The length of each of two sides is $30\frac{1}{6}$ ft. What is the length of the third side in feet?

16 A window is square in shape. The length of a side is 96 in. What is the area of the window in square inches?

17 Paul made a table top out of a piece of wood shaped like a rhombus. The length of an altitude of the piece was 20.5 in. The length of the base was 24 in. What was the area of the table top in square inches?

CHECKING UP

If you need help with either of these tests, you can find it in lessons 160 through 163.

Test 148

For each of exercises 1 through 6, express the given length in terms of metres. U = D.

1 36 mm. **3** 417 cm. **5** 501.2 mm.
2 15 km. **4** 2.08 km. **6** .0099 cm.

For each of exercises 7, 8, and 9, express the given length in terms of inches. 2.54 cm. is the same as 1 in. U = D.

7 50.0 cm. **8** 3.000 m. **9** 44 mm.

For each of exercises 10 through 15, name the unit of length, the greatest possible error, and the interval determined by the given physical measure. U = D.

10 $4\frac{1}{2}$ ft. **12** 18.9 mi. **14** $5\frac{13}{16}$ in.
11 69 mm. **13** 32.51 km. **15** 2.048 m.

Test 149

For each of exercises 16 through 21, find the relative error of the given physical measure to the nearer thousandth. U = D.

16 8 yd. **18** .00025 km. **20** 12.0 cm.
17 $16\frac{0}{2}$ ft. **19** 300 in. $\pm$ 5 **21** $\frac{6}{10}$ mi.

For each of exercises 22 through 27, use scientific notation to express the given physical measure.

22 .002 cm. **25** 80 ft.
23 7.09 in. **26** 5,000,000 mi. $\pm$ 50
24 400,000 m. **27** 100.0063 km.

28 Find the sum of the following physical measures: 11.6 inches, 12 inches, 1.85 inches, 162 inches, 32 inches, 45.7 inches.

29 Find the quotient of the approximations 2×10^2 and 4.5×10^1. Use 2×10^2 as the dividend.

30 Find the product of the approximations 2453 and 63,000 $\pm$ 50.

31 Find the difference of the physical measures $7\frac{1}{8}$ feet and $3\frac{1}{2}$ feet. Use $7\frac{1}{8}$ as the minuend.

Similar triangles

In this lesson you will study a new kind of correspondence between triangles. Later you will see how this new relationship can be used in the process of measurement.

△ABC and △DEF are represented in D1. Notice that the lengths of the sides of △ABC and △DEF are also expressed in D1. We will assume that these lengths are mathematical measures.

When we are developing a new mathematical idea, we will often work with mathematical measures so that we do not need to use the rules that apply to approximations. You will be able to tell from the situation whether we are working with mathematical measures or with physical measures.

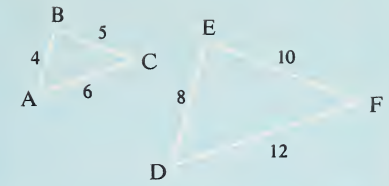
A How do you know that both triangles represented in D1 are scalene?

B Diagram A in D1 represents one way in which the vertices of △ABC and △DEF can be put in one-to-one correspondence. Use the correspondence between the vertices to name the corresponding angles and the corresponding sides of △ABC and △DEF.

C Sentences B, C, and D in D1 express true statements about the corresponding angles of the two triangles. Are the corresponding angles congruent?

D Are the corresponding sides of △ABC and △DEF congruent? Is △ABC ≅ △DEF? Explain your answers.

You know that the corresponding angles of △ABC and △DEF are congruent. However, the two triangles are not congruent because the corresponding sides are not congruent. Now you will see if the corresponding sides of the triangles are related in some other way.



A $ABC \longleftrightarrow DEF$

B $\angle A \cong \angle D$.

C $\angle B \cong \angle E$.

D $\angle C \cong \angle F$.

E $4/8 \sim 5/10$.

F $5/10 \sim 6/12$.

G $4/8 \sim 6/12$.

D1

E Read sentence E in D1. You know that $\overline{AB}$ and $\overline{DE}$ are corresponding sides of △ABC and △DEF. Does 4/8 represent the comparison of $m(\overline{AB})$ with $m(\overline{DE})$? $\overline{BC}$ and $\overline{EF}$ are also corresponding sides. What rate pair represents the comparison of $m(\overline{BC})$ with $m(\overline{EF})$? How do you know that statement E is true?

F Read sentence F. You know that 5/10 represents the comparison of $m(\overline{BC})$ with $m(\overline{EF})$. What rate pair represents the comparison of $m(\overline{AC})$ with $m(\overline{DF})$? Is statement F true?

G Look at sentence G. What comparison is represented by 4/8? By 6/12? Is statement G true?

H Are 4/8, 5/10, and 6/12 members of the same proportional relation?

> REMINDER
>
> A proportional relation is a set of ordered pairs in which each member is equivalent to each of the other members.
>
> *See lesson 41, page 170, Book 1.*

You know that each of the rate pairs 4/8, 5/10, and 6/12 represents a comparison of the measures of two corresponding sides of △ABC and △DEF. Because these rate pairs are members of the same proportional relation, we say that the measures of the corresponding sides of △ABC and △DEF are *proportional.*

Since the corresponding angles of △ABC and △DEF are congruent and the measures of the corresponding sides are proportional, the correspondence between the vertices, represented by diagram A in D1, is a *similarity correspondence*. If there is a similarity correspondence between the vertices of two triangles, then the triangles are *similar* (sim′ə lər).

Mathematicians use the symbol below at the right to express the idea of similarity. Notice that the symbol for similarity is the same as the symbol for equivalence and the symbol for the connective "not." When you see this symbol in a mathematical sentence, you will be able to tell which meaning is intended.

D2 shows how to write and read a sentence that expresses a true statement about △ABC and △DEF.

I Now look at D3. Use diagram H to name the corresponding angles and the corresponding sides of △MPQ and △RST.

J How do you know that the corresponding angles of △MPQ and △RST are congruent?

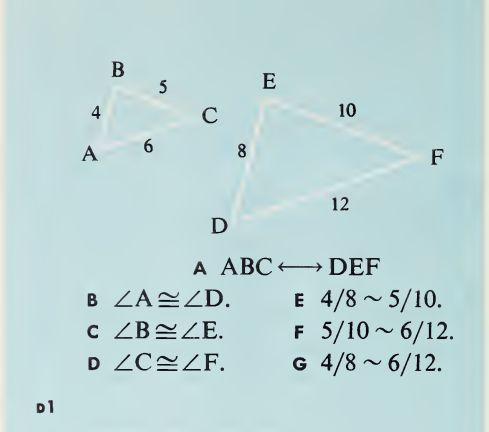
K What rate pairs represent the comparisons between the measures of the corresponding sides of △MPQ and △RST? Are the measures of the corresponding sides proportional? Is △MPQ ~ △RST? Explain your answers.

L How do you know that △MPQ ≅ △RST?

M You know that △MPQ ~ △RST and also that △MPQ ≅ △RST. Is every pair of congruent triangles also a pair of similar triangles?

N Suppose that △UVW ~ △XYZ. The order in which we have named the vertices of the two

sim i lar i ty correspondence (sim′ə lar′ə tē). A correspondence between the vertices of two triangles in which the corresponding angles are congruent and the measures of the corresponding sides are proportional. If there is a similarity correspondence between the vertices of △PQR and △STU, then △PQR and △STU are **similar triangles.**



A ABC ⟷ DEF

B ∠A ≅ ∠D.
C ∠B ≅ ∠E.
D ∠C ≅ ∠F.
E 4/8 ~ 5/10.
F 5/10 ~ 6/12.
G 4/8 ~ 6/12.

D1

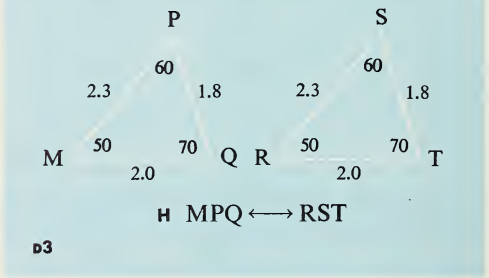
"Triangle ABC

△ABC ~ △DEF.

is similar to

triangle DEF."

D2



H MPQ ⟷ RST

D3

triangles indicates a similarity correspondence between the vertices. How do you know that the corresponding angles of $\triangle UVW$ and $\triangle XYZ$ are congruent? How do you know that the measures of the corresponding sides are proportional?

Next you will see how much information you need to have before you can decide if two given triangles are similar.

A Look at D4. Sentence I expresses a true statement about $\triangle GHI$ and $\triangle JKL$. Name the corresponding angles of the two triangles. How do you know that $\angle G \cong \angle J$, that $\angle H \cong \angle K$, and that $\angle I \cong \angle L$?

B What is the converse of the following statement? "If two triangles are similar, then the corresponding angles of the two triangles are congruent."

The converse that you expressed for exercise B is a true statement and is a property of similarity for triangles. We will accept this property without proof. The property is expressed below.

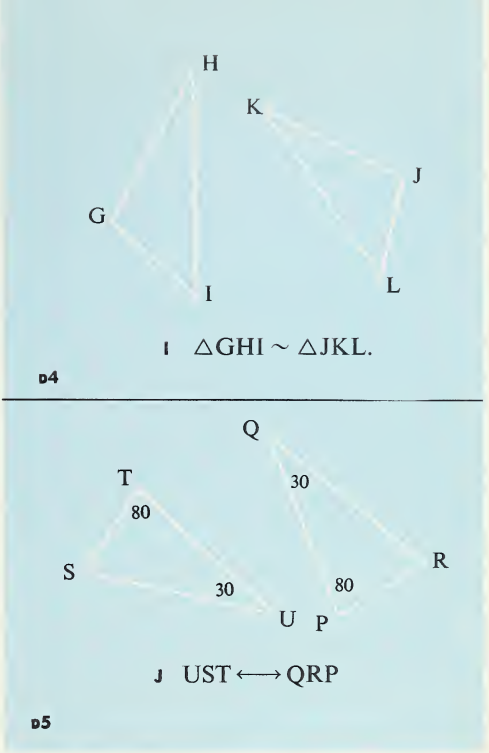
For any two triangles, if the corresponding angles are congruent, then the two triangles are similar.

The property just expressed is often called the *a-a-a* (angle, angle, angle) *property of similarity for triangles.*

C Look again at D4. How do you know that the measures of the corresponding sides of $\triangle GHI$ and $\triangle JKL$ are proportional?

D What is the converse of the following statement? "If two triangles are similar, then the measures of the corresponding sides are proportional."

The converse that you expressed for exercise D is a true statement and is another property of similarity for triangles. We will also accept without proof the *proportional-measures property of similarity for triangles.* The property is expressed below.



For any two triangles, if the measures of the corresponding sides are proportional, then the two triangles are similar.

E Now look at D5. Use diagram J to name the corresponding angles and the corresponding sides of $\triangle UST$ and $\triangle QRP$.

F What is $\angle S°$? What is $\angle R°$? Is $\angle S \cong \angle R$? Explain your answers.

G How do you know that $\triangle UST \sim \triangle QRP$?

In exercises E, F, and G, you derived another property of similarity for triangles. The property is expressed below.

For any two triangles, if two angles of one triangle are congruent to the corresponding angles of the other triangle, then the two triangles are similar.

The property just expressed can be called the *a-a* (angle, angle) *property of similarity for triangles*.

Next you will use the definition of similarity correspondence and the definition of equivalent rate pairs to develop a property of similar triangles.

H Look at D6. Sentence K expresses a true statement about the two triangles represented. Are the measures of the corresponding sides of $\triangle ABC$ and $\triangle DEF$ proportional?

I What rate pair represents the comparison of $m(\overline{AB})$ with $m(\overline{BC})$? What sides of $\triangle DEF$ correspond to $\overline{AB}$ and $\overline{BC}$? What rate pair represents the comparison of $m(\overline{DE})$ with $m(\overline{EF})$?

J How do you know that 6/7.5 is equivalent to 4/5?

K What rate pair represents the comparison of $m(\overline{BC})$ with $m(\overline{AC})$? What sides of $\triangle DEF$ correspond to $\overline{BC}$ and $\overline{AC}$? What rate pair represents the comparison of $m(\overline{EF})$ with $m(\overline{DF})$?

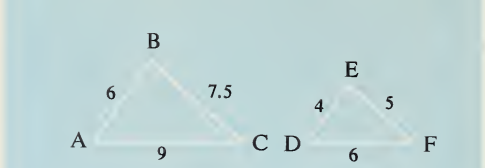
L Are the rate pairs that you named for exercise K equivalent?

M Is $m(\overline{AB})/m(\overline{AC}) \sim m(\overline{DE})/m(\overline{DF})$? How do you know?

In exercises H through M, you have seen that, for the similar triangles ABC and DEF, the rate pair that represents the comparison of the measures of any two sides of $\triangle ABC$ is equivalent to the rate pair that represents the comparison of the measures of the corresponding sides of $\triangle DEF$.

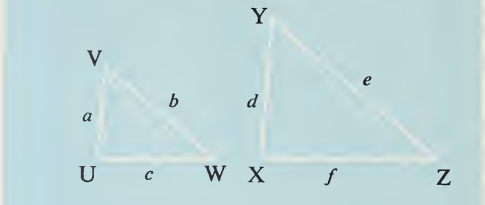
Now look at D7. Sentence L expresses a true statement about $\triangle UVW$ and $\triangle XYZ$. Notice that *a*, *b*, *c*, *d*, *e*, and *f* are used to represent the measures of the sides of the triangles.

N Why can you use the rate pair a/d to represent the comparison of the measures of sides UV and XY?



K $\triangle ABC \sim \triangle DEF$.

D6



L $\triangle UVW \sim \triangle XYZ$.

D7

$U = D$.

M $a/d \sim b/e$.

N $a \times e = b \times d$.

O $a \times e = d \times b$.

P $a/b \sim d/e$.

D8

O Which rate pair, b/e or c/f, can you use to represent the comparison of $m(\overline{VW})$ with $m(\overline{YZ})$?

P Look at sentence M in D8. You know that $\triangle UVW \sim \triangle XYZ$. How do you know that $a/d \sim b/e$?

Q How can you obtain condition N from condition M? How can you obtain condition O from condition N?

R What definition can you use to obtain condition P from condition O? Is condition P equivalent to condition M?

S For $\triangle UVW$ and $\triangle XYZ$, is $a/b \sim d/e$? Explain your answer.

T a/b represents the comparison of the measures of which two sides of $\triangle UVW$? d/e represents the comparison of the measures of which

two sides of $\triangle XYZ$? Is the rate pair that represents the comparison of the measures of sides UV and VW of $\triangle UVW$ equivalent to the rate pair that represents the comparison of the measures of the corresponding sides of $\triangle XYZ$?

U Show that the rate pair that represents the comparison of the measures of sides VW and UW of $\triangle UVW$ is equivalent to the rate pair that represents the comparison of the measures of sides YZ and XZ of $\triangle XYZ$. Use steps like those given in D8.

V Use steps like those given in D8 to show that $m(\overline{UV})/m(\overline{UW}) \sim m(\overline{XY})/m(\overline{XZ})$.

In exercises N through V, you used the definition of similarity correspondence between two triangles, the commutative property of multiplication, and the definition of equivalent rate pairs to develop a *property of similar triangles*. The property is expressed below.

For any two similar triangles, the rate pair that represents the comparison of the measures of any two sides of one triangle is equivalent to the rate pair that represents the comparison of the measures of the corresponding sides of the other triangle.

In this lesson you learned what is meant by a similarity correspondence between two triangles. You also studied certain important properties of similarity for triangles.

On your own

Use the pictures in D9 in connection with exercises 1 through 8. U = D.

1 Look at picture A. What is $\angle B°$? What is $\angle C°$? What is $\angle D°$? What is $\angle F°$?

2 Look at picture B. What condition can you use to determine $m(\overline{JL})$? What is $m(\overline{JL})$?

3 What is $m(\overline{HI})$?

4 Look at picture C. What is $\angle M°$? What is $\angle P°$?

D9

5 What is $\angle O°$? What is $\angle Q°$? What is $\angle N°$?

6 How do you know that $m(\overline{QR}) = 3\sqrt{2}$? What is $m(\overline{PR})$?

7 Now look at picture D. How do you know that $\angle T° = 60$? What is $\angle U°$? What is $m(\overline{ST})$?

8 What is the measure of each angle of $\triangle VWX$? What is the measure of $\overline{WX}$? What is the measure of $\overline{VX}$?

In each of exercises 9 through 17, true statements are expressed about $\triangle UVW$ and $\triangle XYZ$. For each exercise, tell whether or not $\triangle UVW \sim \triangle XYZ$. If the two triangles are similar, tell what property you used to make your decision. In some cases, you may not be able to decide whether the two triangles are similar or not. If this is the case, your answer should be that you cannot tell.

9 $\angle U \cong \angle X$. $\angle W \cong \angle Z$.

10 $m(\overline{UV}) = 6$. $m(\overline{XY}) = 9$.
$m(\overline{VW}) = 4$. $m(\overline{YZ}) = 6$.
$m(\overline{UW}) = 7$. $m(\overline{XZ}) = 10$.

11 $\overline{UV} < \overline{XY}$. $\overline{VW} < \overline{YZ}$. $\overline{UW} < \overline{XZ}$.

12 $\angle U^\circ = 90$. $\angle Y^\circ = 15$.
$\angle V^\circ = 15$. $\angle X^\circ = 90$.
$\angle W^\circ = 75$. $\angle Z^\circ = 75$.

13 $\angle V^\circ = 32$. $\angle Y^\circ = 32$.
$m(\overline{UV}) = 4$. $m(\overline{XY}) = 8$.

14 $m(\overline{UV}) = 5$. $m(\overline{XY}) = 2\frac{1}{2}$.
$m(\overline{VW}) = 6\frac{1}{2}$. $m(\overline{YZ}) = 3\frac{1}{4}$.
$m(\overline{UW}) = 8$. $m(\overline{XZ}) = 4$.

15 $\angle U \cong \angle X$. $\angle V \cong \angle Y$. $\angle W \cong \angle Z$.

16 $m(\overline{UV}) = 2.5$ $m(\overline{XY}) = 5.0$.
$m(\overline{UW}) = 4.8$. $m(\overline{XZ}) = 9.6$.

17 $\triangle UVW \cong \triangle XYZ$.

KEEPING SKILFUL

In the following exercises, the numbers that are expressed are approximations. For each exercise, find the sum, difference, product, or quotient.

1 $3.76 \div .72$
2 $85.01 - 69.1$
3 $.296 \times 176.3$
4 $52.644 \div 1.64$
5 $.750 \times .65$
6 $.94067 - .342$
7 254×3112
8 $1.592 \div .75$
9 $3451 - 2872.7$
10 $12.4 + 18 + 1.9$
11 $.006 + 1.34 + 8.91$
12 $630.1 + 80.52 + 65.3$
13 $169.4 - 3.21$
14 $256 + 7.89 + .49$
15 $(1.6 \times 10^3) \div (3.4 \times 10^2)$
16 $(6.4 \times 10^4) \times (3.6 \times 10^5)$

Trigonometric rate pairs

In this lesson you will study rate pairs that represent comparisons of the measures of two sides of a right triangle. First you need more information about right triangles.

A Look at D1. $\overline{BC} \perp \overline{AC}$. Which angle of $\triangle ABC$ is a right angle? Is $\triangle ABC$ a right triangle?

B Tabulate the intersection of $\angle C$ and $\overline{AB}$. ? Which side of $\triangle ABC$ is opposite $\angle C$?

$\overline{AB}$ is the *hypotenuse* of $\triangle ABC$. The hypotenuse of a right triangle is the side that is opposite the right angle of the triangle.

C In $\triangle ABC$, how do you know that $\angle A$ is ? an acute angle?

D Which side of $\triangle ABC$ is opposite $\angle A$?

E Which sides of $\triangle ABC$ are subsets of $\angle A$?

F Which of the sides that you named for exercise E is not the hypotenuse of $\triangle ABC$?

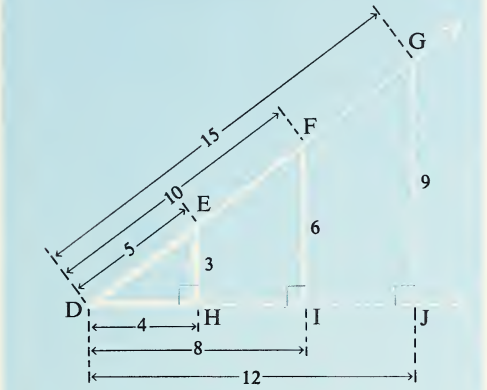
$\overline{AC}$ is the side of $\triangle ABC$ that is *adjacent* to $\angle A$. For either acute angle of a right triangle, the side of the triangle that is a subset of the angle, and that is not the hypotenuse of the triangle, is the side adjacent to the acute angle.

hy pot e nuse (hī pot′ə nüz or hī pot′ə nūz). The side of a right triangle that is opposite the right angle.

com ple men ta ry angles (kom′plə men′tə rē). A pair of angles the sum of whose measures in degrees is 90. Each angle of the pair is a **complement** of the other angle.

Definitions of tangent, sine, and cosine of acute angles of right triangles

D1



A $\triangle DEH \sim \triangle DFI$.
B $\triangle DFI \sim \triangle DGJ$.
C $\triangle DEH \sim \triangle DGJ$.

D2

G What kind of angle is $\angle B$? Which side of $\triangle ABC$ is opposite $\angle B$? Adjacent to $\angle B$?

H What is $\angle A° + \angle B° + \angle C°$? What is $\angle C°$? What is $\angle A° + \angle B°$?

$\angle A$ and $\angle B$ are *complementary angles*. Complementary angles are a pair of angles the sum of whose measures in degrees is 90.

I Explain why the two acute angles of any right triangle are complementary angles.

Next you will study rate pairs that can be associated with right triangles.

A Name the three right triangles represented in D2. Is $\angle D$ an angle of each triangle?

B How do you know that each sentence in D2 expresses a true statement?

C For each triangle, name the hypotenuse and the sides opposite and adjacent to $\angle D$.

D The measures of the sides of each triangle are expressed in D2. For $\triangle DGJ$, does 9/12 represent the comparison of $m(\overline{GJ})$ with $m(\overline{DJ})$? For each of the other triangles, name the rate pair that represents the comparison of the measure of the side opposite $\angle D$ with the measure of the side adjacent to $\angle D$.

The rate pairs that you named for exercise D are *trigonometric* (trig′ə nə met′rik) *rate pairs*. A trigonometric rate pair represents the comparison of the measures of any two sides of a right triangle.

E Is each rate pair that you named for exercise D equivalent to each of the other rate pairs? Explain your answer. Is each rate pair a member of {3/4, . . .}? Is this set infinite?

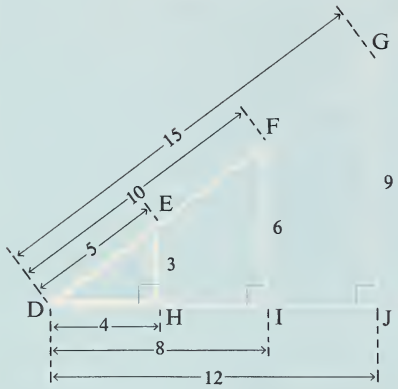
F How do you know that infinitely many right triangles have $\angle D$ as an acute angle? Is each of these triangles similar to each triangle represented in D2?

G For each right triangle that has $\angle D$ as an acute angle, does {3/4, . . .} contain the rate pair that represents the comparison of the measures of the side opposite $\angle D$ and the side adjacent to $\angle D$? Explain your answer.

Now you are going to find the first component of the member of {3/4, . . .} that has a second component of 1. You will use the condition $y/1 \sim 3/4$ to help you. U = D.

H Tabulate $\{y | y/1 \sim 3/4\}$. What number is the first component of the trigonometric rate pair that has a second component of 1 and that is a member of {3/4, . . .}?

For any right triangle that has $\angle D$ as an acute angle, the number .75, or $\frac{3}{4}$, is the *tangent* of $\angle D$. The tangent of an acute angle of a right triangle is the first component of a rate



A $\triangle DEH \sim \triangle DFI$.
B $\triangle DFI \sim \triangle DGJ$.
C $\triangle DEH \sim \triangle DGJ$.

D2

tan gent of an acute angle of a right triangle (tan′jənt). The first component of a rate pair that has a second component of 1 and that is equivalent to the rate pair that represents the comparison of the measure of the side opposite the angle with the measure of the side adjacent to the angle.

sine of an acute angle of a right triangle (sīn). The first component of a rate pair that has a second component of 1 and that is equivalent to the rate pair that represents the comparison of the measure of the side opposite the angle with the measure of the hypotenuse.

pair that has a second component of 1. This rate pair is equivalent to the rate pair that represents the comparison of the measure of the side opposite the angle with the measure of the side adjacent to the angle.

Now you will study two other trigonometric rate pairs that have second components of 1.

I Look again at D2. For $\triangle DGJ$, what rate pair represents the comparison of $m(\overline{JG})$ with $m(\overline{DG})$? For each of the other triangles represented in D2, name the rate pair that represents the comparison of the measures of the side opposite $\angle D$ and the hypotenuse.

J Explain why each rate pair that you named for exercise I is a member of {3/5, . . .}.

K Find the first component of the member of {3/5, . . .} that has a second component of 1.

For any right triangle that has $\angle D$ as an acute angle, the number .6, or $\frac{3}{5}$, is the *sine* of $\angle D$. The sine of an acute angle of a right triangle is the first component of a rate pair that has a second component of 1. This rate pair is equivalent to the rate pair that represents the comparison of the measure of the side opposite the angle with the measure of the hypotenuse.

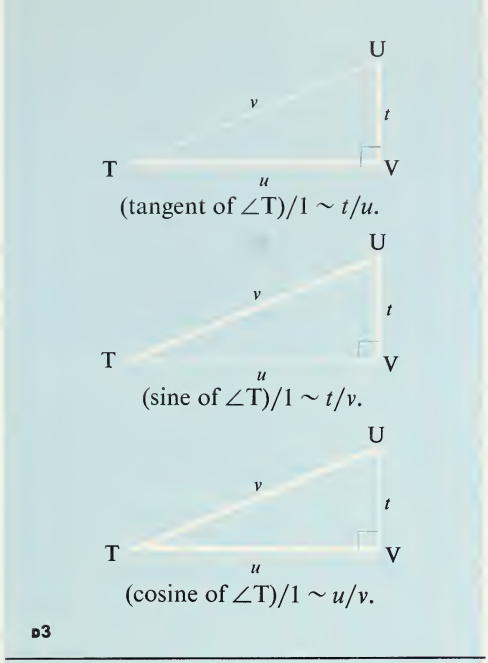
L Look again at D2. For $\triangle DGJ$, what rate pair represents the comparison of $m(\overline{DJ})$ with $m(\overline{DG})$? For each of the other triangles represented in D2, name the rate pair that represents the comparison of the measures of the side adjacent to $\angle D$ and the hypotenuse.

M Explain why each rate pair that you named for exercise L is a member of {4/5, . . .}.

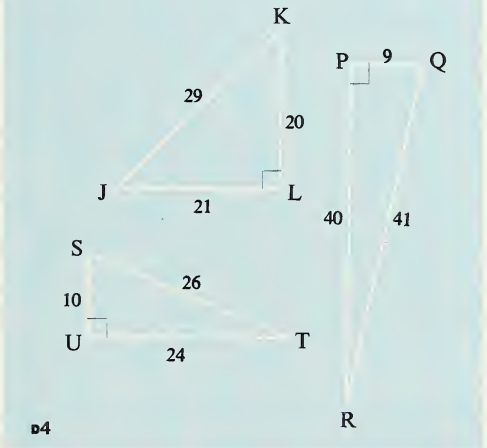
N Is each member of {4/5, . . .} equivalent to .8/1? Explain your answer.

For any right triangle that has $\angle D$ as an acute angle, the number .8, or $\frac{4}{5}$, is the *cosine* of $\angle D$. The cosine of an acute angle of a right triangle is the first component of a rate pair that has a second component of 1. This rate pair is equivalent to the rate pair that represents the comparison of the measure of the

co sine of an acute angle of a right triangle (kō′sīn). The first component of a rate pair that has a second component of 1 and that is equivalent to the rate pair that represents the comparison of the measure of the side adjacent to the angle with the measure of the hypotenuse.



D3



D4

side adjacent to the angle with the measure of the hypotenuse.

In this lesson you learned to identify the hypotenuse of a right triangle and the sides opposite and adjacent to an acute angle of the triangle. You also studied the tangent, sine, and cosine of an acute angle of a right triangle.

D3 contains pictures and sentences that will help you remember what you have learned in this lesson. Notice that, in △TUV, t is used to represent the measure of the side opposite ∠T, u is used to represent the measure of the side opposite ∠U, and v is used to represent the measure of the side opposite ∠V.

On your own

The questions in exercises 1 through 11 concern the three right triangles represented in D4.

1 Which angle of △JKL is the complement of ∠J? Which angle of △JKL is the complement of ∠K?

2 What common fraction numeral expresses the tangent of ∠J? The tangent of the complement of ∠J? How do you know that the tangent of ∠J is the reciprocal of the tangent of ∠K?

3 What common fraction numeral expresses the tangent of ∠Q?

4 The reciprocal of the tangent of ∠Q is the tangent of which angle of △PQR?

5 What common fraction numeral expresses the tangent of ∠S?

6 The reciprocal of the tangent of ∠S is the tangent of which angle of △UST?

7 What common fraction numeral expresses the sine of ∠J? Does this numeral also express the cosine of ∠K?

8 What common fraction numeral expresses the sine of ∠K? The sine of ∠K is the same as the cosine of which angle of △JKL?

9 Find the decimal that expresses the sine of ∠R. (Ten-thousandths) This decimal expresses the cosine of which angle of △PQR?

10 What decimal expresses both the sine of ∠Q and the cosine of ∠R? (Ten-thousandths)

11 For each acute angle of △UST, find its sine and its cosine. (Ten-thousandths)

Using a table of tangents, sines, and cosines

In the last lesson you used the measures of the sides of a right triangle, together with conditions for equivalence, to find the tangent, the sine, and the cosine of an acute angle of a right triangle. Now you will learn how to use a table to find these numbers for any acute angle of a given measure.

Suppose you know that $\triangle TUV$ is a right triangle and that $\angle T^\circ$ is 30. Although you may not know the measures of the sides of $\triangle TUV$, you can find the tangent, the sine, and the cosine of $\angle T$ by using a table.

The part of this table that you will use immediately precedes the index in this book. Find the table now. Notice that angle measures in degrees from 1 through 89 are expressed under the heading shown at the right. The heading is read "the measure in degrees of $\angle A$ is x." x is a variable for a number of degrees. In this table, the universe for x is the set of integers from 1 through 89.

$\angle A^\circ = x.$

A Under the heading $\angle A^\circ = x$, find the numeral 30 and the numeral .5774 directly to the right. Find the heading under which the numeral .5774 appears. This heading is read "the tangent of x." It means the tangent of an angle whose measure in degrees is x.

.5774 is an approximation of the tangent of the angle whose measure in degrees is 30. D1 shows how to read and write a sentence that is a short way of expressing this idea. Study D1.

"The tangent of 30 is approximately equal to .5774."

$\tan 30 \approx .5774.$

D1

B What numeral is directly to the right of the numeral for tan 30? The heading under which the numeral .5000 appears is read "the sine of x."

C What numeral is directly to the right of the numeral for sin 30? The heading under which the numeral .8660 appears is read "the cosine of x."

Notice that the numbers named under the headings tan x, sin x, and cos x are expressed to the nearer ten-thousandth. In most cases, the numbers named in the table are approximations of the tangents, sines, and cosines. We will think of all the numbers expressed in the table as approximations.

D Using the table, find sin 29. Find cos 55. Find tan 89. In each case, write a sentence like the one in D1 to express a true statement about the number you found.

E Using the table, find sin 62. Find cos 28. ? Use a sketch of a right triangle to help you explain why the sine of an angle is equal to the cosine of the complement of the angle.

F Now suppose that you want to find the measure of an angle whose tangent is .2679. Under the heading tan x, find the numeral .2679. What numeral is to the left of the numeral .2679 and under the heading $\angle A^\circ = x$? What is the measure of an angle whose tangent is .2679?

G What is the measure of an angle whose sine is .8090? Of an angle whose cosine is .6157?

H Suppose that the measure of each of two ? angles, $\angle W$ and $\angle X$, is a number from 1 through 89. If $\angle W^\circ > \angle X^\circ$, is the tangent of $\angle W$ greater than the tangent of $\angle X$? Is the sine of $\angle W$ greater than the sine of $\angle X$? Is the cosine of $\angle W$ greater than the cosine of $\angle X$? Explain your answers.

In the work that follows, you will solve problems that are concerned with the measures of the angles and sides of right triangles.

Read the problem in D2 and study the picture of $\triangle BCG$. The universe for each variable is D.

A Is $\overline{CG}$ opposite $\angle B$ of $\triangle BCG$? Use a as a variable for $m(\overline{CG})$.

B What is the measure of the side of $\triangle BCG$ that is adjacent to $\angle B$?

C Think about the first rate pair expressed at the right. With which is this rate pair associated, the tangent, the sine, or the cosine of $\angle B$? The rate pair whose first component you want to find must be equivalent to the second rate pair expressed at the right above.

$a/4$

$(\tan 23)/1$

D Does the sentence below in blue express a condition for the problem in D2?

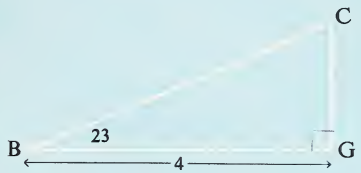
$$a/4 \sim (\tan 23)/1.$$

E How can you obtain $a = (\tan 23)4$ from $a/4 \sim (\tan 23)/1$? To find the a that satisfies $a = (\tan 23)4$, you must first use the table to help you find tan 23. Is the numeral 23 in the table under the heading $\angle A^\circ = x$?

F What is tan 23?

G How do you obtain $a = (.4245)4$ from $a = (\tan 23)4$?

$\angle G$ of $\triangle BCG$ is a right angle. $\angle B^\circ = 23$. $m(\overline{BG}) = 4$. What is the measure of $\overline{CG}$?



D2

$\angle N$ of $\triangle LMN$ is a right angle. $m(\overline{LN}) = 10$. $m(\overline{LM}) = 25$. What is the measure of $\angle L$?

D3

H Tabulate the solution set of $a = (.4245)4$. What is the measure of side CG of $\triangle BCG$?

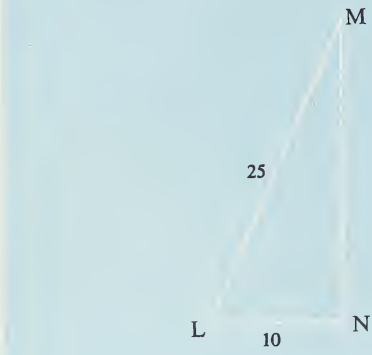
Next read the problem in D3 and study the picture. The universe for each variable is D.

I Which side of $\triangle LMN$ has a measure of 10? Is this side opposite $\angle L$, or is it adjacent to $\angle L$? Name the hypotenuse of $\triangle LMN$. What is the measure of the hypotenuse?

J Think about the rate pair expressed at the right. With which is it associated, the tangent, the sine, or the cosine of $\angle L$?

$10/25$

$\angle N$ of $\triangle LMN$ is a right angle. $m(\overline{LN}) = 10$. $m(\overline{LM}) = 25$. What is the measure of $\angle L$?



D3

You want to find the measure of $\angle L$. Use x as a variable for the measure of $\angle L$. The cosine of $\angle L$ is the first component of a rate pair that has a second component of 1 and that is equivalent to 10/25. Therefore, you want the rate pair expressed at the right to be equivalent to 10/25. Remember that "cos x" means the cosine of an angle whose measure in degrees is x.

$(\cos x)/1$

The condition expressed below in blue will help you find the answer to the problem in D3.

$(\cos x)/1 \sim 10/25.$

K How can you obtain $\cos x = .4000$ from $(\cos x)/1 \sim 10/25$? What is the cosine of $\angle L$?

L The table will help you find the measure of an angle whose cosine is .4000. Is the numeral .4000 in the table under the heading $\cos x$?

M Is .4000 between .4067 and .3907?

N From the table you know that $\cos 66 \approx .4067$, and $\cos 67 \approx .3907$. What is the absolute difference of .4067 and .4000? Of .3907 and .4000?

O Since the absolute difference of .4067 and .4000 is less than the absolute difference of .3907 and .4000, you can say that .4000 is nearer to cos 66 than to cos 67. Therefore, $.4000 \approx \cos 66$. Now you can give the answer to the problem in D3. What is the measure of $\angle L$ to the nearer degree?

P Read the problem in D4 and study the picture. What is the measure of the hypotenuse of $\triangle DEF$? What is $\angle D°$?

Q Which are you asked to find in the problem, the measure of the side opposite $\angle D$ or the measure of the side adjacent to $\angle D$?

R Use x as a variable for $m(\overline{EF})$. Does the sentence below in blue express a condition for the problem?

$x/624 \sim (\sin 75)/1.$

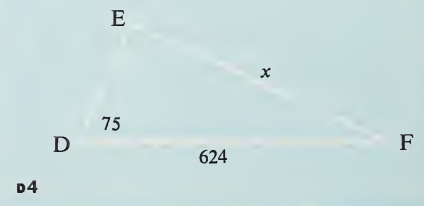
S How can you obtain $x = (\sin 75)624$ from $x/624 \sim (\sin 75)/1$? How can you obtain $x = (.9659)624$ from $x = (\sin 75)624$?

T Tabulate the solution set of $x = (.9659)624$.

U What is $m(\overline{EF})$? Use a numeral that has three significant digits to express your answer.

V Read the problem in D5 and study the picture. What are you asked to find in the problem?

$\angle E$ of $\triangle DEF$ is a right angle. $\angle D° = 75$. $m(\overline{DF}) = 624$. What is $m(\overline{EF})$?



D4

$\angle I$ of $\triangle GHI$ is a right angle. $m(\overline{HI}) = 54$. $m(\overline{GI}) = 228$. What is $\angle H°$?

D5

W In $\triangle GHI$, do you know the measure of the hypotenuse? Of the side opposite $\angle H$? Of the side adjacent to $\angle H$?

X Use r as a variable for $\angle H°$. Which one of the following sentences expresses a condition for the problem in D5?

$(\sin r)/1 \sim 228/54$.

$(\tan r)/1 \sim 228/54$.

$(\cos r)/1 \sim 54/228$.

Y How do you obtain $(\tan r)54 = 228$ from $(\tan r)/1 \sim 228/54$? How do you obtain $\tan r = 4.222\overline{2}$ from $(\tan r)54 = 228$?

Z Use the table to find $\angle H°$ to the nearer degree. Why should you give $\angle H°$ as 77 rather than as 76?

In this lesson you learned to interpret a table of tangents, sines, and cosines of angles. You also learned to find the measure of an angle whose sine, cosine, or tangent you know, and you learned how to solve problems about the sides and angles of right triangles.

On your own

For each of exercises 1 through 9, make a sketch to help you answer the questions.

1 $\angle T$ of $\triangle RST$ is a right angle. $\angle S° = 34$. $m(\overline{TR}) = 12$. What is $m(\overline{RS})$? (Tenths)

2 $\angle E$ of $\triangle CDE$ is a right angle. $m(\overline{CD}) = 34$. $m(\overline{CE}) = 17$. What is $\angle D°$? What is $\angle C°$?

3 $\angle L$ of $\triangle JKL$ is a right angle. $m(\overline{KL}) = \sqrt{6}$. $m(\overline{JL}) = \sqrt{6}$. What is $\angle J°$? What is $\angle K°$?

4 $\angle W$ of $\triangle UVW$ is a right angle. $\angle U° = 55$. $m(\overline{UW}) = 20$. What is $m(\overline{UV})$? What is $m(\overline{VW})$? (Hundredths)

5 $\angle Q$ of $\triangle MNQ$ is a right angle. $m(\overline{MN}) = 30$. $\angle M° = 73$. What is $m(\overline{NQ})$? (Hundredths)

6 $\angle R$ of $\triangle PQR$ is a right angle. $m(\overline{PQ}) = 18$. $\angle Q° = 82$. What is $m(\overline{QR})$? What is $m(\overline{RP})$? (Thousandths)

7 $\angle U$ of $\triangle SRU$ is a right angle. $m(\overline{RU}) = 13$. $m(\overline{SU}) = 52$. What is $\angle S°$? What is $\angle R°$?

8 $\angle X$ of $\triangle VWX$ is a right angle. $m(\overline{VW}) = 45$. $m(\overline{VX}) = 15$. What is $\angle V°$? What is $\angle W°$?

9 $\angle K$ of $\triangle IJK$ is a right angle. $m(\overline{IK}) = 30$. $\angle I° = 62$. What is $m(\overline{JK})$? What is $m(\overline{IJ})$? (Hundredths)

KEEPING SKILFUL

For each of exercises 1 through 12, use a decimal to express the rational number named.

1 $\frac{5}{100}$	**4** $\frac{7}{11}$	**7** $\frac{67}{8}$	**10** $\frac{-25}{9}$
2 $\frac{-9}{4}$	**5** $\frac{65}{500}$	**8** $\frac{-55}{3}$	**11** $\frac{7}{80}$
3 $\frac{-16}{50}$	**6** $\frac{-11}{16}$	**9** $\frac{37}{5}$	**12** $\frac{-13}{1000}$

Each exercise below involves rational numbers. For each exercise, find the sum, difference, product, or quotient.

13 $70.37 + 8.149$	**20** $-71.91 \div 1.7$
14 $65 - -29.7$	**21** $-6.52 + -13.004$
15 $115.7 \div 13$	**22** $325.65 - 118.38$
16 85.3×59	**23** $.075 \times -3.25$
17 $-16.35 + 14.39$	**24** $-68.31 \div -.009$
18 -7.8×-7.95	**25** $-42.35 - 64.4$
19 $-15.832 - -22.79$	**26** $32.14 + -17.289$

Problems involving indirect measurement

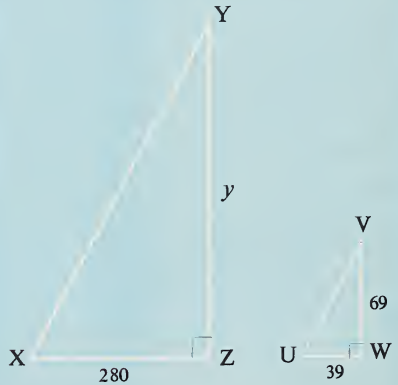
In this lesson you will use your knowledge of similar triangles and trigonometric rate pairs to solve problems that involve measurement. Since the measures given in the problems in this lesson are physical measures, you will use the rules that apply to approximations that you studied in lesson 163.

There are many measures that you cannot find directly by using a common measuring instrument such as a rule. To find measures like the average distance from the earth to the sun or the length of a diameter of a nucleus of an atom, it is necessary to use *indirect measurement*.

Now you will learn how to solve problems by using certain kinds of indirect measurement. Read the problem in D1 and study the sketches. The sketches will help you understand the problem.

Don wants to find the height of a radio tower. At a certain time of day, the tower casts a shadow 2.80×10^2 ft. long. At the same time Don, who is 69 in. tall, casts a shadow 39 in. long. What is the height of the radio tower?

D1



D2

A Two triangles are suggested in the problem. One triangle is determined by the points suggested by the top of the tower, its base, and the spot on the ground where its shadow ends. Describe the other triangle that is suggested.

Look at D2, which contains pictures of the two triangles discussed in exercise A.

B Side XZ of $\triangle XYZ$ is suggested by the length of the shadow of the tower. What is $m(\overline{XZ})$? What suggests side YZ? Do you know $m(\overline{YZ})$?

C Which side of $\triangle UVW$ is suggested by the length of Don's shadow? Which side of $\triangle UVW$ is suggested by Don's height? What are the measures of these two sides? How are the measures given, in feet or in inches?

D Assume that the tower and its shadow are perpendicular, and that Don and his shadow

are perpendicular. What, then, should you assume about $\angle Z$ and $\angle W$?

E For a given time of day, the angle suggested by a ray of the sun and the shadow of an object is the *angle of elevation* (el′ə vā′shən) of the sun from the tip of the shadow. $\angle X$ of $\triangle XYZ$ is the angle of elevation of the sun. Which angle of $\triangle UVW$ also is the angle of elevation of the sun?

You know that the length of the tower's shadow and the length of Don's shadow were determined at the same time of day. Therefore, $\angle X \cong \angle U$.

F How do you know that $\triangle XYZ \sim \triangle UVW$?

G Name the corresponding sides of $\triangle XYZ$ and $\triangle UVW$.

Now you can develop a condition for the problem in D1. The universe for each variable is D.

H You know that $m(\overline{UW}) = 39$ and that $m(\overline{VW}) = 69$. Does the rate pair expressed at the right represent the comparison of these two measures? 39/69 also represents the comparison of the length of Don's shadow with his height.

39/69

I You know that $m(\overline{XZ}) = 280$. Use y as a variable for the measure of side YZ of $\triangle XYZ$. Does the rate pair expressed at the right represent the comparison of these two measures? $280/y$ also represents the comparison of the length of the tower's shadow with the tower's height.

$280/y$

J Remember that $\triangle XYZ$ and $\triangle UVW$ are similar. How do you know that 39/69 is equivalent to $280/y$? Does the sentence below in blue express a condition for the problem?

$$39/69 \sim 280/y.$$

K Tabulate $\{y \mid 39/69 \sim 280/y\}$.

A surveyor wants to determine how far a fire hydrant is from the base of a certain building, but he cannot measure the distance directly. He finds that the angle of elevation of the top of the building from the fire hydrant has a measure of 9 degrees. The building is 235 ft. high. How far is the fire hydrant from the base of the building?

D3

D4

L How do you know that the answer to the problem in D1 should be expressed by using a decimal that has two significant digits?

M Give the answer to the problem.

You solved the problem in D1 by using a property of similar triangles. Now you will solve a problem by using a trigonometric rate pair.

Read the problem in D3. The universe for each variable is D.

A Look at D4. The information given in the problem in D3 was used to make the picture of $\triangle PQR$. Side PR is suggested by the distance from the base of the building to the fire hydrant. Which side of $\triangle PQR$ is suggested by the height of the building?

B Which angle of $\triangle PQR$ is the angle of elevation of the top of the building from the fire hydrant? What is $\angle P°$? Is $\angle P$ an acute angle?

C You should assume that the fire hydrant is on the same level as the base of the building and that the walls of the building are perpen-

A surveyor wants to determine how far a fire hydrant is from the base of a certain building, but he cannot measure the distance directly. He finds that the angle of elevation of the top of the building from the fire hydrant has a measure of 9 degrees. The building is 235 ft. high. How far is the fire hydrant from the base of the building?

D3

D4

dicular to the ground. What is $\angle R°$? What kind of triangle is $\triangle PQR$?

D You know the measures of an acute angle and a side of $\triangle PQR$. For any right triangle, if you know the measures of one acute angle and a side, you can use trigonometric rate pairs to find the measures of the other acute angle and the other two sides. Name the side of $\triangle PQR$ whose measure you must find before you can give an answer to the problem in D3.

E Use a as a variable for the measure of $\overline{PR}$. Is a the measure of the side of $\triangle PQR$ that is adjacent to $\angle P$? What is the measure of the side of $\triangle PQR$ that is opposite $\angle P$?

F The rate pair expressed at the right represents the comparison of the measure of the side opposite $\angle P$ with the measure of the side adjacent to $\angle P$. Which of the rate pairs named below is equivalent to $235/a$?

$235/a$

(sin 9)/1 (tan 9)/1 (cos 9)/1

G Can you use the condition expressed in the next column in blue to find the measure of $\overline{PR}$? Is the condition also a condition for the problem in D3?

$$235/a \sim (\tan 9)/1.$$

H How do you obtain $235 = (\tan 9)a$ from $235/a \sim (\tan 9)/1$?

Use the table of tangents, sines, and cosines to determine the tangent of 9. We will agree that, when you use this table, you should express the sine, the cosine, or the tangent of an angle by a numeral with four significant digits to the right of the decimal point.

I Is $\tan 9 \approx .1584$? When you replace tan 9 by .1584 in $235 = (\tan 9)a$, what condition do you obtain? To find a, you must find the quotient of what two numbers?

J Before you can give the answer to the problem, you must decide how accurately you should express the answer. How many significant digits are in the decimal .1584? How many significant digits are in the decimal 235? How many significant digits should there be in the decimal that expresses the quotient of 235 and .1584?

Now you can give the answer to the problem in D3. The fire hydrant is 1.48×10^3 ft. from the base of the building.

Read the problem in D5 and study the picture of $\triangle JKL$, which is suggested by the problem. The universe for each variable is D.

K Is $\triangle JKL$ a right triangle? Which angle of $\triangle JKL$ is the angle of elevation of the top of the ladder from the bottom?

L Which side of $\triangle JKL$ is suggested by the length of the ladder? What is the measure of this side?

M Why is 10.8 given as the measure of $\overline{JL}$?

N Notice that you know the measures of two sides of $\triangle JKL$. For any right triangle, if you know the measures of two sides, you can use trigonometric rate pairs to find the measure of

A 48-foot ladder is placed against a wall so that the bottom of the ladder is 10.8 ft. from the base of the wall. What is the measure of the angle of elevation of the top of the ladder from the bottom of the ladder?

D5

What is the distance from the base of the wall to the top of the ladder?

D6

the third side and the measure of each of the two acute angles. Name the angle of $\triangle JKL$ whose measure you must find before you can give the answer to the problem in D5.

O Use x as a variable for $\angle J°$. Which of the ? rate pairs named below should you use in a condition for the problem?

$(\sin x)/1$ $(\tan x)/1$ $(\cos x)/1$

P How do you know that the rate pair you named for exercise O is equivalent to the rate pair $10.8/48$ expressed at the right?

The sentence below in blue expresses a condition for the problem in D5.

$$10.8/48 \sim (\cos x)/1.$$

Q How do you obtain $10.8 = (\cos x)48$ from $10.8/48 \sim (\cos x)/1$? To find $\cos x$, you must find the quotient of what two numbers?

R Since the trigonometric table in this book expresses the cosine of an angle to the nearer ten-thousandth, you should find the quotient of 10.8 and 48 to ten-thousandths. What is this quotient to the nearer ten-thousandth?

S Use the trigonometric table in this book to find the measure of an angle whose cosine is .2250. What is $\angle J°$? Give the answer to the problem.

T Look again at D5. You know that $\angle J° = 77$. What is $\angle K°$? Explain your answer.

Now that you know the measures of the acute angles in $\triangle JKL$, you can obtain the answer to the problem in D6. Read the problem in D6.

U Which side of $\triangle JKL$ is suggested by the distance from the base of the wall to the top of the ladder?

V Use d as a variable for the measure of $\overline{KL}$.

? Which two of the rate pairs expressed below can you use in conditions for the problem in D6? Explain your answer.

$(\sin 77)/1$ $(\tan 77)/1$ $(\cos 77)/1$

W Suppose that you use $(\sin 77)/1$ in a condition for the problem. Is $(\sin 77)/1$ equivalent to the rate pair that represents the comparison of the measure of the side opposite $\angle J$ with the measure of the hypotenuse? Now suppose that you use $(\tan 77)/1$ in a condition for the problem. What rate pair is equivalent to $(\tan 77)/1$?

First you will use the condition expressed below in blue to determine $m(\overline{KL})$.

$$d/48 \sim (\sin 77)/1.$$

X How do you obtain $d = (\sin 77)48$ from $d/48 \sim (\sin 77)/1$? How do you obtain $d = (.9744)48$ from $d = (\sin 77)48$?

Y The measure that you obtain for $\overline{KL}$ cannot be more accurate than the less accurate of the physical measures 48 and 10.8. Which is a less accurate physical measure, 48 or 10.8? What is $m(\overline{KL})$? What is the distance from the base of the wall to the top of the ladder?

Z Now use $d/10.8 \sim (\tan 77)/1$ to determine the distance from the base of the wall to the top of the ladder. How does this measure compare with the measure you found for exercise Y?

Now you know that it is sometimes necessary or convenient to find measures indirectly. You have learned how to use your knowledge of similar triangles and trigonometric rate pairs to develop conditions for some problems that involve indirect measurement.

On your own

For each problem, write a sentence that expresses a condition and give the answer to the problem. You may find it helpful to make a sketch for each problem. The universe for each variable is D.

1 At a certain time of day, a 75-foot monument casts a 115-foot shadow. At the same time of day and in the same place, how long is the shadow cast by a 135-foot building?

2 The distance along a straight road from point P at the bottom of a hill to point Q at the top of the hill is 162 ft. The angle of elevation of point Q from P has a measure of 18 degrees. What is the vertical height of the hill in feet?

3 Two pennants are different sizes, but their shapes suggest similar triangles. The lengths of the edges of the larger pennant are 18.0 in., 20.0 in., and 7.0 in. The length of the shortest edge of the smaller pennant is 4.0 in. How long is the longest edge of the smaller pennant?

4 How long is the third side of the smaller pennant described in problem 3?

5 What is the measure of the angle of elevation of the top of a 120-foot building from a point that is 75 ft. from the base of the building?

6 The bottom of a ladder is placed 7.5 ft. from the base of a building. The measure of the angle of elevation of the top of the ladder from the bottom of the ladder is 75. What is the length of the ladder?

7 At a given moment, the angle of elevation of an airplane from point S on the ground has a measure of 58. At the same time, the airplane is directly above point T on the ground, and point T is 1.500×10^3 ft. from point S. How high above point T is the airplane?

8 A ramp that leads to the delivery entrance of a building is 12.5 ft. long. The entrance is 4.25 ft. above the base of the building. What is the measure of the angle of elevation of the entrance from the foot of the ramp?

9 A guy wire extends from the ground at point D to the top of a pole that is 50 ft. high. The measure of the angle of elevation of the top of the pole from point D is 41. What is the length of the guy wire?

For problems 10 through 13, you can think of the geometric figures as being suggested by physical situations. Therefore, the measures given in the problems are approximations.

10 In $\triangle ABC$, $\angle C$ is a right angle. $\angle A^\circ = 5$. $m(\overline{BC}) = 325$ ft. What is $m(\overline{AC})$?

11 In $\triangle DEF$, $\angle F$ is a right angle. $\angle D^\circ = 38$. $m(\overline{DE}) = 1.36$ mi. What is $m(\overline{DF})$?

12 In $\triangle GHI$, $\angle H$ is a right angle. $\angle I^\circ = 73$. $m(\overline{GH}) = 7.50 \times 10^5$ mm. What is $m(\overline{HI})$?

13 In $\triangle JKL$, $\angle J$ is a right angle. $m(\overline{KL}) = 5.5$ yd. $m(\overline{JK}) = 3.495$ yd. What is $\angle K^\circ$?

APPLYING MATHEMATICS

For each problem, first write a sentence that expresses a condition. Then give the answer to the problem. The universe for each variable is D.

Use the rules for computation involving approximations. You may also find it helpful to make a sketch for each problem.

1 Howard is 68 in. tall. At a certain time of day, he casts a shadow 42.5 in. long. At the same time of day, Ruth casts a shadow 40 in. long. What is Ruth's height in inches?

2 Mr. Jacobs cut out two pieces of canvas whose shapes suggested similar triangles. The lengths of the edges of the larger piece were 23.5 ft., 16.75 ft., and 13.0 ft. The length of the longest edge of the smaller piece was 14.0 ft. What was the length in feet of the shortest edge of the smaller piece?

3 The measure of the angle of elevation of the top of a lighthouse from a small boat was 15 degrees. The top of the lighthouse was 135 ft. above the surface of the water. What was the distance in feet from the base of the lighthouse to the boat?

4 A workman attached one end of a rope at point X on a building. Point X was 23.75 ft. above the ground. He attached the other end of the rope to a stake in the ground so that the rope was taut. The length of the rope was 47.5 ft. What was the measure of the angle of elevation of point X from the stake?

5 Use the answer to problem 4 to help you find the distance in feet from the stake to point Y at the base of the building. Point Y is directly below point X.

6 The top of a 26.5-foot ladder was placed against a wall. The measure of the angle of elevation of the top of the ladder from the bottom was 68 degrees. What was the distance in feet from the base of the wall to the top of the ladder?

7 An office building is 112.75 ft. tall. At a certain time of day, the building casts a shadow 62.5 ft. long. What is the measure of the angle of elevation of the sun?

8 A rectangular swimming pool is 18.5 ft. wide. The measure of an angle whose sides include the width of the pool and a diagonal of the pool is 70. What is the distance in feet from one corner of the pool to the opposite corner?

9 A wire extends from point T on the ground to point U directly above the base of a building. The wire is 125 ft. long. The distance from point T to the base of the building is 108.25 ft. What is the measure of the angle of elevation of point U from point T?

10 A television antenna is attached to the roof of a house. The measure of the angle of elevation of the top of the antenna from point R on the ground is 35 degrees. The top of the antenna is directly above point S on the ground. The distance from point R to point S is 55.3 ft. What is the distance in feet from point S to the top of the antenna?

11 In $\triangle ABC$, $\angle C$ is a right angle. $\angle A^\circ = 62$. The length of side AB is 94.5 in. What is the length of side BC in inches?

12 In $\triangle DEF$, $\angle E$ is a right angle. The length of side DE is 2.70×10^3 cm. $\angle F^\circ = 13$. What is the length of side EF in centimetres?

13 In $\triangle GHI$, $\angle G$ is a right angle. The length of side GI is 4975 km. $\angle I^\circ = 52$. What is the length of side GH in kilometres?

14 In $\triangle JKL$, $\angle L$ is a right angle. The length of side JL is 12.5 ft. The length of side KL is 5 ft. What is $\angle J^\circ$?

15 In $\triangle MNO$, $\angle O$ is a right angle. The length of side MN is 11 in. The length of side MO is 4 in. What is $\angle M^\circ$?

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you need it.

Test 150

What words or symbols best complete exercises 1 through 12?

1 A physical measure of 5.0 ft. is ~~~ precise than a physical measure of 5.0 in. (314)

2 A physical measure of ~~~ feet determines $\{x \mid 6.15 \leqq x \leqq 6.25\}$. U = D. (313)

3 In $\triangle GHI$, if $\angle H^\circ = 90$, then side ~~~ is the hypotenuse. (330)

4 The sine of an angle whose measure in degrees is 18 equals the cosine of an angle whose measure in degrees is ~~~. (332)

5 Of two physical measures, 4.5 m. and 223 m., ~~~ is the more accurate. (315)

6 A length of 1 kilometre is the same as a length of ~~~ metres. (304)

7 The greatest possible error of a physical measure of 2.360 centimetres is ~~~. (313)

8 The sum of the physical measures .39 ft. and 1.7 ft. should be rounded to the nearer ~~~. (319)

9 The unit of measure of a physical measure of $3\frac{0}{4}$ feet is ~~~. (311)

10 When $\frac{2}{3}$ is a physical measure, the tolerance of $x \approx \frac{2}{3}$ is ~~~. U = D. (313)

11 The complement of an angle whose measure in degrees is 13 is an angle whose measure in degrees is ~~~. (330)

12 The unit of measure of a physical measure of 45,000 in. $\pm$ 50 is ~~~. (313)

Test 151

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

13 If $\angle A$ of $\triangle ABC$ is a right angle and $m(\overline{AB}) = m(\overline{AC})$, then the cosine of $\angle B$ equals the sine of $\angle B$. (332)

14 A segment with endpoints whose coördinates are 7.5 and 10 has a measure of $\bar{}2.5$. (303)

15 Any two right triangles are similar triangles. (326)

16 The relative error of a physical measure of $2\frac{1}{2}$ ft. is the average error per foot. (315)

17 If $\angle A$ is congruent to $\angle D$ and $\angle B$ is congruent to $\angle E$, then triangle ABC is similar to triangle DEF. (327)

18 The numeral $2.80 \times 10^{\bar{}3}$ has two significant digits. (317)

19 If $\triangle ABC$ and $\triangle DEF$ are two isosceles triangles that are similar, there is more than one similarity correspondence between their vertices. (326)

20 The product of the approximations 2.80 and 3.26 should be rounded so that the numeral for the product has exactly two significant digits. (321)

21 Any two triangles that are congruent are also similar. (326)

22 In $\triangle JKL$, if $\angle L^\circ = 90$, then $\overline{JK}$ is the side adjacent to $\angle J$. (330)

23 A rate pair that represents the comparison of the measures of two corresponding sides of two triangles is a trigonometric rate pair. (331)

24 Any two equilateral triangles are similar. (327)

25 If $\triangle ABC \sim \triangle RST$, then $m(\overline{AB})/m(\overline{BC}) \sim m(\overline{RS})/m(\overline{ST})$. (329)

26 If 7, 5, and 3 are the measures of the sides of $\triangle UVW$, and $3\frac{1}{2}$, $2\frac{1}{2}$, and $1\frac{1}{2}$ are the measures of the sides of $\triangle XYZ$, then $\triangle UVW$ and $\triangle XYZ$ are similar. (327)

27 For any two physical measures, the more precise of the two measures is also the more accurate measure. (316)

28 The unit of measure of a physical measure of 20,000,000 mi. $\pm$ 500 is 500 ft. (311)

Test 152

For each problem below, write a sentence that expresses a condition. Then give the answer to the problem. You may use tables whenever necessary. The measures given are physical measures. The universe for each variable is D.

29 In Toronto, the precipitation over a period of 30 days was 2.56 inches. What was the average daily precipitation? (321)

30 The altitude of McArthur Peak in the Yukon is 1.44×10^4 ft. The altitude of Mount Augusta is 1.407×10^4 ft. McArthur Peak is how many feet higher than Mount Augusta? (317)

31 A driveway slopes upward toward a garage from a level that is 3.2 ft. lower than the level of the garage floor. The driveway is 24 ft. long. What is the measure of the angle of elevation of the top of the driveway from the bottom of the driveway? (339)

32 One kind of watch mainspring has a width of .520 centimetres. What is the width of the mainspring in inches? (306)

33 A world altitude record was once set by an airplane. This airplane reached an altitude of 25,644 m. What was this altitude in kilometres? (305)

34 $\angle B$ of $\triangle ABC$ is a right angle. The measure of $\overline{AB}$ is 25.5. The measure of $\overline{AC}$ is 51. What is the measure in degrees of $\angle A$? (334)

35 $\angle M^\circ$ of $\triangle KML$ is a right angle. The measure of $\overline{KM}$ is 15. The measure in degrees of $\angle K$ is 37. What is the measure of $\overline{KL}$? (334)

36 $\triangle DEF \sim \triangle GHI$. $m(\overline{DE}) = 35.2$. $m(\overline{EF}) = 29.54$. $m(\overline{GH}) = 43$. What is $m(\overline{HI})$? (326)

37 The length of a side of a square is 58 ft. What is the length of a diagonal of the square? (341)

38 The length of a diagonal of rectangle ABCD is 75.000 in. $\angle ABD^\circ = 22$. What is $m(\overline{AB})$? (341)

CHECKING UP

The small numeral within parentheses tells what page to turn to for help if you need it.

Test 153

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 If ℓ_3 and $\odot J$ are coplanar and ℓ_3 intersects $\odot J$ in point M, then M is a point of tangency. (115)

2 Three points in a circle determine exactly one central angle of the circle. (117)

3 In a real-number line, there is a point whose coördinate is $\sqrt{2}$. (263)

4 The set of quadrilaterals is a subset of the set of all trapezoids. (122)

5 If you use two different unit segments to measure a third segment, you will obtain two different measures for the third segment. (304)

6 If $\overline{MG}$ is a median of $\triangle MLN$, then G is the midpoint of $\overline{ML}$. (127)

7 In the real plane, point $(^-3, 0)$ is located in the second quadrant. (285)

8 The area of a parallelogram is 2 times the area of a triangle with the same base and altitude as the parallelogram. (143)

9 The graph of $\{(x, y) \mid y = x \wedge x + y = 4\}$ is a dot. The universe for (x, y) is $D \times D$. (288)

10 If the corresponding angles of two triangles are congruent, then the triangles must be congruent. (139)

Test 154

For each of exercises 11 through 19, tabulate the solution set of the condition expressed, if possible. Otherwise, write a standard description of the solution set. The universe for each variable is D.

11 $x + {}^-\frac{1}{2} = \frac{3}{4}$. (276)

12 $5y + 2y = 231$. (277)

13 $^-3m < 17$. (281)

14 $m \approx_{.05} 12.7$. (90)

15 $2.4/3 \sim (n-2)/.7$. (279)

16 $x+4=\frac{2}{3} \vee x+3>{}^{-}\frac{1}{3}$. (282)

17 $\sim(4c<.8)$. (283)

18 $a=3b \wedge a-b=12$. (289)

19 $(x+2y-z=0) \wedge (8y=z) \wedge (\frac{4}{3}-x={}^{-}2)$. (290)

Test 155

What words or symbols best complete each of the following exercises?

20 The decimal for $\sqrt{9}$ is a ~~~ infinite decimal. (262)

21 The side of a right triangle that is opposite the right angle is the ~~~. (330)

22 A standard name of $(\frac{3}{4})^0 \times (\frac{3}{4})^3$ is the numeral ~~~. (230)

23 If $S=\{\frac{1}{2}, 2, 0, {}^{-}4, 3\}$, then the intersection of S and the set of positive integers is ~~~. (168)

24 The repetend of the decimal for $\frac{5}{32}$ is ~~~. (218)

25 A standard name of the rational number $\{(x, y) | (x, y) \sim (5, 2)\}$ is the numeral ~~~. The universe for (x, y) is $R_a \times R_a$. (157)

26 $(\frac{4}{3})\frac{3}{4}y=(\frac{4}{3})12$ and $\frac{3}{4}y=12$ are equivalent conditions because of the ~~~ property of multiplication of real numbers. U = D. (268)

27 The numeral ~~~ expresses the product of 44,300 and .0036 in scientific notation. (234)

28 $10-{}^{-}3=10+3$ because of the ~~~ property of rational numbers. (207)

29 The multiplicative inverse of ${}^{-}1.4$ is ~~~. (209)

30 $\sqrt{312}$ to the nearer tenth is ~~~. (258)

31 If U = D, the complement of the set of rational numbers is the set of ~~~ numbers. (262)

32 There is at least one real number between $\sqrt{7}$ and any given real number because of the ~~~ property of the set of real numbers. (272)

Unit 13

Formulas

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170	Relations determined by $y=\frac{k}{x}$	354
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Relations determined by $y = kx$

Up to now, you have used many kinds of conditions to help you solve problems. In this unit you will study certain kinds of conditions more thoroughly. These conditions are useful in solving business and scientific problems.

It is often helpful to keep a record of certain kinds of data, or information. You can then decide if the recorded data are related in some way. If the quantities expressed in the data are related, you can use the relation to make predictions about quantities that are different from those you have already recorded.

Study the chart in D1. This chart is a record of gas purchases and mileages that Mr. Miller kept on a cross-country trip.

A How many miles did Mr. Miller travel on 10 gal. of gasoline? On 8 gal. of gasoline?

B Did Mr. Miller travel the same number of miles each time he bought 12 gal. of gasoline? How many gallons of gasoline did he use when he traveled 195 mi.? When he traveled 225 mi.?

C Look at D2. Each ordered pair in set S was obtained from the chart in D1. How many members are there in S? To what does the first component of each member of S refer? To what does the second component refer?

D Is $(10, 150) \in S$? How do you know that $10/150 \sim 1/15$?

E Is each of the other ordered pairs in S equivalent to 1/15? How do you know?

F Now look at D3. What condition is expressed by sentence A? What is the universe for (x, y)?

Number of miles traveled

Number of gallons bought

10	150
12	180
15	225
8	120
13	195
11	165
12	180
9	135

D1

$S = \{(10, 150), (12, 180), (15, 225), (8, 120), (13, 195), (11, 165), (9, 135)\}$.

D2

Universe for $(x, y) = D \times D$.

A $x/y \sim 1/15$.

B $y = 15x$.

D3

G How can you obtain $x \cdot 15 = y$ from condition A? How can you obtain condition B from $x \cdot 15 = y$?

H Is each member of S, tabulated in D2, a solution of condition B? Is (1, 15) a solution? Is $(^-6, ^-90)$ a solution?

I Is $\{(x, y) | y = 15x\}$ a finite set? Is it an infinite set? Is S a subset of $\{(x, y) | y = 15x\}$?

J How do you know that $\{(x, y) | y = 15x\}$ is a proportional relation?

K Look again at D2. For each member of S, is the second component equal to the product of 15 and the first component?

L In $\{(x, y) | y = 15x\}$, the second component of each element is equal to the product of what number and the first component?

For each member of $\{(x, y) | y = 15x\}$, y is equal to the product of 15 and x. Therefore, we say that y changes directly, or *varies directly*, as x changes, or varies. In $y = 15x$, notice that 15 does not vary, but remains the same, or constant. 15 is the *constant* (kon′stənt) *of variation* (vãr′ē ā′shən). $y = 15x$ is a *condition for direct variation.*

Suppose that the solution set of a given condition is a set of ordered pairs. Suppose also that the second component of each ordered pair in this set is equal to the product of a constant of variation and the first component. Then the solution set of the condition is a proportional relation, and the given condition is a condition for direct variation.

M Can you use $(x, 15x)$ to represent each member of $\{(x, y) | y = 15x\}$? Explain your answer. The universe for (x, y) is D × D.

N Look at D4. Is condition C a condition for equivalence? Name the variables in the condition. How was condition D obtained from condition C? Are the two conditions equivalent?

Any condition that is obtained from condition D by making a replacement for k is a condition for direct variation. A condition that is obtained in this way is a condition of the form $y = kx$. In the condition $y = kx$, the variable k is a variable for the constant of variation.

O What replacement was made for k in $y = kx$ to obtain condition E, expressed in D4? What is the constant of variation in condition E? Why can you use $(x, 3x)$ to represent each member of the solution set of this condition?

P Is the solution set of condition E a proportional relation? How do you know that condition E is a condition for direct variation?

Universe for each variable = D.

C $x/y \sim 1/k$.

D $y = kx$.

E $y = 3x$.

F $y = {}^{-}\frac{3}{4}x$.

G $y = {}^{-}2x$.

H $y = \frac{1}{2}x$.

D4

Universe for (x, y) = D × D.

I $y = \frac{1}{3}x$.

D5

Q Now look at sentences F, G, and H in D4. Name the constant of variation in the condition expressed by each of these sentences.

R How do you know that conditions F, G, and H are conditions for direct variation?

S Look again at sentence D. Why can you use (x, kx) to represent each member of the solution set of condition D?

In the following exercises, you will learn how to make a graph of the solution set of a condition of the form $y = kx$.

A What condition is expressed in D5?

B How do you know that, in condition I, y varies directly as x varies? Name the constant of variation in condition I.

C Make a chart to help you find some of the solutions of $y = \frac{1}{3}x$. Use ${}^{-}3$, ${}^{-}2$, 0, 1, and 3 as replacements for x.

D Study your chart. As the replacements for x in $y = \frac{1}{3}x$ become greater, what do you notice about the replacements for y? What ordered pairs can you obtain from your chart?

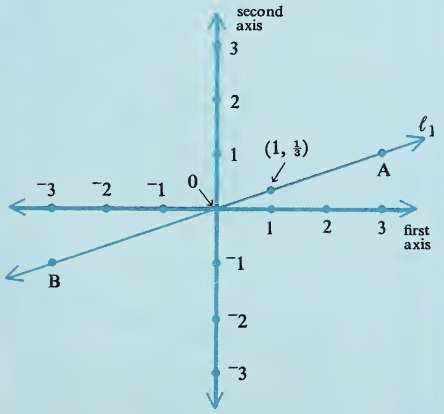
E Now look at the graph in D6. What are the coördinates of the origin? Of point A? Of point B? Is each of these points associated with an ordered pair that you named for exercise D?

F Are the dots for point A, point B, and point O in the graph of $\{(x, y) | y = \frac{1}{3}x\}$? Is each of the points contained in ℓ_1?

The picture of ℓ_1 is a graph of the solution set of $y = \frac{1}{3}x$. Line ℓ_1 is the *locus* of $\{(x, y) | y = \frac{1}{3}x\}$. The locus of the solution set of a condition is the set of points associated with the elements of the solution set. Each element of the solution set is associated with a point in the locus, and each point in the locus is associated with an element of the solution set. The locus of the solution set of any condition of the form $y = kx$ is a line that contains the origin.

locus of the solution set of a condition (lō′kəs). The set of points that are associated with the elements of the solution set of a condition.

Universe for $(x, y) = D \times D$.



$\{(x, y) | y = \frac{1}{3}x\}$
Incomplete graph

D6

Number of cents of cost	
Number of pencils bought	
3	15
2	10
5	25
10	50
4	20

D7

In this lesson you learned that, if a condition is of the form $y = kx$, then y varies directly as x varies. You also learned that k is a variable for the constant of variation in $y = kx$ and that the locus of the solution set of a condition of the form $y = kx$ is a line that contains the origin.

On your own

Use the chart in D7 in connection with exercises 1 through 5.

1 What is the cost of 5 pencils? Of 10 pencils? How many pencils can be bought for 20¢? For 15¢?

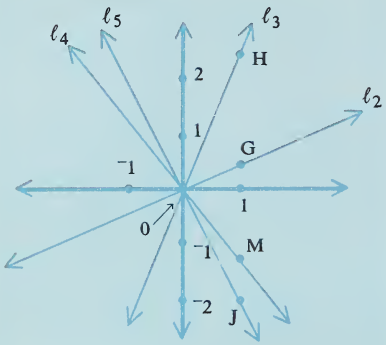
2 Tabulate the set of ordered pairs that you can obtain from the chart. Use the number of pencils bought as the first component of each rate pair. Use the cost in cents as the second component. Use T as a name of the set.

3 Is each member of T equivalent to 1/5? The second component of each member is the product of what number and the first component?

4 Is T a subset of $\{(x, y) | y = 5x\}$? The universe for (x, y) is $D \times D$. What is the constant of variation in $y = 5x$?

5 Make a graph of $\{(x, y) | y = 5x\}$. What is the locus of $\{(x, y) | y = 5x\}$?

Universe for $(x, y) = D \times D$.



Incomplete graphs

D8

Use the graphs in D8 for exercises 6 through 10. Each line represented in D8 contains the origin. The universe for (x, y) is $D \times D$.

6 ℓ_2 is the locus of the solution set of a condition. The coördinates of point G are $(1, \frac{1}{2})$. Is ℓ_2 the locus of $\{(x, y) | y = \frac{1}{2}x\}$? Explain your answer.

7 In the condition $y = \frac{1}{2}x$, does y vary directly as x varies? What is the constant of variation in $y = \frac{1}{2}x$?

8 The coördinates of point H are $(1, 2\frac{1}{2})$. ℓ_3 is the locus of the solution set of what condition?

9 The coördinates of point M are $(1, {}^{-}1\frac{1}{3})$. ℓ_4 is the locus of the solution set of what condition?

10 The coördinates of point J are $(1, {}^{-}2)$. ℓ_5 is the locus of the solution set of what condition?

Make a graph of the solution set of each condition expressed in exercises 11 through 16. The universe for (x, y) is $D \times D$.

11 $y = {}^{-}\frac{1}{4}x$. **13** $y = 4x$. **15** $y = {}^{-}x$.

12 $y = \frac{3}{5}x$. **14** $y = {}^{-}3x$. **16** $y = 2x$.

Relations determined by $y = kx^2$

In the last lesson you studied sets of ordered pairs that can be obtained from certain kinds of data. From these sets you developed conditions of the form $y = kx$. In this lesson you will study sets of ordered pairs from which you can develop another kind of condition.

For example, think about a situation in which an object such as a ball is released from a height. It is possible to determine how many feet the ball will drop in 1 second, in 2 seconds, in 3 seconds, and so on. The chart in D1 gives the distances the ball will drop if it falls for 1 second, for 2 seconds, for 3 seconds, and for 4 seconds. x is a variable for the number of seconds; y is a variable for the total number of feet.

A Study the chart. How many feet will the ball drop in the first second? In the first 2 seconds? In the first 3 seconds? In the first 4 seconds?

B As the ball drops, does its speed increase, decrease, or remain the same? How do you know?

C Tabulate the set of all (x, y) that you can obtain from the chart in D1.

D Look at D2. Is M the same set as the set that you tabulated for exercise C?

Set M is a *relation*. A relation is any set of ordered pairs. In the last lesson you discovered that $\{(x, y) | y = kx\}$ is a proportional relation.

E Look again at D2. Is M a proportional relation? How do you know?

You know that, for each (x, y) that satisfies a condition of the form $y = kx$, the compo-

x	1	2	3	4
y	16	64	144	256

D1

$M = \{(1, 16), (2, 64), (3, 144), (4, 256)\}$.

D2

---

x	1	2	3	4
x^2	1	4	9	16
y	16	64	144	256

D3

Universe for $(x, y) = D \times D$.

A $y = 16x^2$.

D4

Universe for x and $y = D$.

B $y = kx^2$.

C $y = 16x^2$.

D $y = -\frac{1}{2}x^2$.

E $y = (\sqrt{2})x^2$.

D5

---

nents are related in a special way: y is equal to the product of a constant of variation and x.

F Think about the members of M. By what number can you multiply the first component of (1, 16) to obtain the second component? By what number can you multiply the first component of (2, 64) to obtain the second component?

G For both of your answers to exercise F, did you obtain the same number? Is there one number by which you can multiply the first component of each member of M to obtain the second component?

Now you will see how, for each member of M, y is related to the *square* of x.

H Study the chart in D3. How does this chart differ from the chart in D1?

I Look at the numerals in the row opposite the letter y in D3. Look also at the numerals in the row opposite the symbol x^2. 16 is the product of what number and 1? 64 is the product of what number and 4? 144 is the product of what number and 9? 256 is the product of what number and 16?

J Remember that the set tabulated in D2 is the set of all (x, y) that you can obtain from the charts in D1 and D3. For each (x, y) in set M, is y equal to the product of 16 and x^2?

K What condition is expressed in D4? How do you know that set M is a subset of $\{(x, y) | y = 16x^2\}$?

For each ordered pair in the solution set of $y = 16x^2$, y is equal to the product of 16 and x^2. For this reason, we say that, in $y = 16x^2$, y varies directly as x^2 varies. 16 is the constant of variation.

L Can you use $(x, 16x^2)$ to represent each member of $\{(x, y) | y = 16x^2\}$?

M Now look at sentence B in D5. What is the universe for x and y?

N The universe for k in $y = kx^2$ is the set of non-zero real numbers. This set contains every real number except 0. What replacement was made for k in $y = kx^2$ to obtain condition C? To obtain condition D? To obtain condition E?

O Conditions C, D, and E are conditions of the form $y = kx^2$. In condition C, you know that y varies directly as x^2 varies, and 16 is the constant of variation. In conditions D and E, does y vary directly as x^2 varies? What is the constant of variation in each condition?

In the work that follows, you will study graphs of solution sets of conditions of the form $y = kx^2$.

A Suppose that you replace k by 2 in $y = kx^2$. What condition do you obtain?

B Make a chart from which you can obtain some members of $\{(x, y) | y = 2x^2\}$. Use $^-2$, $^-1\frac{1}{2}$, $^-1$, 0, 1, $1\frac{1}{2}$, and 2 as replacements for x. Then tabulate the set of (x, y) that you can obtain from your chart.

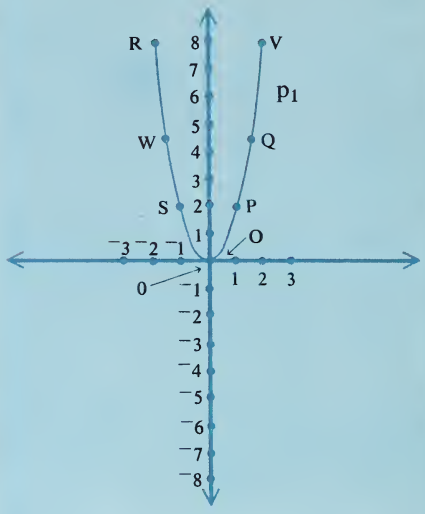
C Now look at the graph in D6. Each point named by a letter is associated with an (x, y) that you can obtain from your chart. For example, point R is associated with $(^-2, 8)$. What are the coördinates of point V? What are the coördinates of point W? Of point Q? Of point O?

If you continue to locate points associated with members of $\{(x, y) | y = 2x^2\}$, you will find that all these points are contained in the curve represented in D6. Notice that the curve is named "p_1."

D Does p_1 contain the origin?

Curve p_1 is the locus of $\{(x, y) | y = 2x^2\}$. The locus of the solution set of a condition of the form $y = kx^2$ is a special kind of curve known as a *parabola* (pə rab′ə lə). The universe for x and y is D. The universe for k is the set of non-zero real numbers. The locus of $\{(x, y) | y = 2x^2\}$, represented in D6, is an example of a parabola.

Universe for $(x, y) = D \times D$.



Incomplete graph
$\{(x, y) | y = 2x^2\}$

D6

In lesson 168 you learned that the solution set of a condition of the form $y = kx$ is a proportional relation. You also learned that, if $y = kx$, then y varies directly as x varies. In this lesson you discovered that the solution set of a condition of the form $y = kx^2$ is *not* a proportional relation. Thus, y does *not* vary directly as x varies, but y does vary directly as x^2 varies. You also learned that the locus of $\{(x, y) | y = kx^2\}$ is a parabola.

On your own

For each of exercises 1 through 5, tabulate the set of (x, y) that you can obtain from the chart. Then tell whether the ordered pairs satisfy a condition of the form $y = kx$ or a condition of the form $y = kx^2$. Finally, write a sentence that expresses the condition. For some of the exercises, it will be helpful to find replacements for x^2.

1

x	$^-2$	$^-1$	0	1	2
y	4	2	0	$^-2$	$^-4$

2

x	$^-3$	$\frac{1}{2}$	5	0	12
y	9	$\frac{1}{4}$	25	0	144

3

x	3	4	5	6	7
y	$4\frac{1}{2}$	8	$12\frac{1}{2}$	18	$24\frac{1}{2}$

4

x	5	10	15	20	25
y	$^{-}3\frac{1}{3}$	$^{-}6\frac{2}{3}$	$^{-}10$	$^{-}13\frac{1}{3}$	$^{-}16\frac{2}{3}$

5

x	$^{-}2$	0	3	5	7
y	24	0	54	150	294

6 Look at D7. Each point named by a letter is associated with one of the ordered pairs expressed below. Name the point associated with each of these ordered pairs.

$(^{-}3, 3)$ $(^{-}2, 1\frac{1}{3})$ $(^{-}1, \frac{1}{3})$ $(0, 0)$
$(1, \frac{1}{3})$ $(2, 1\frac{1}{3})$ $(3, 3)$

7 Think about the ordered pairs named in exercise 6. In each case, is the second component the product of $\frac{1}{3}$ and the square of the first component? What condition of the form $y = kx^2$ is satisfied by each of these ordered pairs? What is the constant of variation in the condition?

8 p_2, represented in D7, is the locus of a solution set of a condition. Write a standard description of this solution set.

9 Now look at D8. Each point named by a letter is associated with one of the ordered pairs expressed below. Name the point associated with each of these ordered pairs.

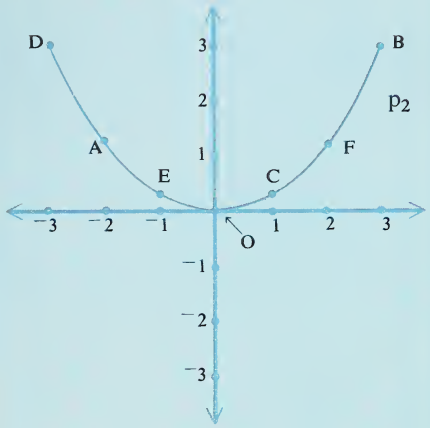
$(^{-}3, ^{-}3)$ $(^{-}2, ^{-}1\frac{1}{3})$ $(^{-}1, ^{-}\frac{1}{3})$ $(0, 0)$
$(1, ^{-}\frac{1}{3})$ $(2, ^{-}1\frac{1}{3})$ $(3, ^{-}3)$

10 What condition of the form $y = kx^2$ is satisfied by each of the ordered pairs named in exercise 9? What is the constant of variation in the condition?

11 p_3, represented in D8, is the locus of a solution set of a condition. Write a standard description of this solution set.

12 How do you know that p_2, represented in D7, and p_3, represented in D8, are parabolas?

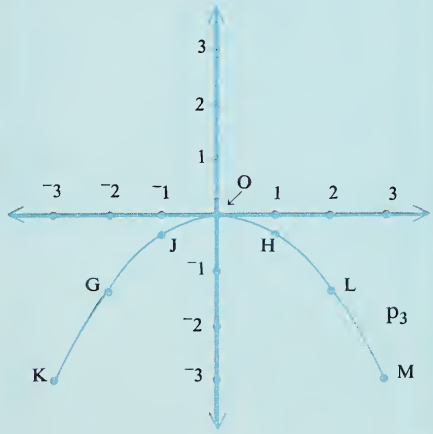
Universe for $(x, y) = D \times D$.



Incomplete graph

D7

Universe for $(x, y) = D \times D$.



Incomplete graph

D8

Relations determined by $y = \frac{k}{x}$

So far in this unit, you have studied conditions of the form $y = kx$ and $y = kx^2$. In this lesson you will study another kind of condition. As in the two preceding lessons, you will first obtain a set of ordered pairs from certain data. Then you will find a condition whose solution set contains these ordered pairs.

Mr. Dirkman plans to have the outside of his house painted. The time required to paint the house depends on how many painters are hired. The chart in D1 tells how many hours it will take a given number of painters to paint the house.

A Study the chart in D1. x refers to the number of painters hired, and y refers to the number of hours required to paint the house. You can see that 1 painter takes 40 hours to paint the house. How many hours will 2 painters take? How many hours will 3 painters take? How many hours will 4 painters take?

B How many painters are needed to paint the house in 20 hours? In 8 hours?

C Tabulate the set of all (x, y) that you can obtain from the chart in D1.

D Look at D2. Is set S the same as the set that you tabulated for exercise C?

In the following exercises, you will find a condition whose solution set includes set S. First you will discover that the form of this condition is neither $y = kx$ nor $y = kx^2$.

E Is $(1, 40) \in S$? By what number can you multiply the first component of $(1, 40)$ to obtain the second component?

x	1	2	3	4	5
y	40	20	$13\frac{1}{3}$	10	8

D1

$S = \{(1, 40), (2, 20), (3, 13\frac{1}{3}), (4, 10), (5, 8)\}$.

D2

x	1	2	3	4	5
$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
y	40	20	$13\frac{1}{3}$	10	8

D3

F Is $(3, 13\frac{1}{3}) \in S$? By what number can you multiply the first component of $(3, 13\frac{1}{3})$ to obtain the second component?

G How do you know that set S cannot be determined by a condition of the form $y = kx$?

H For each member of set S, find the square of the first component.

I Is $(2, 20) \in S$? By what number can you multiply the square of the first component of $(2, 20)$ to obtain the second component?

J Is $(4, 10) \in S$? By what number can you multiply the square of the first component of $(4, 10)$ to obtain the second component?

K How do you know that set S cannot be determined by a condition of the form $y = kx^2$?

You can see that set S is not included in the solution set of either a condition of the form $y = kx$ or a condition of the form $y = kx^2$. To help you find a condition whose solution set does include set S, you will next decide how y is related to the *multiplicative inverse* of x in each member of set S. Remember that the

multiplicative inverse of x is the same as the reciprocal of x.

L Is $\frac{1}{x}$ the multiplicative inverse of x? Is 1 the multiplicative inverse of 1? Is $\frac{1}{2}$ the multiplicative inverse of 2?

M What is the multiplicative inverse of 3? Of 4? Of 5?

N Now study the chart in D3. How does this chart differ from the chart in D1?

O By what number can you multiply the multiplicative inverse of 1 to obtain 40? By what number can you multiply the multiplicative inverse of 2 to obtain 20?

P Is there one number by which you can multiply the multiplicative inverse of each x that is expressed in the chart in D3 to obtain the corresponding y? What is this number?

Q Does each (x, y) expressed in the chart in D3 satisfy $y = 40(\frac{1}{x})$?

R Now look at D4. What condition is expressed by sentence A? By sentence B? What is the universe for (x, y)?

S Is the product of 40 and $\frac{1}{x}$ the same as the quotient of 40 and x? Does condition B have the same solution set as condition A? Is this set a finite set, or is it an infinite set?

T Remember that set S, tabulated in D2, is the set of all (x, y) that you obtained from the chart in D1. Is set S a subset of the solution set of condition B? Explain your answer.

U How do you know that (8, 5) is a member of $\{(x, y) | y = \frac{40}{x}\}$? Is the second component of (8, 5) equal to the product of 40 and the multiplicative inverse of the first component?

V For each member of $\{(x, y) | y = \frac{40}{x}\}$, is the second component equal to the product of 40 and the multiplicative inverse of the first component?

Universe for $(x, y) = D \times D$.

A $y = 40(\frac{1}{x})$. **B** $y = \frac{40}{x}$.

D4

Universe for x and $y = D$.

C $y = \frac{k}{x}$. **E** $y = \frac{31.5}{x}$.

D $y = \frac{40}{x}$. **F** $y = \frac{-25}{x}$.

D5

For each (x, y) that satisfies $y = \frac{40}{x}$, y is equal to the product of 40 and $\frac{1}{x}$. For this reason, we say that y varies directly as $\frac{1}{x}$ varies. Since $\frac{1}{x}$ is the multiplicative inverse of x, we also say that y *varies inversely as* x. The constant of variation is 40. If, for each (x, y) that satisfies a condition, y is the product of the constant of variation and $\frac{1}{x}$, then y varies inversely as x.

You have studied one condition in which y varies inversely as x. Now you will study other conditions of this type.

A Sentence C in D5 expresses what condition? What is the universe for x and y? The universe for k is the set of non-zero real numbers.

B What replacement was made for k in condition C to obtain condition D? To obtain condition E? To obtain condition F?

You already know that, in condition D, y varies inversely as x, and 40 is the constant of variation.

C In each of conditions E and F, what is the constant of variation? Does y vary inversely as x?

Conditions D, E, and F are conditions of the form $y = \frac{k}{x}$. The variable for the constant of variation is k.

Next you will study the locus of the solution set of a condition of the form $y = \frac{k}{x}$.

D Suppose that you replace k by 2 in $y = \frac{k}{x}$. What condition do you obtain?

E What is the constant of variation in $y = \frac{2}{x}$?

F Make a chart to help you obtain some of the members of $\{(x, y) | y = \frac{2}{x}\}$. Use $^{-}3$, $^{-}2$, $^{-}1$, 1, 2, and 3 as replacements for x. Then tabulate the set of (x, y) that you can obtain from your chart. U = D × D.

G Now look at the graph in D6. Each point named by a letter is associated with an (x, y) that you can obtain from your chart. What are the coördinates of point H? Of point K? Of point M?

If you continue to locate points associated with members of $\{(x, y) | y = \frac{2}{x}\}$, you will find that all these points are contained in the two-part curve represented in D6. This curve is the locus of $\{(x, y) | y = \frac{2}{x}\}$. A locus such as this is a *hyperbola* (hī pėr′bə lē′). The locus of the solution set of any condition of the form $y = \frac{k}{x}$ is a hyperbola.

H Is (0, 0) an element of $\{(x, y) | y = \frac{2}{x}\}$? Is (0, 2) an element? Is (2, 0) an element?

I Explain why there is no solution of $y = \frac{2}{x}$ that has 0 as either of its components.

J How do you know that $\{(x, y) | xy = 2\}$ is the same set as $\{(x, y) | y = \frac{2}{x}\}$? The universe for (x, y) is D × D.

K Look again at D6. Which quadrant includes the part of the hyperbola that is associated with the solutions of $y = \frac{2}{x}$ that have positive components? Which quadrant includes the part of the hyperbola that is associated with the solutions that have negative components?

Notice that the first coördinate of point I is greater than the first coördinate of point H, and that the first coördinate of point J is greater than the first coördinate of point I.

L Explain how the second coördinate of point I compares with the second coördinate of point H. Compare the second coördinate of point J with the second coördinate of point I.

Universe for x and y = D.

C $y = \frac{k}{x}$.

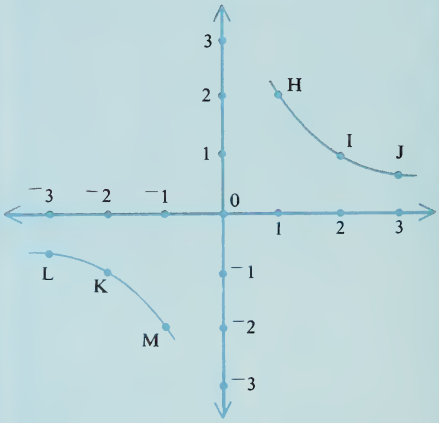
D $y = \frac{40}{x}$.

E $y = \frac{31.5}{x}$.

F $y = \frac{^{-}25}{x}$.

D5

Universe for (x, y) = D × D.



$\{(x, y) | y = \frac{2}{x}\}$

Incomplete graph

D6

M ? Suppose that points (a, b) and (c, d) are members of that part of the locus of the solution set of $y = \frac{2}{x}$ that is included in the first quadrant. If a is greater than c, how does b compare with d? Explain your answer.

In this lesson you studied relations determined by conditions in which y varies inversely as x. You learned that the locus of the solution set of a condition of the form $y = \frac{k}{x}$ is a hyperbola.

On your own

For exercises 1, 2, and 3, tabulate the set of (x, y) that you can obtain from the chart. Then tell which kind of condition is satisfied by the ordered pairs: a condition of the form $y = kx$, of the form $y = kx^2$, or of the form $y = \frac{k}{x}$.

1

x	$^-3$	$^-2$	1	4	5
y	$^-\frac{1}{3}$	$^-\frac{1}{2}$	1	$\frac{1}{4}$	$\frac{1}{5}$

2

x	$^-8$	$^-4$	1	2	6
y	2	1	$^-\frac{1}{4}$	$^-\frac{1}{2}$	$^-\frac{3}{2}$

3

x	$^-4$	$^-3$	$^-2$	1	5
y	$^-80$	$^-45$	$^-20$	$^-5$	$^-125$

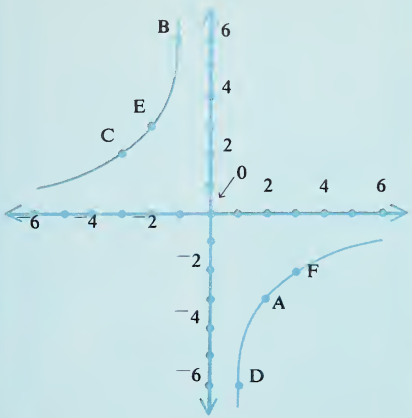
4 $Q = \{(^-3, 2), (^-2, 3), (^-1, 6), (1, ^-6), (2, ^-3), (3, ^-2)\}$. Which of the conditions expressed below is satisfied by each member of Q? The universe for (x, y) is $D \times D$.

$y = {}^-6x$. $\quad y = \frac{6}{x}$. $\quad y = 6x$.

$y = \frac{^-6}{x}$. $\quad y = 6x^2$. $\quad y = {}^-6x^2$.

5 Look at D7. Each of points A, B, C, D, E, and F is associated with a member of set Q. Name the coördinates of each of these points. The hyperbola that contains these points is the locus of what solution set?

Universe for $(x, y) = D \times D$.



$\{(x, y) | y = \frac{^-6}{x}\}$

Incomplete graph

D7

6 Does $(0, 0)$ satisfy $y = \frac{^-6}{x}$? Is the origin contained in the locus of $\{(x, y) | y = \frac{^-6}{x}\}$?

7 Why does neither the first quadrant nor the third quadrant contain points in the locus of the solution set of $y = \frac{^-6}{x}$?

8 Now make a chart to help you find some solutions of $y = \frac{6}{x}$. Use $^-6$, $^-3$, $^-2$, $^-1$, 1, 2, 3, and 6 as replacements for x. $U = D \times D$.

9 Tabulate the set of (x, y) that you can obtain from the chart you made for exercise 8.

10 Which quadrants contain points in the locus of $\{(x, y) | y = \frac{6}{x}\}$?

For each condition expressed below, decide if y varies directly as x varies, or if y varies inversely as x.

11 $y = 12x$. $\quad$ 13 $y = {}^-11x$. $\quad$ 15 $y = \frac{32}{x}$.

12 $y = \frac{^-4}{x}$. $\quad$ 14 $xy = 20$. $\quad$ 16 $xy = {}^-100$.

Relations determined by $y = \frac{k}{x^2}$

This lesson concerns a fourth kind of condition for variation. First you will study such a condition. Then you will examine the ordered pairs that satisfy the condition.

A Study condition A, expressed in D1. What is the universe for (x, y)?

Condition A tells you that, for each (x, y), y must be equal to the quotient of 16 and x^2.

B Is $\frac{1}{x^2}$ the multiplicative inverse of x^2? For each x, is $16(\frac{1}{x^2}) = \frac{16}{x^2}$?

C For each (x, y) that satisfies $y = \frac{16}{x^2}$, is y equal to the product of 16 and the multiplicative inverse of x^2?

For each member of $\{(x, y) | y = \frac{16}{x^2}\}$, y is equal to the product of 16 and $\frac{1}{x^2}$. This means that, in $y = \frac{16}{x^2}$, y varies directly as $\frac{1}{x^2}$ varies. Since $\frac{1}{x^2}$ is the multiplicative inverse of x^2, we say that y varies inversely as the square of x, or y varies inversely as x^2. The constant of variation in $y = \frac{16}{x^2}$ is 16.

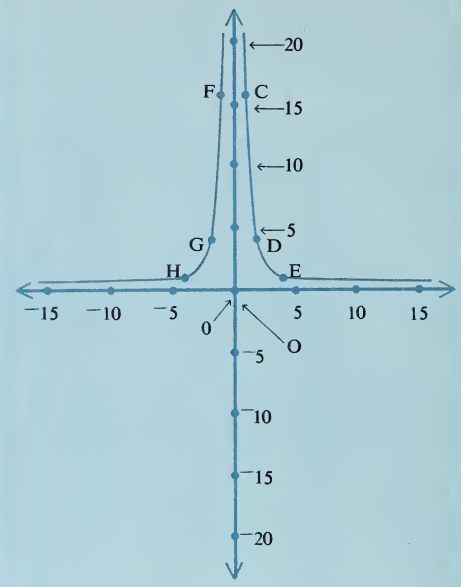
Universe for $(x, y) = D \times D$.

A $y = \frac{16}{x^2}$.

x	$^-4$	$^-2$	$^-1$	1	2	4
y	1	4	16	16	4	1

D1

Universe for $(x, y) = D \times D$.



$\{(x, y) | y = \frac{16}{x^2}\}$

Incomplete graph

D2

D How can you obtain $y = \frac{16}{x^2}$ from $y = \frac{k}{x^2}$? The universe for x and y is D. The universe for k is the set of non-zero real numbers.

E In any condition of the form $y = \frac{k}{x^2}$, y varies inversely as the square of x. What is the variable for the constant of variation in $y = \frac{k}{x^2}$?

F Now look at the chart in D1. Tabulate the set of (x, y) that you can obtain from the chart. Use the letter "B" to name the set you tabulated.

G For each member of set B, is y equal to the product of 16 and the multiplicative inverse of the square of x?

H Is each member of B a solution of $y = \frac{16}{x^2}$? How do you know that B is a subset of $\{(x, y) | y = \frac{16}{x^2}\}$?

Now you will use the members of B to help you make a graph of the solution set of $y = \frac{16}{x^2}$.

I Look at the graph in D2. Each of points C through H is associated with a member of B. Give the coördinates of each point. In which quadrants are these points contained?

Each member of the solution set of $y = \frac{16}{x^2}$ is associated with a point in the two-part curve represented in D2; and each point in this curve is associated with a member of the solution set of $y = \frac{16}{x^2}$. Therefore, the two-part curve represented in D2 is the locus of $\{(x, y) | y = \frac{16}{x^2}\}$.

J Explain why there is no solution of $y = \frac{16}{x^2}$ that has 0 as either of its components.

For any condition of the form $y = \frac{k}{x^2}$, the locus of the solution set is a two-part curve like the curve represented in D2.

In this lesson you studied a set of ordered pairs that satisfy a condition of the form $y = \frac{k}{x^2}$. You also studied the locus of the solution set of a condition of the form $y = \frac{k}{x^2}$.

On your own

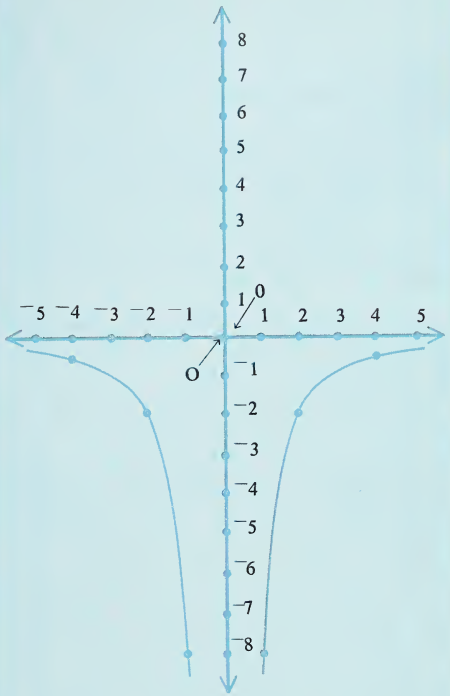
In each of the following exercises, the universe for x and y is D.

1 Which conditions expressed below are of the form $y = \frac{k}{x^2}$?

$y = \frac{1}{x^2}$. $y = {}^-14(\frac{1}{x^2})$. $y = x^2$. $y = \frac{2}{x}$.

2 Make a chart to help you find members of the solution set of $y = \frac{{}^-8}{x^2}$. Use $^-4$, $^-2$, $^-1$, 1, 2, and 4 as replacements for x.

Universe for $(x, y) = D \times D$.



$\{(x, y) | y = \frac{{}^-8}{x^2}\}$

Incomplete graph

D3

3 The graph in D3 is a graph of the solution set of $y = \frac{{}^-8}{x^2}$. Is the two-part curve represented in D3 the locus of $\{(x, y) | y = \frac{{}^-8}{x^2}\}$?

4 Name the additive inverse of $^-8$.

5 Make a chart to help you find members of the solution set of $y = \frac{8}{x^2}$. Use $^-4$, $^-2$, $^-1$, 1, 2, and 4 as replacements for x.

6 In which quadrants is the locus of the solution set of $y = \frac{8}{x^2}$?

Relations determined by conditions for variation

So far in this unit, you have worked with conditions of the following forms: $y = kx$, $y = kx^2$, $y = \frac{k}{x}$, and $y = \frac{k}{x^2}$. In this lesson you will review these conditions.

A What condition is expressed by sentence A in D1? How many times is x used as a factor in condition A? Is condition A of the form $y = kx$?

B Does y vary directly as x varies in $y = {}^-5x$? y is equal to the product of what number and x? What is the constant of variation in $y = {}^-5x$?

C You can use your knowledge of the products of real numbers to decide what kinds of solutions condition A has. If x is replaced by a negative number, how do you know that the y that satisfies $y = {}^-5x$ is positive? If x is replaced by a positive number, what do you know about the y that satisfies $y = {}^-5x$?

D What replacements for x are named in the chart in D1? If x is replaced by $^-\frac{1}{2}$, what y satisfies condition A? If x is replaced by 1, what y satisfies the condition?

A graph of $\{(x, y) | y = {}^-5x\}$ is shown in D1. Study the graph.

E Is $(0, 0)$ an element of $\{(x, y) | y = {}^-5x\}$? Is the dot for the origin in the graph of the solution set of condition A?

F Each of points M and N is in ℓ_1. Use the chart in D1 to help you name the coördinates of points M and N. Is ℓ_1 the locus of the solution set of $y = {}^-5x$?

G Look at D2. What condition is expressed by sentence B? What power of x is in condition B? Is condition B of the form $y = kx^2$?

H Does y vary directly as x^2 varies in condition B? Name the constant of variation in condition B.

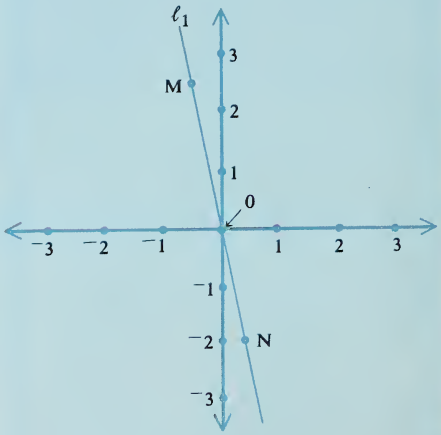
I For each x, is x^2 a non-negative number? ? How do you know? If x is replaced by a number other than 0, what do you know about the y that satisfies $y = {}^-3x^2$?

J Now look at the chart in D2. If x is replaced by $^-1$, what y satisfies $y = {}^-3x^2$? If x is replaced by 1, what y satisfies $y = {}^-3x^2$?

Universe for $(x, y) = D \times D$.

A $y = {}^-5x$.

x	$^-2$	$^-1$	$^-\frac{1}{2}$	0	$\frac{2}{5}$	1	$\frac{3}{2}$
y	10	5	$2\frac{1}{2}$	0	$^-2$	$^-5$	$^-7\frac{1}{2}$



$\{(x, y) | y = {}^-5x\}$

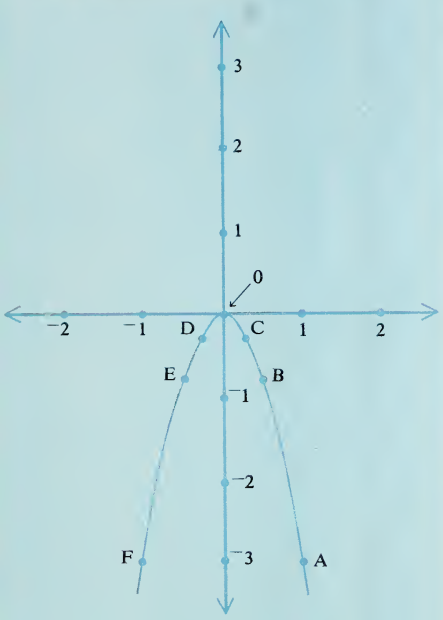
Incomplete graph

D1

Universe for $(x, y) = D \times D$.

B $y = {}^{-}3x^2$.

x	$^{-}1$	$^{-}\frac{1}{2}$	$^{-}\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{2}$	1
y	$^{-}3$	$^{-}\frac{3}{4}$	$^{-}\frac{1}{3}$	0	$^{-}\frac{1}{3}$	$^{-}\frac{3}{4}$	$^{-}3$



$\{(x, y) | y = {}^{-}3x^2\}$
Incomplete graph

D2

K What y satisfies condition B if x is replaced by $^{-}\frac{1}{2}$? By $\frac{1}{2}$? What y satisfies condition B if x is replaced by $^{-}\frac{1}{3}$? By $\frac{1}{3}$?

L Suppose that you replace x in $y = {}^{-}3x^2$ first by a given number, and then by the additive inverse of the given number. What can you say about the y that satisfies the condition in each case?

Now study the graph of $\{(x, y) | y = {}^{-}3x^2\}$ shown in D2.

M Use the chart in D2 to help you name the coördinates of points A through F. Each of these points is in the locus of the solution set of $y = {}^{-}3x^2$.

N Is the locus of the solution set of $y = {}^{-}3x^2$ a parabola? Does the locus contain the origin?

O Explain why none of the points in the first and second quadrants are in the locus of $\{(x, y) | y = {}^{-}3x^2\}$.

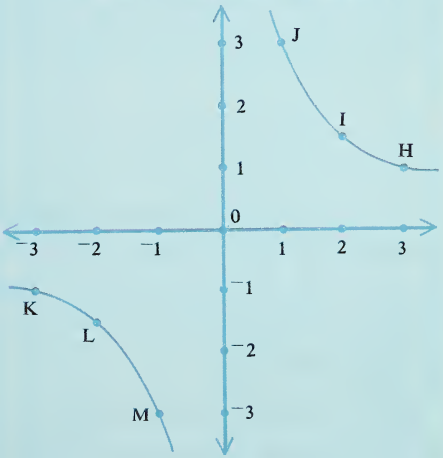
In the exercises that follow, you will review conditions of the forms $y = \frac{k}{x}$ and $y = \frac{k}{x^2}$.

A Look at D3. Is condition C of the form $y = \frac{k}{x}$?

Universe for $(x, y) = D \times D$.

C $y = \frac{3}{x}$.

x	$^{-}3$	$^{-}2$	$^{-}1$	1	2	3
y	$^{-}1$	$^{-}1\frac{1}{2}$	$^{-}3$	3	$1\frac{1}{2}$	1



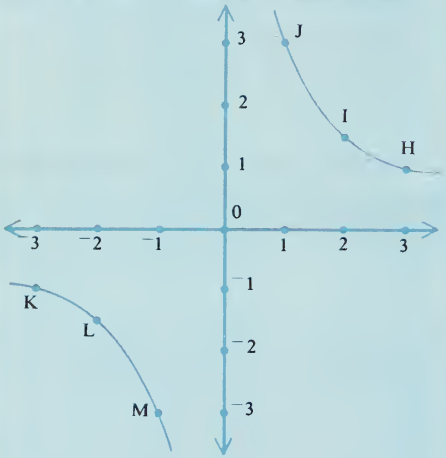
$\{(x, y) | y = \frac{3}{x}\}$
Incomplete graph

D3

Universe for $(x, y) = D \times D$.

C $y = \frac{3}{x}$.

x	$^-3$	$^-2$	$^-1$	1	2	3
y	$^-1$	$^-1\frac{1}{2}$	$^-3$	3	$1\frac{1}{2}$	1



$\{(x, y) | y = \frac{3}{x}\}$

Incomplete graph

D3

B In condition C, does y vary inversely as x? How do you know? What is the constant of variation in $y = \frac{3}{x}$?

C How do you know that, for each positive replacement of x in $y = \frac{3}{x}$, the y that satisfies the condition is also positive? If x is replaced by a negative number, what can you say about the y that satisfies $y = \frac{3}{x}$?

D Explain why there is no solution of $y = \frac{3}{x}$ that has 0 as either its first component or its second component.

E Now look at the chart in D3. What negative replacements for x are named? What positive replacements for x are named?

F As you make greater positive replacements for x, what will be true of the corresponding replacements for y? Give examples to explain your answer.

A graph of $\{(x, y) | y = \frac{3}{x}\}$ is shown in D3. Study the graph.

G Use the chart in D3 to help you name the coordinates of points H through M. Each of these points is in the locus of $\{(x, y) | y = \frac{3}{x}\}$.

H How do you know that the locus of the solution set of $y = \frac{3}{x}$ is a hyperbola? Name the quadrants that contain points in the locus of $\{(x, y) | y = \frac{3}{x}\}$. Why are these the only quadrants that contain points in the locus?

I Now look at sentence D in D4. Is condition D of the form $y = \frac{k}{x^2}$?

J In condition D, does y vary inversely as x, or does y vary inversely as x^2?

K How do you know that each y that satisfies $y = \frac{4}{x^2}$ is positive? Does any solution of condition D have 0 as either of its components?

L Suppose that you replace x in $y = \frac{4}{x^2}$ by $^-3$ and then by 3. What can you say about the y that satisfies the condition in each case? What is true of y if $^-2$ and 2 are used as replacements for x? If $^-1$ and 1 are used as replacements for x?

M As you make greater positive replacements for x, what will be true of the corresponding replacements for y? Give examples to explain your answer.

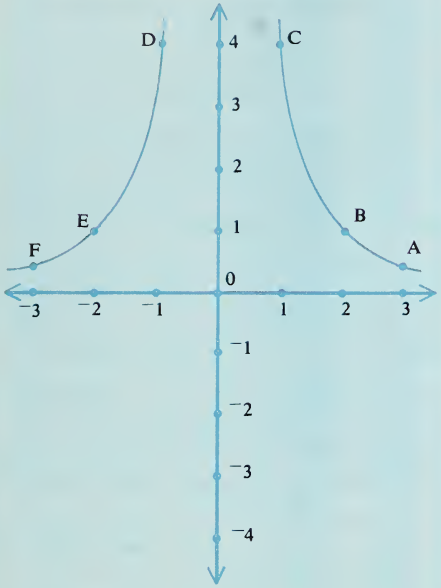
Study the graph of $\{(x, y) | y = \frac{4}{x^2}\}$ in D4.

N Use the chart in D4 to help you name the coördinates of points A through F. Each of these points is in the locus of the solution set of condition D.

Universe for $(x, y) = D \times D$.

D $y = \frac{4}{x^2}$.

x	$^-3$	$^-2$	$^-1$	1	2	3
y	$\frac{4}{9}$	1	4	4	1	$\frac{4}{9}$



$\{(x, y) | y = \frac{4}{x^2}\}$

Incomplete graph

D4

o Name the quadrants that contain points in the locus of $\{(x, y) | y = \frac{4}{x^2}\}$.

In this lesson you reviewed conditions of the forms $y = kx$, $y = kx^2$, $y = \frac{k}{x}$, and $y = \frac{k}{x^2}$. You also reviewed the locus of the solution set of each kind of condition.

On your own

For each condition expressed in exercises 1 through 9, tell in which way y varies: directly as x; directly as x^2; inversely as x; or inversely as x^2. The universe for (x, y) is $D \times D$.

1 $y = \frac{^-1}{x^2}$.

2 $y = 3x$.

3 $y = 5x^2$.

4 $y = \frac{30}{x}$.

5 $y = \frac{^-5}{2}x$.

6 $y = x^2$.

7 $xy = 8$.

8 $y = \frac{^-6}{x^2}$.

9 $y = ^-6x^2$.

Copy and complete each chart given in exercises 10 through 15. The universe for (x, y) is $D \times D$.

10

$y = ^-10x$.							
x	$^-4$	$^-2$	$^-1$	0	1	2	4
y							

11

$y = \frac{3}{x^2}$.						
x	$^-3$	$^-2$	$^-1$	1	2	3
y						

12

$y = ^-4x^2$.							
x	$^-2$	$^-1$	$^-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y							

13

$y = \frac{5}{x}$.						
x	$^-15$	$^-10$	$^-5$	5	10	15
y						

14

$y = \frac{2}{3}x$.							
x	$^-9$	$^-6$	$^-3$	0	3	6	9
y							

15

$y = \frac{1}{2}x^2$.							
x	$^-8$	$^-6$	$^-4$	0	4	6	8
y							

16 What geometric figure is the locus of the solution set of $y = \frac{^-21}{x}$?

17 What geometric figure is the locus of the solution set of $y = 2x^2$?

SPECIAL CHALLENGE

In the Special Challenge on page 129 and also that on page 199, you studied finite number systems and the operations of addivision and muldivision. You can use a finite number system and these two operations to answer the following question: "On which day of the week were you born?"

A Set W, tabulated below, contains all the possible remainders that can be obtained when you divide natural numbers by 7. Make an addivision table and a muldivision table for W.

$W = \{0, 1, 2, 3, 4, 5, 6\}$.

B The display below describes a correspondence between the days of the week and the members of W. Which day corresponds to 4? Which member of W corresponds to Monday?

C A correspondence can also be set up between W and the set of natural numbers. What remainder do you obtain when you divide 68 by 7? The natural number 68 corresponds to 5. What member of W corresponds to the natural number 11? To 49? To 300?

In most years February has only 28 days. However, in certain years, February has 29 days. The years in which February has 29 days are called leap years.

D 1944 and 1948 were leap years. Name the other leap years from 1940 through 1960. There were how many February 29's between March 3, 1940, and March 3, 1960?

E A year that is not a leap year has how many days? A leap year has how many days?

0 ⟷ Sunday	4 ⟷ Thursday
1 ⟷ Monday	5 ⟷ Friday
2 ⟷ Tuesday	6 ⟷ Saturday
3 ⟷ Wednesday	

F Which member of W corresponds to 365? Which member of W corresponds to 366?

Now you are going to answer this question: "On which day of the week did May 16, 1961, fall?" Use x to represent the member of W that corresponds to the day of the week on which May 16, 1961, fell.

G Was there a February 29 between May 16, 1961, and May 16, 1962? Which member of W corresponds to the number of days from May 16, 1961 to May 16, 1962?

H May 16, 1962, fell on a Wednesday. What member of W corresponds to Wednesday? Tabulate the solution set of $x \oplus 1 = 3$. On which day of the week was May 16, 1961?

Next you are going to answer this question: "On which day of the week did August 10, 1932, fall?" Use x to represent the member of W that corresponds to the day of the week on which August 10, 1932, fell.

I How many February 29's were there between August 10, 1932, and August 10, 1944? Does the muldiv $3 \otimes 2$ correspond to the number of days in the leap years between August 10, 1932, and August 10, 1944?

J How many Februarys between August 10, 1932, and August 10, 1944, did not have 29 days? Which member of W corresponds to 9? Does the muldiv $2 \otimes 1$ correspond to the number of days in years that were not leap years between August 10, 1932, and August 10, 1944?

K August 10, 1944, fell on a Thursday. Which member of W corresponds to Thursday? Tabulate the solution set of $x \oplus (3 \otimes 2) \oplus (2 \otimes 1) = 4$. On which day of the week was August 10, 1932?

L March 13, 1950, fell on a Wednesday. Using this fact, find the day of the week on which March 13, 1914, fell. Find the day of the week on which March 13, 1998, will fall.

M On which day of the week were you born?

APPLYING MATHEMATICS

For each problem, first write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem.

Some of the problems may seem unusually easy. However, each problem concerns physical measures. Therefore, you must remember to use the rules for computing with approximations that you learned in lesson 163. The universe for each variable is D.

1 On a flight from London to Toronto, the average speed of the airplane was 550 mi. per hour. On the return flight, the average speed was 485 mi. per hour. The average speed of this airplane on the flight from London to Toronto was how many miles more per hour than the average speed on the return flight?

2 A certain building is 475 ft. high. A television sending tower 352.1 ft. high was installed on the roof of the building. How many feet above the ground is the top of the television tower?

3 In a recent year, the total production of iron ore in Quebec was 1.2×10^7 T. In the same year, the total production of iron ore in Ontario was 6×10^6 T. That year, how many times as much iron ore was produced in Quebec as in Ontario?

4 A mixture of chocolate-covered raisins and nuts weighed at most 25 lb. The mixture contained $10\frac{3}{4}$ lb. of nuts. How many pounds of raisins did the mixture contain?

5 Mr. Page bought 12.5 lb. of grass seed. After he had used some of it to seed his lawn, he had more than 5.25 lb. of the seed left. How many pounds of the seed did he use?

6 Lake Champlain is 95 ft. above sea level. The elevation of Lake Superior is 6.09 times the elevation of Lake Champlain. What is the elevation of Lake Superior in feet?

7 The area of Manitoba is 251,000 sq. mi. The area of Saskatchewan is 251,700 sq. mi. The area of Alberta is 255,285 sq. mi. What is the total area of the three provinces in square miles?

8 Mr. Ryan bought a rectangular piece of land that was 80.5 rd. long and 35 rd. wide. How many square rods of land did he buy?

9 Mr. Simon replaced the glass in a window that is shaped like a rhombus. The length of an altitude is 26.5 in. The length of the base is 30.25 in. What is the area of the window in square inches?

For each of problems 10 through 13, make a sketch to help you solve the problem.

10 The top of a chimney was sighted from point Z on level ground. The measure of the angle of elevation of the top of the chimney from point Z was 48. The distance from point Z to the point on the ground directly below the chimney was 45.6 ft. What is the distance in feet from the top of the chimney to the point on the ground directly below the chimney?

11 The length of a wire that extends from point X on level ground to point Y at the top of a pole is 25.6 ft. The measure of the angle of elevation of point Y from point X is 63. What is the height of the pole in feet?

12 At a given moment, an airplane is 965 ft. directly above point T on the ground. The distance from point T to point U on level ground is 2036 ft. At that moment, what is the measure of the angle of elevation of the airplane from point U?

13 The top of a post rests against a building so that the bottom of the post is 3.75 ft. from the base of the building. The measure of the angle of elevation of the top of the post from the bottom of the post is 75. What is the length of the post in feet?

CHECKING UP

If you have trouble with these two tests, you can find help in lessons 168 through 172.

Test 156

Use the chart in the display for exercises 1 through 4. The universe for (x, y) is D × D.

1 Tabulate the set of all ordered pairs that you can obtain from the chart. Is this set a proportional relation?

2 The second component of each element of the set that you tabulated is equal to the product of what number and the first component?

3 What condition is satisfied by each element of the set that you tabulated for exercise 1?

4 The locus of the solution set of the condition that you expressed for exercise 3 is what geometric figure? Make a graph of the locus.

Test 157

Make a chart to help you find some solutions of each condition expressed in exercises 5 through 10. Use 5, 3, 1, $^{-}1$, and $^{-}3$ as replacements for x. Tabulate the set of all ordered pairs that you can obtain from the chart. The universe for (x, y) is D × D.

5 $y = {}^{-}2x^2$. **7** $y = {}^{-}7x$. **9** $y = \frac{6}{5}x^2$.

6 $y = \frac{14}{x}$. **8** $y = \frac{1}{2}x$. **10** $y = \frac{1}{x^2}$.

For each condition expressed below, tell whether y varies directly as x varies, directly as x^2 varies, inversely as x, or inversely as x^2. Then name the constant of variation. The universe for (x, y) is D × D.

11 $y = {}^{-}14x$. **12** $y = \frac{5}{x^2}$. **13** $y = \frac{{}^{-}19}{x}$.

x	$^{-}2$	$^{-}1$	1	2	3
y	$^{-}8$	$^{-}4$	4	8	12

Problems involving conditions for variation

In this lesson you will learn how to use conditions for variation to solve problems. Assume that all measures given in the problems are mathematical measures, and not physical measures. The universe for each variable is D.

Read the problem in D1. You have often developed conditions for equivalence to solve problems like the one in D1. In lesson 168 you learned that a condition for direct variation can be derived from a condition for equivalence. Therefore, you can develop a condition for direct variation for this problem.

The condition for the problem will be a condition of the form expressed below. Use x as a variable for the number of miles of road. Use y as a variable for the number of months.

$$y = kx.$$

A You know that the crew can build 10 mi. of road in 4 mo. You can use this information to help you find the k that satisfies the condition for direct variation. For what is k a variable?

B Why do you use 10 as a replacement for x in $y = kx$? By what number do you replace y? What condition do you obtain?

C Look at D2. Is condition A the same as the condition that you obtained for exercise B?

D Use definitions and properties of real numbers to explain how each of the other conditions expressed in D2 was obtained.

E What number satisfies $\frac{2}{5} = k$? Is this number the constant of variation in a condition of the form $y = kx$?

A road crew can build 10 mi. of road in 4 mo. At this rate, how many miles of road can the crew build in 12 mo.?

D1

Universe for each variable = D.

A $4 = k(10)$.

B $4 \times \frac{1}{10} = k(10) \times \frac{1}{10}$.

C $4 \times \frac{1}{10} = k(10 \times \frac{1}{10})$.

D $\frac{2}{5} = k(1)$.

E $\frac{2}{5} = k$.

D2

Universe for x and y = D.

For each x, $x^2 \geqq 0$.

For each x and y, if $x = y$ or if $x = {}^{-}y$, then $x^2 = y^2$. If $x^2 = y^2$, then $x = y$ or $x = {}^{-}y$.

For each x and y, if $x \geqq 0$ and $y \geqq 0$, and if $x < y$, then $x^2 < y^2$. If $x \geqq 0$ and $y \geqq 0$, and if $x^2 < y^2$, then $x < y$.

D3

F If you replace k by $\frac{2}{5}$ in $y = kx$, do you obtain the condition expressed at the right?

$y = \frac{2}{5}x$.

G Remember that the problem in D1 asks you to find how many miles of road the crew can build in 12 mo. If you replace y by 12 in $y = \frac{2}{5}x$, do you obtain the condition expressed at the right?

$12 = \frac{2}{5}x$.

H What number satisfies $12 = \frac{2}{5}x$?

You can use the solution of $12 = \frac{2}{5}x$ to get the answer to the problem. The crew can build 30 miles of road in 12 months.

Think again about the problem in D1. This time use x as a variable for the number of months and y as a variable for the number of miles.

I You know that in 4 mo. the road crew can build 10 mi. of road. Why do you use 4 as a replacement for x in $y = kx$? What number do you use as a replacement for y? What condition do you obtain?

J What k satisfies $10 = k(4)$?

K If you replace k by $2\frac{1}{2}$ in $y = kx$, what condition do you obtain?

L Why do you use 12 as the replacement for x in $y = 2\frac{1}{2}x$?

M Tabulate $\{y \mid y = 2\frac{1}{2}(12)\}$ and give the answer to the problem.

N Is the number that satisfies the condition $y = 2\frac{1}{2}(12)$ the same as the number that satisfies the condition $12 = \frac{2}{5}x$?

Now think about what you have done. You have used a condition of the form $y = kx$ to solve a problem that involves direct variation. First you found k, the constant of variation, by using replacements for x and y that were given in the problem. Then you used the constant of variation, together with further information obtained from the problem, to develop another condition for the problem.

Next you will study certain properties of squares and square roots of real numbers. These properties are helpful in solving problems that involve conditions of the form $y = kx^2$.

In lesson 149 you studied three properties of squares of rational numbers. We will assume that these same properties also hold true for real numbers. Study D3, where these properties of real numbers are expressed.

Two more properties that involve square roots of real numbers are developed in the work that follows.

O Read sentence F in D4. Is $(7)^2 = 49$? What is $\sqrt{49}$? Remember that $\sqrt{49}$ is the positive square root of 49. Is statement F true?

P Read sentence G. What is $(\frac{2}{3})^2$? What is $\sqrt{\frac{4}{9}}$? Is statement G true?

Q How do you know that sentence H expresses a true statement?

The statements expressed in D4 are examples of a property of square roots of real numbers. The property is expressed below.

The universe for x *is D. For each* x, *if* $x \geq 0$, *then* $\sqrt{x^2} = x$.

R Use an example to show that, if $x < 0$, then $\sqrt{x^2} \neq x$. The universe for x is D.

S Now look at D5. How can you obtain statement I from $x = y$? How can you obtain statement K from $\sqrt{x} = \sqrt{y}$? Were the same replacements made for x and y in each case?

T Is statement I true? Is statement K true?

U What replacements were made for x and y in $x = y$ and $\sqrt{x} = \sqrt{y}$ to obtain statement J and statement L? Are both of these statements true?

Statements I and K and statements J and L are examples of another property of square roots of real numbers. The property is expressed below.

The universe for x *and* y *is D. For each* x *and* y, *if* $x \geq 0$ *and* $y \geq 0$, *and if* $x = y$, *then* $\sqrt{x} = \sqrt{y}$. *If* $x \geq 0$ *and* $y \geq 0$, *and if* $\sqrt{x} = \sqrt{y}$, *then* $x = y$.

The method you used for the problem in D1 has wide applications. You will see some of these applications in the following problems.

Read the problem in D6.

A From the problem, you know that the distance a car travels after the brakes are applied varies directly as the square of the speed of the car. Use x as a variable for the speed of the car in miles per hour. Use y as a variable for the distance in feet that the car traveled after the brakes were applied. How do you know that the condition for the problem is a condition of the form expressed at the right?

$y = kx^2$.

B You know that the first car, going 30 mi. per hour, traveled a distance of 63 ft. after the brakes were applied. What number do you use as a replacement for x in $y = kx^2$? What number do you use as a replacement for y? What condition do you obtain?

C What is a standard name of 30^2? To find the k that satisfies $63 = k(900)$, you must find

F $\sqrt{(7)^2} = 7$.

G $\sqrt{(\frac{2}{3})^2} = \frac{2}{3}$.

H $\sqrt{(\frac{8}{5})^2} = \frac{8}{5}$.

D4

Universe for x and y = D.

$x = y$.	$\sqrt{x} = \sqrt{y}$.
I $16 = \frac{64}{4}$.	K $\sqrt{16} = \sqrt{\frac{64}{4}}$.
J $9 = \frac{225}{25}$.	L $\sqrt{9} = \sqrt{\frac{225}{25}}$.

D5

The distance that a car travels after the brakes are applied varies directly as the square of the speed of the car. A first car, going 30 mi. per hour, traveled a distance of 63 ft. after the brakes were applied. A second car traveled a distance of 175 ft. after the brakes were applied. How fast was the second car going when the brakes were applied?

D6

the quotient of what two numbers? What decimal expresses this quotient?

D If you replace k by .07 in $y = kx^2$, what condition do you obtain?

E The problem asks you to find how fast the second car was going when the brakes were applied. You know that the car traveled a distance of 175 ft. after the brakes were applied. Why do you replace y by 175 in $y = .07x^2$? What condition do you obtain?

F Explain how you can obtain $2500 = x^2$ from $175 = .07x^2$.

G How do you know that $x \geqq 0$? What properties of square roots of real numbers can you use to obtain $50 = x$ from $2500 = x^2$?

H What number satisfies $50 = x$? How fast was the second car going when the brakes were applied?

Sometimes it is a little hard to decide what kind of variation is involved in a problem. Read the problem in D7. Use x as a variable for the number of pipes. Use y as a variable for the number of hours.

I Suppose that you were to double the number of pipes used to fill the reservoir. How would the time required to fill the reservoir be affected? Suppose that you were to divide the original number of pipes by 3. How would the time required to fill the reservoir be affected?

J In a condition for the problem in D7, does y vary directly as x? Does y vary inversely as x?

Since y varies inversely as x, you know that the condition for the problem is a condition of the form expressed at the right.

$$y = \frac{k}{x}.$$

K You know that 12 pipes can fill the reservoir with water in 20 hr. If you replace x by 12 and y by 20 in $y = \frac{k}{x}$, do you obtain the condition $20 = \frac{k}{12}$?

12 pipes of a certain size can fill a reservoir with water in 20 hr. How many hours will it take 10 of these pipes to fill the reservoir?

D7

The electrical resistance of wire is measured in ohms. The electrical resistance of wire varies inversely as the square of the diameter of the wire. A wire whose diameter is .04 in. has an electrical resistance of .2 ohms. The diameter of another wire of the same length is .05 in. What is the electrical resistance of the second wire in ohms?

D8

L What number satisfies the condition that you obtained for exercise K? What number do you use as a replacement for k in $y = \frac{k}{x}$?

M The problem in D7 asks you to find how many hours it will take 10 pipes to fill the reservoir. What number satisfies the condition expressed at the right?

$$y = \frac{240}{10}.$$

N How many hours will it take 10 pipes to fill the reservoir?

Read the problem in D8. Use y as a variable for the electrical resistance of the wire in ohms. Use x as a variable for the diameter of the wire in inches.

From the problem, you know that the electrical resistance of a wire varies inversely as the square of the diameter of the wire. This means that the condition for the problem is a condition of the form expressed at the right.

$$y = \frac{k}{x^2}.$$

The electrical resistance of wire is measured in ohms. The electrical resistance of wire varies inversely as the square of the diameter of the wire. A wire whose diameter is .04 in. has an electrical resistance of .2 ohms. The diameter of another wire of the same length is .05 in. What is the electrical resistance of the second wire in ohms?

D8

O Replace y by .2 in $y = \frac{k}{x^2}$. What number do you use as a replacement for x? What condition do you obtain?

P Does the decimal .0016 express the same number as the decimal $(.04)^2$?

Q Is $.2(.0016) = k$ equivalent to $.2 = \frac{k}{.0016}$? What number satisfies $.2(.0016) = k$?

R .00032 is the constant of variation for the problem in D8. If you replace k by .00032 in $y = \frac{k}{x^2}$, what condition do you obtain?

S The problem asks you to find the electrical resistance in ohms of a second wire with a diameter of .05 in. When you replace x by .05 in $y = \frac{.00032}{x^2}$, what condition do you obtain?

T Is .0025 the same number as $(.05)^2$?

U To find the y that satisfies $y = \frac{.00032}{.0025}$, you must find the quotient of what two numbers?

V What number satisfies $y = \frac{.00032}{.0025}$?

W Give the answer to the problem in D8.

In this lesson you used conditions for variation to solve problems.

On your own

For each problem, first choose a condition for variation. Determine the constant of variation. Next write a sentence that expresses a condition that will help you obtain the answer to the problem. Then give the answer to the problem. The universe for each variable is D.

1 A clock gains $2\frac{1}{2}$ min. in a period of 12 hr. At this rate, how many minutes does the clock gain in 48 hr.?

2 Three boys can wax a car in $2\frac{3}{4}$ hr. In how many hours can two boys wax the car?

3 For each inch of precipitation in Churchill last year, Halifax had 3.6 inches of precipitation. Churchill had 15 inches of precipitation last year. How many inches of precipitation did Halifax have?

4 Three printing presses can print 270,000 newspapers in 2 hours. How many hours will it take 8 presses to print 270,000 newspapers?

5 For all triangles that have a common base, the area varies directly as the altitude. Triangles ABC and ABD have a common base. The measure of the altitude of $\triangle ABC$ is 6.4 in. The measure of the altitude of $\triangle ABD$ is 4.8 in. The area of $\triangle ABC$ is 50 sq. in. What is the area of $\triangle ABD$ in square inches?

6 The length of a simple pendulum varies directly as the square of the time it takes to make one single swing. A pendulum that is 60 cm. long makes one swing in 1.5 sec. What is the length in centimeters of a pendulum that makes one swing in 2 sec.?

7 The distance traveled by a falling object, starting from rest, varies directly as the square of the time traveled. A rock falling from the top of a cliff traveled 400 ft. in the first 5 sec. that it fell. How many feet did the rock travel in the first 10 sec. that it fell?

8 The weight of an object varies inversely as the square of its distance from the center of the earth. A rocket that is 4000 mi. from the center of the earth weighs 600 lb. If the rocket were 5000 mi. from the center of the earth, what would be its weight in pounds?

KEEPING SKILFUL

For each of exercises 1 through 6, express the given unit in the unit indicated in parentheses.

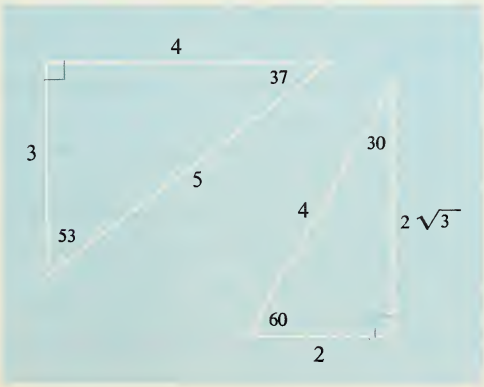
1 5 T. (pounds)
2 68 in. (feet)
3 3 mi. (yards)
4 $2\frac{1}{2}$ A. (square rods)
5 45 sq. ft. (square yards)
6 8 sq. ft. (square inches)

Use the display below for exercises 7 through 16. For each exercise, find a rate pair that is equivalent to the trigonometric rate pair named. Then write a sentence expressing a true statement about the two rate pairs. For exercise 7, you should write the following sentence: $2/4 \sim (\sin 30)/1$.

7 $(\sin 30)/1$
8 $(\tan 37)/1$
9 $(\cos 30)/1$
10 $(\sin 53)/1$
11 $(\tan 30)/1$
12 $(\sin 37)/1$
13 $(\cos 53)/1$
14 $(\tan 60)/1$
15 $(\tan 53)/1$
16 $(\sin 60)/1$

For each condition expressed below, tabulate the solution set if possible. Otherwise, give a standard description of the solution set. $U = D$.

17 $^{-}1\frac{1}{3} + x = 6$.
18 $20/(y - \frac{7}{9}) \sim 45/5$.
19 $.5h + .8h > 39$.
20 $\sim (6t - 3 \leqq {}^{-}12)$.
21 $\frac{5}{6}m > {}^{-}10 \wedge m - 5 < {}^{-}3$.
22 $.25n > {}^{-}1.5 \vee n + 1.2 > 10$.



Formulas derived from $y/x \sim k/1$

You have learned to develop conditions for direct variation from conditions for equivalence. In this lesson you will again start with a condition for equivalence, from which you will develop special conditions to solve many different kinds of problems. The universe for each variable is D.

A Read problem A in D1. Do you know the number of miles traveled in all? Use y as a variable for this number. What rate pair can you use to represent the distance traveled per 5 hours? What rate pair can you use to represent the distance traveled per 1 hour? The sentence above at the right expresses a condition for problem A.

$y/5 \sim 20/1$.

B Read problem B in D1. Why does the sentence at the right express a condition for this problem?

$156/3 \sim k/1$.

A How many miles does a freight train travel in 5 hr. at a rate of 20 mi. per hour?

B How many miles per hour is a car traveling if it travels 156 mi. in 3 hr.?

C How many hours does it take an airplane to fly 1500 mi. at a rate of 600 mi. per hour?

D1

C Read problem C in D1. The sentence at the right expresses a condition for this problem. To what does the first component of $1500/x$ refer? For what is x a variable? To what does each component of $600/1$ refer?

$$1500/x \sim 600/1.$$

D Read sentence D in D2. Is the condition for each problem in D1 of the same form as condition D?

For each problem in D1, y is a variable for the *distance* traveled; x is a variable for the *time* traveled; and k is a variable for the first component of the rate pair that represents the distance traveled per 1 hour. Instead of using the letters y, x, and k to name the variables in $y/x \sim k/1$, we will choose other letters that are appropriate for the problems in D1.

E Study sentence E in D2. Since the letter d is the first letter of the word *distance*, the letter d is used to name the variable for the distance traveled. Why is the letter t used to name the variable for the time traveled?

Notice that the letter r is used in sentence E to name the variable for the first component of the rate pair that represents the distance traveled per 1 hour. The distance traveled per hour is a rate, but we often use the word *rate* to refer to only the *first* component of the rate pair that represents the distance traveled per 1 hour.

F Does sentence E in D2 express the same condition as sentence D? How do you obtain condition F from condition E?

G Can condition F be used to solve all three problems in D1?

A condition that can be used to solve many problems of a particular kind is a *formula* (fôr′myù lə). The condition $d = rt$ is called the *distance* formula because it is used to solve many problems that involve distances.

A How many miles does a freight train travel in 5 hr. at a rate of 20 mi. per hour?

B How many miles per hour is a car traveling if it travels 156 mi. in 3 hr.?

C How many hours does it take an airplane to fly 1500 mi. at a rate of 600 mi. per hour?

D1

Universe for each variable = D.

D $y/x \sim k/1$.

E $d/t \sim r/1$.

F $d = rt$.

D2

H Look again at D1. If you use $d = rt$ to solve problem A, you should replace t by what number? You should replace r by what number?

I Do you obtain the condition $d = 20(5)$?

J For problem B, what replacements should you make for d and t in the formula $d = rt$? What condition do you obtain?

K For problem C, what condition can you obtain from the formula $d = rt$?

In the exercises that follow, you will derive other useful formulas from $y/x \sim k/1$.

A Read problem G in D3. Do you know the cost of 4 rolls of film? Use y as a variable for the total cost. Can you use $y/4$ to represent the cost per 4 rolls?

B What rate pair can you use to represent the cost per 1 roll? The sentence below at the right expresses a condition for problem G. Is this condition of the form $y/x \sim k/1$?

$$y/4 \sim 1.25/1.$$

G Sam bought 4 rolls of film priced at \$1.25 per roll. What was the cost of the 4 rolls of film?

H 80% of the members of the Oakdale Student Council attended a council meeting. If 12 members attended the meeting, how many members are on the council?

I In a baseball park, the distance from home plate to the centre-field fence is 426 ft. What is this distance in yards?

D3

Problem G involves three different quantities: the *cost* of 1 item, the *number* of items, and the *total cost*. We will refer to the cost of 1 item as *price*. Sometimes price refers to a rate pair; at other times it refers to a component of a rate pair.

C Think about $y/x \sim k/1$. For what quantity in problem G is y a variable? For what quantity is x a variable? For what is k a variable?

You can choose letters other than y, x, and k to name the variables in $y/x \sim k/1$. Use the letter c to name the variable for the total cost. Use the letter n to name the variable for the number of items. Use the letter p to name the variable for the price of 1 item.

D Does the sentence $c/n \sim p/1$ express the same condition as the sentence $y/x \sim k/1$?

E How can you obtain $c = np$ from $c/n \sim p/1$?

The condition $c = np$ is a *cost* formula.

F How can you use the formula $c = np$ for problem G in D3?

G Read problem H in D3. The sentence below at the right expresses a condition for the problem. Is this condition of the form $y/x \sim k/1$?

$12/x \sim 80/100.$

H How do you know that $80/100 \sim .8/1$? Is $12/x \sim .8/1$ also a condition for problem H? Is $12/x \sim .8/1$ of the form $y/x \sim k/1$?

Notice that you can develop a condition of the form $y/x \sim k/1$ for problem H if you use a rate pair that is equivalent to 80% and that has a second component of 1.

Every per cent problem, like problem H, involves three numbers: a total quantity, or *base*; a *percentage* (pər sen′tij) of the base; and a *rate*. The word *rate* in a per cent problem refers to the first component of a rate pair whose second component is 1. The rate pair is equivalent to the given per cent. If the rate in a problem is less than 1, then the percentage is less than the base; but if the rate is greater than 1, then the percentage is greater than the base. Now you can use different letters to name the variables in $y/x \sim k/1$.

I Read problem H again. In $y/x \sim k/1$, is y a variable for the base or for a percentage of the base? Use the letter p to name this variable. Use the letter b instead of the letter x to name the variable for the base. Use the letter r instead of the letter k to name the variable for the rate.

J Does the sentence $p/b \sim r/1$ express the same condition as the sentence $y/x \sim k/1$? How can you obtain $p = br$ from $p/b \sim r/1$?

The condition $p = br$ is called a *percentage* formula. This formula can be used to solve per cent problems. You should keep in mind, however, that r is a variable for a number that is $\frac{1}{100}$ times the first component of a per cent. For example, if a problem involves 65%, then the replacement for r is obtained by dividing 65 by 100.

K How can you use the formula $p = br$ for problem H in D3?

L Now read problem I in D3. What does the problem ask you to find? Use x as a variable

for the number you want to find. What comparison is represented by $426/x$?

M What comparison is represented by $3/1$? The sentence below at the right expresses a condition for problem I.

Is the condition of the form $y/x \sim k/1$?

$426/x \sim 3/1.$

Problem I concerns changing, or *converting*, from one unit of measure to another. Every problem of this type involves a *lesser* unit of measure; a *greater* unit of measure; and a *constant* that represents the number of lesser units in 1 of the greater units.

N Think about problem I. In $y/x \sim k/1$, is y a variable for the number of lesser units or for the number of greater units? For what is x a variable? For what is k a variable?

O Use the letters l and g instead of the letters y and x to name the variables for the number of lesser units and the number of greater units. Is $l/g \sim k/1$ the same condition as $y/x \sim k/1$?

P How can you obtain $l = kg$ from $l/g \sim k/1$?

The condition $l = kg$ is called a *conversion* (kən vėr′zhən) formula. This formula is useful for solving problems that involve different units of measure.

Q How can you use $l = kg$ for problem I?

R Now read problem J in D4. Does the sentence below at the right express a condition for the problem? Is the condition of the form $y/x \sim k/1$?

$90/8 \sim k/1.$

Problems about averages involve a *total* for a certain *number* of items and also an *average* per 1 item. Now you can develop a formula for such problems.

S In $y/x \sim k/1$, y is a variable for the total in problem J. For what is x a variable in problem J? For what is k a variable?

T Suppose that, instead of the letters y, x, and k, you use the letters t, n, and a, respectively, to name the three variables. Is $t/n \sim a/1$ the same condition as $y/x \sim k/1$?

U How can you obtain $a = \frac{t}{n}$ from $t/n \sim a/1$?

To what does each variable in $a = \frac{t}{n}$ refer?

The condition $a = \frac{t}{n}$ is called the *average* formula because it is used in solving problems that concern averages.

V How can you use $a = \frac{t}{n}$ to solve problem J?

G Sam bought 4 rolls of film priced at \$1.25 per roll. What was the cost of the 4 rolls of film?

H 80% of the members of the Oakdale Student Council attended a council meeting. If 12 members attended the meeting, how many members are on the council?

I In a baseball park, the distance from home plate to the centre-field fence is 426 ft. What is this distance in yards?

D3

J In 8 basketball games, Ralph scored a total of 90 points. What was the average number of points Ralph scored per game?

D4

Next you will develop trigonometric formulas for finding the tangent, the sine, and the cosine of an acute angle of a right triangle. You will use t as the variable for the measure of the acute angle, x as the variable for the measure of the side adjacent to the acute angle, y as the variable for the measure of the

side opposite the acute angle, and r as the variable for the measure of the hypotenuse.

A Explain why the condition expressed below in blue can be used to determine the tangent of an acute angle of a right triangle.

$$y/x \sim (\tan t)/1.$$

B Show the steps that you can use to obtain the condition $\tan t = \frac{y}{x}$ from $y/x \sim (\tan t)/1$.

C Look at D5. How do you know that $\triangle ABC$ is a right triangle? Which angle is a right angle? Which angles are acute?

D To find the tangent of $\angle A$, can you use the formula $\tan t = \frac{y}{x}$? For what is $\tan t$ a variable? Why should you replace y in $\tan t = \frac{y}{x}$ by 6? By what number should you replace x?

E How can you use the formula $\tan t = \frac{y}{x}$ to find the tangent of $\angle B$?

F How do you know that the condition expressed below in blue can be used to find the sine of an acute angle of a right triangle?

$$y/r \sim (\sin t)/1.$$

G Show the steps that you can use to obtain the condition $\sin t = \frac{y}{r}$ from $y/r \sim (\sin t)/1$.

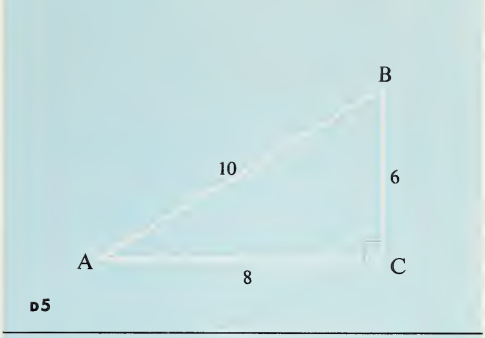
H Look again at D5. How can you use the formula $\sin t = \frac{y}{r}$ to find the sine of $\angle B$?

I To find the sine of $\angle A$, what replacements should you make for y and r in $\sin t = \frac{y}{r}$?

J Is the solution of $\sin t = \frac{6}{10}$ the sine of $\angle A$?

K Can you use the condition expressed below in blue to determine the cosine of an acute angle of a right triangle? Show the steps that you can use to obtain the formula $\cos t = \frac{x}{r}$ from $x/r \sim (\cos t)/1$.

$$x/r \sim (\cos t)/1.$$



D5

Universe for each variable = D.

$d = rt.$

$c = np.$

$p = br.$

$l = kg.$

$a = \frac{t}{n}.$

$\tan t = \frac{y}{x}.$

$\sin t = \frac{y}{r}.$

$\cos t = \frac{x}{r}.$

D6

L Look again at D5. Explain how you can use $\cos t = \frac{x}{r}$ to determine the cosine of $\angle A$. To determine the cosine of $\angle B$.

In this lesson you have derived the formulas that are expressed in D6.

On your own

For each of the following problems, first choose the appropriate formula from D6. Then use the information given in the problem to make replacements in the formula and write a sentence that expresses the condition you obtain. Do not find the solution set of the condition or attempt to give an answer for the problem. The universe for each variable is D.

1 What is the length in centimetres of a curtain rod that is .75 m. long?

2 How many 5-cent pencils can Judy buy for 35¢?

Universe for each variable = D.

$d = rt.$ $\quad \tan t = \frac{y}{x}.$

$c = np.$ $\quad \sin t = \frac{y}{r}.$

$p = br.$ $\quad \cos t = \frac{x}{r}.$

$l = kg.$

$a = \frac{t}{n}.$

D6

3 In $\triangle ABC$, $\angle C^\circ = 90$ and $\angle A^\circ = 40$. $m(\overline{AC}) = 3.5$. What is $m(\overline{BC})$?

4 26 is 48% of what number?

5 Clarence caught 5 fish. The average weight per fish was $2\frac{1}{3}$ lb. What was the total weight of the 5 fish in pounds?

6 In $\triangle ABC$, $\angle C$ is a right angle. What is the cosine of $\angle A$ if $m(\overline{AC}) = 14$ and $m(\overline{AB}) = 22$?

7 Hal paid 60¢ for 4 batteries of the same kind. What was the price of each battery?

8 How far can a plane fly in 15 min. if it is traveling at a rate of 7.5 mi. per minute?

9 Amy is $5\frac{1}{2}$ ft. tall. What is Amy's height in inches?

10 In $\triangle RST$, $\angle R^\circ = 90$. $\angle S^\circ = 30$. What is the measure of the hypotenuse if $m(\overline{TR}) = 8.2$?

11 \$40 is what per cent of \$8500?

12 How long does it take a car to travel 450 km. if its speed is 95 km. per hour?

13 In $\triangle XYZ$, $\angle Z^\circ = 90$. What is the tangent of $\angle X$ if $m(\overline{XZ}) = 2.6$, and $m(\overline{YZ}) = 9.45$?

14 In $\triangle ABC$, $\angle C^\circ = 90$. What is the cosine of $\angle A$ if $m(\overline{AB}) = 12$, and $m(\overline{AC}) = 5$?

15 What is the length in millimetres of a screw that is .125 m. long?

16 How much must Mrs. Karroll pay for $4\frac{1}{2}$ yd. of material priced at \$1.65 per yard?

17 A bar of soap weighs 4 oz. What fraction of a pound does the soap weigh?

18 What number is 175% of 8.2?

The use of formulas

In lesson 174 you learned how to derive certain formulas from $y/x \sim k/1$. In this lesson you will learn how to use these formulas to solve problems. Think of the measures given in the problems as mathematical measures, not as physical measures. The universe for each variable is D.

Read the problem in D1. First you will develop a compound condition for the problem.

A You are asked to find how many miles the plane flew. You can use the formula expressed at the right as one of the simple conditions for the problem. For what is d a variable?

$d = rt.$

B You know that the plane flew at a rate of 400 mi. per hour. You can use the condition expressed at the right as another simple condition for the problem. For what is r a variable?

$r = 400.$

C How many hours did the plane fly? You can use the condition expressed below at the right as another simple condition for the problem. For what is t a variable?

$t = 3.$

The sentence below in blue expresses the compound condition for the problem.

$$d = rt \wedge r = 400 \wedge t = 3.$$

Now you will find (r, t, d) that satisfies the compound condition for the problem.

D How do you know that 400 is the first component of the ordered triple that satisfies the

A jet plane travelling from Winnipeg to Vancouver flew at a rate of 400 mi. per hour for 3 hr. How many miles did the plane fly in 3 hr.?

D1

Mr. Haynes bought 110 stoves for his apartment buildings. The price was the same for each stove. Altogether, the stoves cost \$16,995. What was the price of each stove?

D2

Miss Allen received a discount of \$2.39 per square yard on carpet regularly priced at \$11.95 per square yard. What rate of discount did Miss Allen receive?

D3

compound condition? What is the second component of the ordered triple that satisfies the compound condition?

E Replace r by 400 and t by 3 in $d = rt$. What condition do you obtain? What number satisfies $d = 400 \times 3$?

F Tabulate the set of ordered triples that satisfies $d = rt \wedge r = 400 \wedge t = 3$.

Now you can give the answer to the problem in D1. Travelling at the rate of 400 mi. per hour for 3 hr., the plane flew 1200 mi.

To solve the problem in D1, you developed a compound condition. You can avoid developing a compound condition by using the information given in the problem to make immediate replacements in the formula $d = rt$.

G You know that $d = rt$ is the formula for the problem in D1. Which variables in $d = rt$ can you replace immediately by using the information given in the problem?

H By what number should you replace r? By what number should you replace t?

I Can you use the condition expressed at the right to solve the problem in D1?

$d = 400 \times 3.$

J What number satisfies $d = 400 \times 3$? Does this number give you the same answer for the problem as the answer that you obtained by using a compound condition?

From now on, to solve problems involving compound conditions, use the short method that has just been described whenever you can.

A Read the problem in D2. Why can you use the formula expressed at the right for the problem?

$c = np.$

B You want to find the price of each stove that Mr. Haynes bought. Which variable in $c = np$ refers to the price per stove?

C By what number should you replace c in $c = np$? By what number should you replace n? Does the sentence at the right express the condition you obtain?

$16995 = 110p.$

D Explain how you can obtain $154.50 = p$ from $16995 = 110p$. What number satisfies $154.50 = p$?

E What was the price of each stove?

F Now read the problem in D3. Can you use the percentage formula expressed at the right for the problem?

$p = br.$

G In $p = br$, which variable refers to percentage? Which variable refers to base? To solve the problem in D3, what replacement do you make for p? For b?

Miss Allen received a discount of \$2.39 per square yard on carpet regularly priced at \$11.95 per square yard. What rate of discount did Miss Allen receive?

D3

The Ritchie High School football team played 6 home games this year. The average attendance per home game was 950 persons. What was the total attendance at the 6 home games?

D4

Mrs. Reynolds ordered $1\frac{3}{8}$ yd. of linen from a department store. When the linen was delivered, she discovered that she had received only 40 in. of cloth. How many inches of linen should she have received?

D5

H You can use the condition expressed at the right to solve the problem. For what is r a variable? $2.39 = 11.95r.$

I Explain how you can obtain $.2 = r$ from $2.39 = 11.95r$. What number satisfies $.2 = r$?

In $p = br$, r is a variable for a number that is the first component of a rate pair whose second component is 1. The rate of discount is usually expressed as a per cent. So multiply .2 by 100 to obtain the answer to the problem.

J What is the product of 100 and .2? What rate of discount did Miss Allen receive?

K Read the problem in D4. Why can you use the formula expressed at the right for the problem? $a = \frac{t}{n}.$

L You want to find the total attendance. Which variable in $a = \frac{t}{n}$ is a variable for this number?

M Which variable in $a = \frac{t}{n}$ refers to the average attendance per home game this year? Which variable refers to the number of home games?

N How can you obtain the condition $950 = \frac{t}{6}$ from the formula $a = \frac{t}{n}$? Tabulate $\{t \mid 950 = \frac{t}{6}\}$.

O What was the total attendance at the 6 home games played by the Ritchie High School football team?

Next you will use a conversion formula to help you solve a problem.

P Read the problem in D5. You are asked to find how many inches of linen Mrs. Reynolds should have received. Why can you use the formula expressed at the right for the problem? $l = kg.$

Q Which variable in $l = kg$ refers to the number of inches in $1\frac{3}{8}$ yd.? Which variable in $l = kg$ should you replace by $1\frac{3}{8}$?

R How many inches are in 1 yd.? By what number should you replace k in $l = kg$?

S How can you obtain $l = 36(1\frac{3}{8})$ from $l = kg$? Tabulate $\{l \mid l = 36(1\frac{3}{8})\}$.

T Mrs. Reynolds should have received how many inches of linen?

Next you will use formulas involving trigonometric rate pairs to help you solve problems. Read problem A in D6.

A Make a sketch of a right triangle ABC to represent the information given in problem A. Let $\angle A$ represent the angle of elevation. Let $\angle C$ represent the right angle suggested by the wall of the building and the ground. What is the measure of the side of $\triangle ABC$ that is oppo-

A A 33-foot ladder is placed against a building so that the vertical distance from the base of the building to the top of the ladder is 16.5 ft. What is the measure of the angle of elevation of the top of the ladder from the bottom?

B What is the distance from the bottom of the ladder to the base of the building? (Hundredths)

D6

site $\angle A$? What is the measure of the hypotenuse?

B Why can you use the formula expressed at the right for problem A?

$$\sin t = \frac{y}{r}.$$

C What number do you use as a replacement for y in $\sin t = \frac{y}{r}$? What number do you use as a replacement for r? What condition do you obtain? Find the quotient of 16.5 and 33.

D Use the table of tangents, sines, and cosines, which precedes the index, to find the measure of an angle whose sine is .5000. Then give the answer to problem A.

Read problem B in D6. You can use your sketch of $\triangle ABC$ and the answer for problem A to help you choose a formula that applies to problem B.

E You are asked to find the distance from the bottom of the ladder to the base of the building. Explain why you can use the formula expressed at the right for problem B.

$$\cos t = \frac{x}{r}.$$

F How can you obtain $\cos 30 = \frac{x}{33}$ from $\cos t = \frac{x}{r}$?

G Look up cos 30 in the table. How can you obtain $.8660 = \frac{x}{33}$ from $\cos 30 = \frac{x}{33}$?

H To find the x that satisfies $.8660 = \frac{x}{33}$, you must find the product of what two numbers?

I What is the product of 33 and .8660 to the nearer hundredth?

J Give the answer to problem B.

K Read problem B again. How can you use $\tan t = \frac{y}{x}$ to obtain a condition for the problem? Tabulate the solution set of $.5774 = \frac{16.5}{x}$

In this lesson you learned how to use formulas to help you solve problems. First you chose a formula that applied to a problem. Then you used the information given in the problem to make replacements for all but one of the variables in the formula. In this way, you obtained a condition whose solution could be used to answer the question in the problem.

On your own

For each problem, first choose a formula that applies to the problem. Next, use the information in the problem to make replacements in the formula and write a sentence that expresses the condition obtained. Then give the answer to the problem. The universe for each variable is D.

1 The Wallace Company bought 32 new typewriters of a certain model. The total cost was $4064. What was the price of each typewriter?

2 Distances on the ocean are measured in *nautical miles*. In Canada, one nautical mile is officially the same as a length of 6076.10 ft. A length of 10 nautical miles is the same as a length of how many feet?

3 A certain species of whale was observed swimming 10 nautical miles in $\frac{1}{3}$ hr. How many nautical miles per hour was it swimming?

4 Paula received a $3 discount on a sweater that was regularly priced at $15.95. What was the rate of discount? (Hundredths)

5 Last month Clyde worked a total of 24 days. He worked an average of $2\frac{1}{2}$ hr. per day last month. How many hours did Clyde work?

6 The area of the surface of Lake Superior is 80,000 sq. mi. The surface area of Lake Ontario is 35,000 sq. mi. The surface area of Lake Superior is what per cent of the surface area of Lake Ontario? (Hundredths)

7 Mr. Hall buys paintings for a store. Recently, he paid $3550 for 7 paintings. What was the average price per painting? (Ones)

8 Two monorails were built for the 1962 Seattle World's Fair. Each of the trains running on these monorails traveled 1.2 miles in 96 seconds. What part of an hour did each train take to travel 1.2 mi.?

9 From problem 8 you know what part of an hour the monorail took to travel 1.2 miles. Now find how many miles per hour the monorail traveled. (Ones)

10 The roadbed for an elevated train slopes upward from the ground to a station platform that is 30 ft. above the ground. The distance along the ground from the beginning of the sloping roadbed to a point directly below the platform is 990 ft. What is the measure in degrees of the angle of elevation of the platform?

KEEPING SKILFUL

For each exercise below, use a mixed numeral or a numeral for a basic fraction to express the sum, difference, product, or quotient.

1 $2\frac{3}{4} + 11\frac{7}{8}$

2 $^{-}\frac{2}{3} \times \frac{6}{7} \times \frac{21}{30}$

3 $\frac{5}{12} - \frac{1}{9}$

4 $^{-}3\frac{5}{8} \div \frac{1}{4}$

5 $^{-}7\frac{1}{2} \times 22\frac{2}{3}$

6 $^{-}\frac{5}{8} + ^{-}5\frac{3}{4} + ^{-}\frac{7}{16}$

7 $^{-}36 \div ^{-}\frac{3}{10}$

8 $\frac{19}{24} - ^{-}\frac{5}{6}$

9 $^{-}12\frac{3}{4} + ^{-}2\frac{9}{10}$

10 $4\frac{3}{8} \div 3\frac{1}{2}$

11 $^{-}18 - ^{-}6\frac{5}{6}$

12 $^{-}15\frac{3}{4} \times ^{-}3\frac{1}{3} \times 4\frac{1}{2}$

13 $1\frac{5}{16} + ^{-}1\frac{1}{2} + ^{-}\frac{3}{4}$

14 $^{-}56\frac{15}{16} - 24$

Business and science formulas

Besides the formulas you derived in lesson 174 and used in lesson 175, there are many other formulas that are used in solving science and business problems. In this lesson you will learn several new formulas for solving such problems. The universe for each variable is D.

Read problem A in D1 and study formula B. In the following exercises, you will first see how formula B can be derived.

A Look at sentence C in D2. What connective is used in the compound condition expressed by sentence C? Is each of the simple conditions a condition for equivalence?

In condition C, r is a variable for the *rate* of interest. The rate of interest is the first com-

D1

A Mr. Mack borrowed $1500 from a bank. The interest on the loan is $5\frac{1}{2}\%$ per year. Mr. Mack will repay this loan at the end of 3 years. How many dollars should he pay in interest?

B $i = prt$.

D2

Universe for each variable = D.

C $r/1 \sim x/p \wedge i/t \sim x/1$.

D $rp = x \wedge i = xt$.

E $i = prt$.

ponent of a rate pair that is equivalent to a per cent and that has a second component of 1. In problem A, the rate of interest is given as $5\frac{1}{2}\%$. However, the replacement for r is .055 since $5\frac{1}{2}\% \sim .055/1$. The rate of interest is given per year in most cases. x is a variable for the amount of interest for 1 year. p is a variable for the *principal*, which is the amount of money upon which the interest is paid. i is a variable for the amount of *interest* for the given period of time. t is a variable for the period of *time*.

B Look again at D2. How do you know that condition D is equivalent to condition C?

C Compare conditions D and E. How can you obtain condition E from condition D? Is condition E the same as formula B expressed in D1?

Formula B is called an *interest* formula. This formula can be used to solve many problems that involve interest.

D Now you can use the information given in problem A to make replacements for variables in formula B. Remember that you replace r by .055, not by 5.5. What replacement should you make for p?

E If you use the information in problem A to make replacements in $i = prt$, do you obtain the condition expressed in blue below?

$$i = (1500)(.055)(3).$$

F What number satisfies $i = (1500)(.055)(3)$?

Now you can give the answer to problem A in D1. Mr. Mack should pay $247.50 interest.

Read problem F in D3 and study formula G.

G In formula G, a is a variable for the *amount* Mr. Holland had in the savings account at the end of the year. For what is p a variable? For what is i a variable? The amount is equal to the sum of what two numbers?

H When you use the information given in problem F and make replacements for p and i in $a = p + i$, do you obtain the condition expressed in blue below?

$$a = 2800 + 113.12.$$

I What number satisfies $a = 2800 + 113.12$? Give the answer to problem F.

Read problem H in D4 and study formula I.

In formula I, s is a variable for the *selling price* of an item; c is a variable for the *cost* of the item to the merchant; and p is a variable for the *profit* the merchant makes on the item. o is a variable for the *overhead*, which is the merchant's operating expenses.

J You can use the information given in problem H to make replacements for which of the variables in formula I? Do you obtain the condition expressed in blue below?

$$s = 1.85 + .15 + .75.$$

K What number satisfies $s = 1.85 + .15 + .75$? Give the answer to problem H.

F Mr. Holland kept $2800 in a savings account for a year. During that year, his savings earned $113.12 in interest. What amount did he have in his savings account at the end of the year?

G $a = p + i$.

D3

H Mr. Rader plans to sell a new brand of stationery in his store. The stationery costs him $1.85 per box, and the overhead on each box is $.15. He wants to make a $.75 profit on each box of stationery. What should the selling price be per box?

I $s = c + o + p$.

D4

L Suppose that a merchant sells an item for exactly as much as the sum of the cost and the overhead. In $s = c + o + p$, what must be true of the replacement for p? Does the merchant make a profit?

M Suppose that a merchant sells an item for less than the sum of the cost and the overhead. In $s = c + o + p$, what must be true of the replacement for p? Does the merchant make any profit? In such a case, we say that the merchant has a *loss* on the item.

Next you will study several formulas that are useful in solving science problems.

Read problem J in D5 and study formula K. First you must understand some of the terms used in the problem. *Work* is the product of a *force* and the *distance* through which the force acts. The force and the distance must be measured in the same direction. If the force is given in pounds and the distance is given in feet, then the unit of measure for the work done is the *foot-pound*.

A In formula K, w is a variable for the work done; f is a variable for the force; and d is a variable for the distance. The work done is equal to the product of what 2 numbers?

B Now you can make replacements for f and d in formula K. A weight of 25 lb. exerts a downward force of 25 lb. Therefore, it takes a 25-lb. force to lift the weight. You should replace f by 25. By what number should you replace d? Do you obtain the condition expressed at the right?

$$w = 25 \times 15.$$

C What number satisfies $w = 25 \times 15$? Give the answer to problem J.

Read problem L in D5 and study formula M. *Power* is the rate at which work is done. If the unit of time is seconds, then the power is expressed in foot-pounds per second.

J How many foot-pounds of work are necessary to raise an object that weighs 25 lb. a distance of 15 ft.?

K $w = fd$.

L What is the power in foot-pounds per second of a machine that does 66,000 foot-pounds of work in 30 sec.?

M $p = \frac{w}{t}$.

D5

N How many amperes of current will flow through a resistance of 15 ohms if there is a pressure of 120 volts across the resistance?

O $I = \frac{E}{R}$.

P How many volts of pressure will produce 5500 watts of power with a current of 50 amperes?

Q $W = EI$.

D6

If the unit of time is minutes, then the power is expressed in foot-pounds per minute.

D In formula M, p is a variable for the power. For what are w and t variables? The power is equal to the quotient of what two numbers?

E How do you obtain the condition expressed at the right from formula M?

$$p = \frac{66000}{30}.$$

F What number satisfies $p = \frac{66000}{30}$? Give the answer to problem L.

G Suppose that problem L had asked you to find the power in foot-pounds per minute. What replacement would you make for t? What is the answer to the problem in foot-pounds per minute?

Next you will study two formulas that are used in solving problems about electricity. First, however, we will discuss some of the words that are used in talking about electricity. The *ampere* (am′pēr) is the unit of electrical current. The *volt* (vōlt) is the unit of electrical pressure. The *ohm* (ōm) is the unit of electrical resistance that must be overcome before a current can flow. 1 volt of pressure is required to cause 1 ampere of current to flow through a wire with a resistance of 1 ohm. The *watt* (wot) is the unit of electrical power.

Read problem N in D6 and study formula O.

H In formula O, I is a variable for the number of amperes of current; E is a variable for the number of volts of pressure; and R is a variable for the number of ohms of resistance. The number of amperes of current is equal to the quotient of what two numbers?

I How do you obtain the condition expressed at the right?

$$I = \frac{120}{15}.$$

J What number satisfies $I = \frac{120}{15}$? Give the answer to problem N.

Read problem P in D6 and study formula Q.

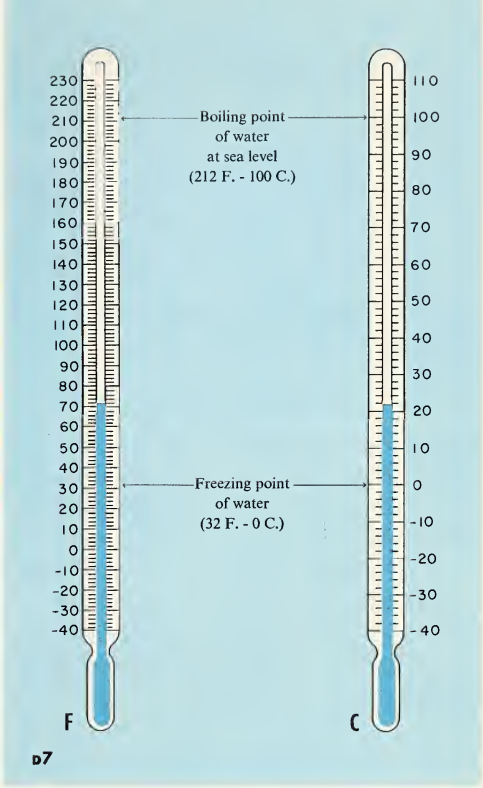
K In formula Q, W is a variable for the number of watts of power. For what is E a variable? For what is I a variable? The number of watts of power is equal to the product of what two numbers?

L Explain how the condition expressed in blue below was obtained. What number satisfies $5500 = E \times 50$?

$$5500 = E \times 50.$$

M Give the answer to problem P.

Next you will learn how temperatures expressed on the scale used in the metric system compare with temperatures expressed on the scale we ordinarily use. Look at D7.



The thermometer shown at the left is a *Fahrenheit* (far′ən hīt) thermometer, like the ones we ordinarily use. A *centigrade* (sen′tə-grād) thermometer is shown at the right. Centigrade thermometers are used in science and in countries that use the metric system.

N On the Fahrenheit scale, 32 degrees (32° F.) is the temperature at which water freezes. On the centigrade scale, 0 degrees (0° C.) is the temperature at which water freezes. What is the boiling temperature of water on each scale?

O What is the absolute difference of the freezing and boiling temperatures of water on the centigrade scale? On the Fahrenheit scale? Is the difference of these two temperatures on the centigrade scale equal to $\frac{5}{9}$ of the difference on the Fahrenheit scale?

Now read problem R in D8 and study formula S.

In formula S, C is a variable for the number of degrees expressed on the centigrade scale, and F is a variable for the number of degrees expressed on the Fahrenheit scale.

P Use formula S for problem R. Which of the variables should be replaced by 77? Can you use the condition expressed in blue below to solve the problem?

$$C = \tfrac{5}{9}(77 - 32).$$

Q What number satisfies $C = \tfrac{5}{9}(77 - 32)$?

You can show that the answer to problem R is given in terms of the centigrade scale by expressing the answer in the following way:

25° C. is the same temperature as 77° F.

Read problem T in D8. You can use formula S to develop a formula for problem T.

R How can you derive the formula expressed in blue below from formula S?

$$F = \tfrac{9}{5}C + 32.$$

S Using the information given in problem T, what replacement should you make for C in $F = \tfrac{9}{5}C + 32$? When you make this replacement, do you obtain the condition expressed in blue below?

$$F = \tfrac{9}{5}(50) + 32.$$

T What number satisfies $F = \tfrac{9}{5}(50) + 32$? Give the answer to problem T.

In this lesson you have seen how certain formulas can be used to help solve problems that concern business and science. These formulas are expressed in D9.

On your own

For each of problems 1 through 13, choose the appropriate formula from those expressed in D9. Then use the information given in the problem to make replacements in the formula. Find the number that satisfies the condition you obtain, and give the answer to the problem. The universe for each variable is D.

1 Mr. Scala sells a certain brand of lawn mower for $69.95. The lawn mower costs him $47.50, and his overhead for each mower is $9.70. What is his profit on a lawn mower?

2 One day in January, the temperature in Lively was $^{-}18°$ F. What temperature shown on the centigrade scale is the same as $^{-}18°$ F.?

3 How many pounds of force acting for a distance of 55 ft. will it take to do 880 foot-pounds of work?

4 Mrs. Carey kept $750 in a savings account for two years. During those two years her money earned interest which, from time to time, was added to the account. At the end of the two years, the amount in her account was

R What temperature is expressed on the centigrade scale when the temperature expressed on the Fahrenheit scale is 77°?

S $C = \tfrac{5}{9}(F - 32)$.

T What temperature is expressed on the Fahrenheit scale when the temperature expressed on the centigrade scale is 50°?

D8

Universe for each variable = D.

$i = prt.$

$a = p + i.$

$s = c + o + p.$

$w = fd.$

$W = EI.$

$p = \frac{w}{t}.$

$I = \frac{E}{R}.$

$C = \tfrac{5}{9}(F - 32).$

$F = \tfrac{9}{5}C + 32.$

D9

$811.83. How much interest did Mrs. Carey's money earn during the two years?

5 What resistance requires 850 volts of pressure to produce 275 amperes of current?

6 Mr. Weaver borrowed $6000 from a bank. At the end of 4 years, he paid the bank $6000 plus $1260 in interest. What was the rate of interest per year?

7 The power of a machine is 165,000 foot-pounds per minute. In how many minutes will the machine do 500,000 foot-pounds of work?

8 During a sale, a merchant sold a transistor radio for $14.79. He had paid $18.00 for the radio, and his overhead on the radio amounted to $4.50. What was the loss on the radio?

9 How many watts of power are produced by 120 volts of pressure and $\frac{1}{2}$ ampere of current?

10 How much interest must Mr. Ott pay in 1 year on a $500 loan if the rate of interest is $6\frac{1}{2}\%$?

11 How many volts of pressure are necessary to force 20 amperes of current through a resistance of 8.5 ohms?

12 Iron melts at 1535° C. What is this temperature expressed on the Fahrenheit scale?

13 What is the selling price of a toaster for which a merchant paid $14 if his overhead for the toaster is $1.95 and his profit is $5.15?

KEEPING SKILFUL

For each exercise below, use a decimal to express the sum, difference, product or quotient.

1 $15.063 - {}^{-}12.794$
2 ${}^{-}24 + 22.61$
3 $74.1 \div 39$
4 ${}^{-}.25 \times 65.3$
5 ${}^{-}28 - {}^{-}12.72$
6 ${}^{-}6.59 - 1.8$
7 $116.8 \div {}^{-}4.8$
8 $12.5 + {}^{-}7.9$
9 ${}^{-}1540 \div {}^{-}61.6$
10 ${}^{-}1.678 + {}^{-}6.65 + 3.82$
11 $72 - 90.9$
12 ${}^{-}410 \div 25$
13 $42.5 \times {}^{-}13.64 \times {}^{-}.1$
14 ${}^{-}149.12 + {}^{-}32.98$
15 ${}^{-}15 \times {}^{-}10.4 \times {}^{-}7$
16 ${}^{-}15.2 \times 4 \times {}^{-}3.2$

Perimeter and area formulas

In unit 8 and again in lesson 125, you learned to find areas and perimeters of several geometric figures. In this lesson, you will see how formulas can be used to find perimeters and areas. Think of all measures given in the problems as mathematical measures.

Read the problem in D1. The universe for each variable is D. Notice that the problem involves a *perimeter*, a *length*, and a *width*.

A Look at D2. In condition A, p is a variable for the perimeter of a rectangle; l is a variable for the length of the rectangle; and w is a variable for the width of the rectangle. Can you use condition A to solve the problem in D1?

B How can you obtain condition B from condition A? Condition C from condition B? Condition D from condition C?

The length of a rectangle is 18 ft. The width is 15 ft. What is the perimeter of the rectangle in feet?

D1

Universe for each variable = D.

A $p = l + w + l + w$.
B $p = l + l + w + w$.
C $p = (1 + 1)l + (1 + 1)w$.
D $p = 2l + 2w$.

D2

Condition D is a formula you can use to solve problems that involve the perimeter of a rectangle.

C If you use formula D for the problem in D1, why do you replace l by 18 and w by 15? Tabulate $\{p \mid p = 2(18) + 2(15)\}$. What is the perimeter of the rectangle in feet?

D If a problem gives you the perimeter and the length of a rectangle, how can you use $p = 2l + 2w$ to determine the width of the rectangle?

E What do you know about the measures of the sides of a rectangle that is a square?

F Look at D3. Use s as a variable for the measure of each side of a square. Why can you use $2s + 2s$ instead of $2l + 2w$, if you are finding the perimeter of a square?

G How can you obtain condition F from condition E? Condition G from condition F?

Condition G is a formula you can use to solve problems that involve the perimeter of a square.

H If a problem gives you the measure of the perimeter of a square, how can you use $p = 4s$ to determine the measure of each side of the square?

I You know that the perimeter of any polygon is the sum of the measures of the sides. Condition H, expressed in D4, is a formula for the perimeter of a triangle. For what is p a variable? For what are a, b, and c variables?

J If a triangle is equilateral, what do you know about the measures of its sides?

K Use s as a variable for the measure of each side of an equilateral triangle. Can you use condition I, expressed in D4, as a formula for the perimeter of an equilateral triangle?

The formulas you have derived so far concern perimeter. Now you will derive some formulas for problems that involve area.

Universe for each variable = D.

A $p = l + w + l + w$.

B $p = l + l + w + w$.

C $p = (1 + 1)l + (1 + 1)w$.

D $p = 2l + 2w$.

D2

Universe for each variable = D.

E $p = 2s + 2s$.

F $p = (2 + 2)s$.

G $p = 4s$.

D3

Universe for each variable = D.

H $p = a + b + c$.

I $p = 3s$.

D4

The measure of an altitude of a parallelogram is $1\frac{1}{2}$ ft. The measure of the base is 3 ft. What is the area of the parallelogram in square feet?

D5

Universe for each variable = D.

J $b/1 \sim A/a$.

K $A = ab$.

D6

Read the problem in D5. The universe for each variable is D. Notice that the problem involves a *base*, an *altitude*, and an *area*.

A In condition J, expressed in D6, A is a variable for the area of a parallelogram; b is a variable for the measure of the base; and a is a variable for the measure of an altitude. Can

you use condition J to solve the problem in D5? How can you obtain condition K from condition J?

In sentences J and K, read the capital letter that names the variable for the area as "cap A," to indicate that it is different from the letter that names the variable for the altitude.

Condition K is a formula for solving problems that involve the area of a parallelogram.

B Think again about the problem in D5. By what do you replace a in $A = ab$? By what do you replace b? Tabulate $\{A \mid A = (1\frac{1}{2})3\}$. Give the answer to the problem in D5.

C Suppose that you are given the width and the length of a rectangle. Can you think of the width as the measure of an altitude of the rectangle? Can you think of the length as a measure of the base? Explain why you can use $A = ab$ to find the area of the rectangle.

When $A = ab$ is used to find the area of a parallelogram that is a rectangle, the formula is sometimes expressed by the sentence $A = lw$.

D Suppose that you know the area and the width of a rectangle. How can you use $A = lw$ to determine the length?

E If a parallelogram is a square, what do you know about the measures of its base and altitude? Is $A = s^2$ a formula for finding the area of a square?

Read the problem in D7. The universe for each variable is D. Notice that the problem in D7 involves an *area*, a *base*, and an *altitude*.

REMINDER

If a parallelogram and a triangle have a common base and a common altitude, then the area of the parallelogram is 2 times the area of the triangle.

See lesson 108, page 55.

The measure of an altitude of a triangle is 2 in. The measure of the base of the triangle is 3.5 in. What is the area of the triangle in square inches?

D7

Universe for each variable = D.

L $b/1 \sim 2A/a$.

M $2A = ab$.

N $A = \frac{1}{2}ab$.

D8

F Look at D8. In condition L, A is a variable for the area of a triangle. For what is b a variable? For what is a a variable? How do you know that you can use condition L to solve the problem in D7?

G How can you obtain condition M from condition L?

H How can you obtain condition N from condition M?

Condition N is a formula that you can use to solve problems involving the area of a triangle.

I Think again about the problem in D7. By what numbers do you replace a and b in $A = \frac{1}{2}ab$?

J Tabulate $\{A \mid A = \frac{1}{2} \times 2 \times 3.5\}$.

K What is the area of the triangle in square inches?

You now have studied formulas for determining the area of a parallelogram, a rectangle, a square, and a triangle. Next you will study a formula for determining the area of a trapezoid. First you will learn more about the measures of segments associated with trapezoids.

A Look at D9. Name the opposite sides of quadrilateral ABCD.

B $\overleftrightarrow{BC} \cap \overleftrightarrow{AD} = \{\ \}$. Is $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$? Is segment BC parallel to segment AD? Explain your answer.

C $\overleftrightarrow{AB}$ is not parallel to $\overleftrightarrow{CD}$. Is $\overline{AB} \parallel \overline{CD}$?

D You know that a trapezoid is a quadrilateral in which two, and only two, sides are parallel. How do you know that quadrilateral ABCD is a trapezoid?

E Tabulate $\overline{GF} \cap \overline{BC}$. Tabulate $\overline{GF} \cap \overline{AD}$.

F Is one endpoint of $\overline{GF}$ contained in the line that includes side BC of the trapezoid? Is the other endpoint of $\overline{GF}$ contained in the line that includes the side of the trapezoid that is parallel to side BC?

G Is $\overline{GF}$ perpendicular to both of the lines that include the parallel sides of the trapezoid?

$\overline{GF}$ is an *altitude* of trapezoid ABCD. An altitude of a trapezoid is a segment that is perpendicular to both of the lines that include the parallel sides of the trapezoid. The endpoints of the altitude are contained in the lines that include the parallel sides. Each of the parallel sides is a *base* of the trapezoid.

H How do you know that $\overline{BE}$ is an altitude of trapezoid ABCD? How do you know that $\overline{HD}$ is also an altitude?

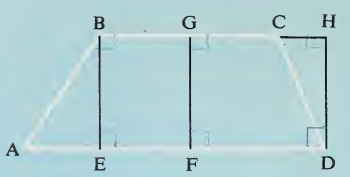
I ? Explain why $\overline{AB}$ is not an altitude of the trapezoid. Explain why $\overline{CD}$ is not an altitude.

J Look again at D9. Polygon EBGF is a parallelogram. How do you know that polygon EBGF is a rectangle?

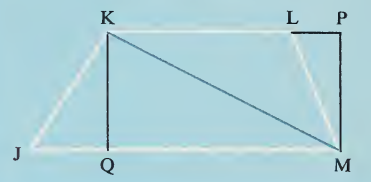
K Is $m(\overline{BE}) = m(\overline{GF})$? How do you know?

L Does each of the altitudes of a given trapezoid have the same measure?

Now you are prepared to find the area of a trapezoid. To do this, you will use what you know about the area of a triangle.



D9



D10

altitude of a trapezoid. A segment that is perpendicular to the lines that include the parallel sides of the trapezoid. The endpoints of the segment are contained in the lines that include the parallel sides. Each of the parallel sides is a **base** of the trapezoid.

Look at D10. Quadrilateral JKLM is a trapezoid.

A Name the pairs of opposite angles of trapezoid JKLM. Name the pairs of opposite vertices of the trapezoid.

REMINDER

Opposite angles of a quadrilateral are two angles of the quadrilateral that intersect in exactly two points. Opposite vertices of a quadrilateral are the vertices of two opposite angles.

See lesson 106, page 42.

B What segment is determined by points K and M?

$\overline{KM}$ is a *diagonal* of trapezoid JKLM. A diagonal of a trapezoid is a segment deter-

diagonal of a trapezoid. A segment determined by two opposite vertices of the trapezoid.

mined by two opposite vertices of the trapezoid.

C Look again at D10. $\triangle JKM$ is the union of what three segments? $\triangle KLM$ is the union of what three segments?

Notice that $\triangle JKM$ is the union of two sides of the trapezoid and diagonal KM. $\triangle KLM$ is the union of the two other sides of the trapezoid and diagonal KM. We say, therefore, that a trapezoid and one of its diagonals form two triangles.

D What closed region is the union of the closed regions determined by $\triangle JKM$ and $\triangle KLM$? Is the area of trapezoid JKLM the sum of the areas of triangle JKM and triangle KLM?

Exercise D involves a property concerning the area of a trapezoid. The property is expressed below.

The area of a trapezoid is equal to the sum of the areas of the two triangles formed by the trapezoid and one of its diagonals.

E $\overline{KQ}$ is perpendicular to $\overleftrightarrow{KL}$ and $\overleftrightarrow{JM}$. Is $\overline{KQ}$ an altitude of trapezoid JKLM? Is $\overline{KQ}$ also an altitude of $\triangle JKM$?

F Which base of the trapezoid is also the base of $\triangle JKM$?

G $\overline{PM}$ is perpendicular to $\overleftrightarrow{KL}$ and $\overleftrightarrow{JM}$. Is $\overline{PM}$ an altitude of trapezoid JKLM? Is $\overline{PM}$ also an altitude of $\triangle KLM$?

H Which base of the trapezoid is also the base of $\triangle KLM$?

Notice that an altitude and one of the bases of trapezoid JKLM form an altitude and the base of $\triangle JKM$. Another altitude and the second base of the trapezoid form an altitude and the base of $\triangle KLM$.

Universe for each variable = D.

O $A = \frac{1}{2}ab_1 + \frac{1}{2}ab_2$.

P $A = \frac{1}{2}a(b_1 + b_2)$.

D11

I Is $m(\overline{KQ}) = m(\overline{PM})$? Is the measure of the altitude of $\triangle JKM$ the same as the measure of the altitude of $\triangle KLM$?

Now you can use what you know about a formula for the area of a triangle to derive a formula for the area of a trapezoid. You can use b_1 to represent the measure of base JM of $\triangle JKM$. Use b_2 to represent the measure of base KL of $\triangle KLM$. The universe for each variable is D. Remember that $\overline{JM}$ and $\overline{KL}$ are also the two bases of trapezoid JKLM.

J Use a to represent the measure of altitude KQ of $\triangle JKM$. Why does a also represent the measure of altitude PM of $\triangle KLM$?

K How do you know that $\frac{1}{2}ab_1$ represents the area of $\triangle JKM$? How do you know that $\frac{1}{2}ab_2$ represents the area of $\triangle KLM$?

L Look at sentence O in D11. You know that the area of trapezoid JKLM is the sum of the areas of $\triangle JKM$ and $\triangle KLM$. If A represents the area of the trapezoid, does sentence O express a condition for finding the area of trapezoid JKLM?

M What property of real numbers can you use to obtain condition P from condition O? In condition P, does a represent the measure of an altitude of the trapezoid? b_1 represents the measure of which side of the trapezoid? b_2 represents the measure of which side?

Condition P is a formula you can use to solve problems involving the area of a trapezoid.

N Suppose that you want to find the area of a trapezoid in square inches. The measure of an

Universe for each variable = D.

$p = 2l + 2w.$ $A = ab.$

$p = 4s.$ $A = lw.$

$p = a + b + c.$ $A = s^2.$

$p = 3s.$ $A = \frac{1}{2}ab.$

$A = \frac{1}{2}a(b_1 + b_2).$

D12

altitude of the trapezoid is 3 in. The measures of the bases of the trapezoid are 5 in. and 7 in. What replacements for a, b_1, and b_2 should you make in $A = \frac{1}{2}a(b_1 + b_2)$?

o Is $A = \frac{1}{2}(3) \cdot (5 + 7)$ the condition you will use to find the area of the trapezoid? What is the area of the trapezoid in square inches?

In this lesson you have derived the formulas expressed in D12.

On your own

For each of the following problems, select an appropriate formula from those expressed in D12. Next, use the information given in the problem to make replacements and write a sentence that expresses the condition that you obtain. Then tabulate the solution set of the condition and give the answer to the problem. Think of the measures given in the problems as mathematical measures. The universe for each variable is D.

1 The measures of the sides of a triangle are 3 mm., 7 mm., and 9 mm. What is the perimeter of the triangle in millimetres?

2 The area of a rectangle is 24 sq. ft. The width is 5 ft. What is the length of the rectangle in feet?

3 The measure of a side of a square is $9\frac{1}{2}$ ft. What is the perimeter of the square in feet?

4 The measure of a side of an equilateral triangle is 5.2 cm. What is the perimeter of the triangle in centimetres?

5 The measure of an altitude of a parallelogram is 4.7 yd. The measure of the base is 6.2 yd. What is the area of the parallelogram in square yards?

6 The length of a rectangle is 32 cm. The width is 20.5 cm. What is the perimeter of the rectangle in centimetres?

7 The measure of a side of a square is $3\frac{1}{4}$ in. What is the area of the square in square inches?

8 The measure of an altitude of a triangle is 5.3 yd. The measure of the base is 6.2 yd. What is the area of the triangle in square yards?

9 A parallelogram has an area of 6.3 square metres. The measure of an altitude is .7 metre. What is the measure of the base of the parallelogram in metres?

10 A rectangle has a perimeter of 39 cm. The length of the rectangle is 14 cm. What is the width of the rectangle in centimetres?

11 The measure of an altitude of a trapezoid is $1\frac{1}{2}$ ft. The measures of the bases are 3 ft. and $5\frac{1}{2}$ ft. What is the area of the trapezoid in square feet?

12 The measure of an altitude of a triangle is $1\frac{1}{2}$ m. The measure of the base is $\frac{7}{10}$ m. What is the area of the triangle in square metres?

13 The perimeter of a square is 13.6 mm. What is the measure of each side of the square in millimetres?

14 The measures of the bases of a trapezoid are 2.2 ft. and 4.3 ft. The area of the trapezoid is 6.5 sq. ft. What is the measure of an altitude of the trapezoid in feet?

15 The perimeter of a square is 35 ft. What is the area of the square in square feet?

16 Each of two sides of a triangle has a measure of 5 ft. An altitude that is perpendicular to the third side has a measure of 4 ft. The perimeter of the triangle is 16 ft. What is the area of the triangle in square feet?

APPLYING MATHEMATICS

For each problem, first choose the appropriate formula from the table of formulas that begins on page 438. Next, use the information given in the problem to make replacements. Then write a sentence that expresses the condition and give the answer to the problem. Think of the given measures as mathematical measures. The universe for each variable is D.

1 At the Modern Furniture Company, the cost of the furniture sold last month was \$10,247. The furniture was sold for \$14,703. The overhead was \$1824.35. What was the profit on the furniture sold last month?

2 The power of a certain machine is 75,000 foot-pounds per second. The machine can do how many foot-pounds of work in 30 seconds?

3 The Hall City National Bank charges $5\frac{1}{2}\%$ interest per year on loans. Mr. Gray borrowed \$2400 for 2 years. How much interest should Mr. Gray have paid the bank for the use of the \$2400?

4 Calcium melts at 810° C. At what temperature on the Fahrenheit scale does calcium melt?

5 Mr. Davis drove his car 250.2 mi. in 4.5 hr. At what average speed was he traveling?

6 The Garcia family uses an average of $\frac{3}{4}$ gal. of milk per day. How many gallons of milk does the Garcia family use in 30 days?

7 The resistance of a certain wire is 9.6 ohms. How many volts of pressure are required so that 10.5 amperes of current will flow through the wire?

8 Mr. Walters had \$4500 in his savings account for 1 year. He made no deposits or withdrawals during the year. During the year, \$146.25 interest was added to his account. How much did he have in his savings account at the end of the year?

9 In one year, 6,049,000 persons were employed in Canada. About 11.1% of these persons were employed in agriculture. How many persons were employed in agriculture?

10 Marvin used $34\frac{1}{2}$ ft. of chicken wire to enclose a rectangular garden. The length of the garden is $10\frac{1}{2}$ ft. What is the width of the garden in feet?

11 A chute extends from the second floor of a building to the basement. The measure in degrees of the angle of elevation of the top of the chute from the bottom is 77. The distance along the basement floor from the bottom of the chute to a point directly below the top of the chute is 5.85 ft. What is the length of the chute in feet?

12 Mr. Allen sold a rectangular piece of land that is 295.5 ft. long and 1572 ft. wide. How many square feet of land did he sell?

13 The current flowing through an electrical heating coil is 2.0×10^1 amperes. The power produced is 1.86×10^5 watts. What is the electrical pressure across the coil in volts?

14 Lynn lifted a 15-lb. bag of groceries a distance of 4 ft. How many foot-pounds of work did she do?

15 Mr. Holt sells a certain make of jacket for \$12.95. Each jacket costs him \$8.75, and the overhead on each jacket is \$.98. What is his profit on each jacket that he sells?

16 Miss Sprague's desk is 69 in. long. Her desk is how many feet long?

17 Alice is making a quilt by sewing together triangular pieces of cloth. The length of each side of each piece is $4\frac{1}{4}$ in. What is the perimeter of each piece in inches?

18 A certain airplane can fly 625 mi. per hour. At this rate, how many hours will it take the airplane to fly 1000 mi.?

19 Miss Newcomb borrowed \$1500 for a period of 3 years. At the end of that time, she

repaid the loan and paid $270 interest. What rate of interest did Miss Newcomb pay?

20 Mr. Schwartz set the thermostat in his living room so that the room temperature would be 68° F. What is a temperature of 68° F. on the centigrade scale?

21 Last year Mr. Willis saved 5.9% of his salary. He saved $395.30. What was his salary last year?

22 The floor of a porch is in the shape of a trapezoid. The length of an altitude of the floor is $12\frac{1}{2}$ ft. The lengths of the bases are 24 ft. and $36\frac{1}{2}$ ft. What is the area of the floor in square feet?

23 Point S on the ground is 155 ft. from the base of a 140-ft. building. What is the measure in degrees of the angle of elevation of the top of the building from point S?

24 Sam used a square piece of leather to cover the top of a table. The length of a side of the piece of leather is $39\frac{1}{2}$ in. What is the perimeter of the piece of leather in inches?

25 An appliance dealer bought 12 electric clocks. The total cost of the clocks was $91.80. Each clock was the same price. What was the price per clock?

26 The portion of the Central School playground that is reserved for baseball is in the shape of a square. The length of a side of this square is 150 ft. How many square feet of the playground are reserved for baseball?

27 A 28-foot ladder rests against a wall so that the top of the ladder is 24.5 ft. from the base of the wall. What is the measure in degrees of the angle of elevation of the top of the ladder from the bottom of the ladder?

28 A machine cuts strips of tin into triangular pieces. The length of an altitude of a piece is 20.4 cm. The length of the base is 30.2 cm. What is the area of the piece of tin in square centimetres?

CHECKING UP

The small numerals within parentheses tell what pages to turn to for help if you need it.

Test 158

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 The solution set of $y = {}^{-}3x$ is a proportional relation. The universe for (x, y) is $D \times D$. (348)

2 $\{(x, y) | y = \frac{1}{2}x^2\}$ is a finite set. $U = D \times D$. (351)

3 In $y = {}^{-}\frac{6}{11}x$, the variable y varies inversely as x. The universe for (x, y) is $D \times D$. (348)

4 The locus of the solution set of a condition is a geometric figure. (349)

5 $({}^{-}3, 63)$ is a solution of $y = 7x^2$. The universe for (x, y) is $D \times D$. (351)

6 In $y = \frac{8}{x^2}$, the variable y varies directly as x^2. The universe for (x, y) is $D \times D$. (358)

7 $\sqrt{(\frac{9}{16})^2} = \frac{3}{4}$. (368)

8 $\sqrt{25} = \sqrt{\frac{625}{25}}$. (368)

9 In $\cos t = \frac{x}{r}$, the variable r is a variable for the measure of the hypotenuse. (375)

10 A merchant's profit is included in his overhead. (381)

Test 159

From the list given below each exercise, choose the word, words, or symbols that correctly complete the exercise.

11 A solution of $y = {}^{-}\frac{2}{3}x$ is ~~~. The universe for (x, y) is $D \times D$. (347)

a $({}^{-}\frac{2}{3}, 1)$ **b** $(1, \frac{2}{3})$ **c** $(2, 3)$ **d** $({}^{-}3, 2)$

12 Each member of $\{(x, y) | y = {}^{-}4x\}$ can be represented by ~~~. $U = D \times D$. (348)

a $({}^{-}4x, x)$ **b** $(x, {}^{-}4x)$ **c** $(\frac{1}{4}x, x)$ **d** $({}^{-}x, \frac{1}{4}x)$

13 The locus of $\{(x, y) | y = 21x^2\}$ is a ~~~. $U = D \times D$. (352)

a line **b** ray **c** parabola **d** hyperbola

14 In ~~~, y varies inversely as x. The universe for (x, y) is $D \times D$. (355)

a $y = 5x$ **c** $y = \frac{5}{x}$

b $y = 5x^2$ **d** $y = \frac{5}{x^2}$

15 The locus of $\{(x, y) | y = \frac{6}{x^2}\}$ is ~~~. $U = D \times D$. (359)

a a closed curve **c** a four-part curve

b a two-part curve **d** the empty set

16 For each x, ~~~. $U = D$. (367)

a $x^2 \geqq 0$ **b** $x^2 > 0$ **c** $x^2 \leqq 0$ **d** $x^2 < 0$

17 $12.435^2 < 12.\overline{4}^2$ is true because ~~~. (367)

a $12.435 > 12.\overline{4}$ **c** $12.4 < 12.435$

b $.43 > .\overline{4}$ **d** $12.435 < 12.\overline{4}$

18 To find the total cost of several items that sell for the same amount of money, you should use the formula ~~~. (373)

a $d = rt$ **b** $c = np$ **c** $p = br$ **d** $l = kg$

19 If a problem involves 40%, the replacement for r in $p = br$ is obtained by dividing 40 by ~~~. (373)

a $\frac{1}{10}$ **b** $\frac{1}{100}$ **c** 10 **d** 100

20 The formula ~~~ is used to find the area of a trapezoid. (389)

a $A = \frac{1}{2}ab$ **c** $A = \frac{1}{2}a(b_1 + b_2)$

b $A = s^2$ **d** $A = p + i$

Test 160

Tell what word, words, or symbols best complete each of the following exercises.

21 The second component of (6, 27) is the product of ~~~ and the first component. (347)

22 In $y = 2\frac{1}{2}x^2$, the variable y varies directly as ~~~ varies. The universe for (x, y) is $D \times D$. (351)

23 In $y = 8x$, 8 is the constant of ~~~. The universe for (x, y) is $D \times D$. (348)

24 The geometric figure that is the locus of $\{(x, y) | y = \frac{9}{x}\}$ is a ~~~. $U = D \times D$. (356)

25 If the first component of a solution of $y = \frac{3}{x^2}$ is 3, then the second component is ~~~. The universe for (x, y) is $D \times D$. (358)

26 The condition $d = rt$ is called the ~~~ formula. (372)

27 The condition $p = br$ is called a ~~~ formula. (373)

28 If you use $l = kg$ to find the number of feet that is the same length as 14 yards, you should replace k by ~~~. (374)

29 The condition $a = \frac{t}{n}$ is called the ~~~ formula. (374)

30 A trapezoid has exactly ~~~ diagonals. (389)

Test 161

For each problem, first write a sentence that expresses a condition. Then give the answer to the problem. The universe for each variable is D.

31 The width of a rectangle is $8\frac{1}{2}$ ft., and the length is $12\frac{1}{2}$ ft. What is the perimeter of the rectangle in feet? (386)

32 The perimeter of a square is 6 in. What is the length of a side in inches? (386)

33 The length of each side of a triangle is 17 m. What is the perimeter in metres? (386)

34 The length of an altitude of a parallelogram is 8 yd., and the area is 12 sq. yd. What is the length of the base in yards? (386)

35 The length of an altitude of a triangle is 4 in., and the area is 32 sq. in. What is the length of the base in inches? (387)

36 The length of an altitude of a rhombus is 5 ft., and the length of the base is 8.6 ft. What is the area of the rhombus in square feet? (387)

37 The length of an altitude of a trapezoid is 4 mi. The lengths of the bases are 8.9 mi. and 6.7 mi. What is the area of the trapezoid in square miles? (389)

CHECKING UP

The small numerals within parentheses tell what pages to turn to for help if you need it.

Test 162

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

1 The set of integers is the union of the set of positive integers and the set of negative integers. (168)

2 The set of rational numbers, together with the operations of multiplication and addition and their properties, forms a number system. (198)

3 A condition that is equivalent to $\sim(p > .5)$ is $p + .3 \leqq .8$. U = R. (238)

4 Each element of the set of non-zero rational numbers has a reciprocal. (211)

5 Any rational number can be expressed by a repeating decimal. (221)

6 $(\frac{17}{23})^2 = (\bar{}\frac{17}{23})^2$ is a true statement. (254)

7 An inch is longer than a centimetre. (306)

8 The two acute angles of any right triangle are complementary angles. (330)

9 The rational number $^+4\frac{2}{3}$ may be indicated by $(7\frac{2}{3}, 3)$. (156)

10 $\sqrt{32} = \sqrt{\frac{96}{3}}$ is a true statement. (368)

Test 163

The names of seven properties of the real numbers are listed below.

Closure property of addition
Commutative property of addition
Associative property of addition
Additive-inverse property
Identity-element property of addition
Well-defined property of addition
Distributive property of multiplication over addition

In each of exercises 11 through 15, a property is expressed. For each exercise, select the name of the property from the list. The universe for each variable is D.

11 $0 + x = x$. (266)

12 $m + n = n + m$. (266)

13 $s + \bar{}s = 0$. (266)

14 $(p + q) + r = p + (q + r)$. (266)

15 If $a = b$ and $c = d$, then $a + c = b + d$. (268)

Test 164

From the list given below each exercise, choose the symbol, word, or words that correctly completes the exercise.

16 The intersection of ⊙T and a chord of ⊙T is a set that contains exactly ~~~. (115)

a zero points **c** two points
b one point **d** three points

17 The measure in degrees of a semicircle is ~~~. (120)

a 90 **b** 180 **c** 270 **d** 360

18 Any equilateral triangle is also ~~~ triangle. (126)

a a scalene **c** an obtuse
b an equiangular **d** a right

19 $(\bar{}\frac{2}{11})^9 \div (\bar{}\frac{2}{11})^3 =$ ~~~. (226)

a $(\bar{}\frac{2}{11})^6$ **b** $(\bar{}\frac{2}{11})^3$ **c** $(\bar{}\frac{2}{11})^{-3}$ **d** $(\bar{}\frac{2}{11})^{12}$

20 An example of a set that is closed under subtraction is the set of ~~~. (267)

a negative rational numbers
b positive rational numbers
c positive integers
d real numbers

21 If $\sqrt{2} < \sqrt{5}$, then $\sqrt{2}(\sqrt{8})$ ~~~. (271)

a $< \sqrt{5}$ **c** $> \sqrt{5}(\sqrt{8})$
b $< \sqrt{8}$ **d** $< \sqrt{5}(\sqrt{8})$

22 One metre is the same length as ~~~. (304)

a $\frac{1}{100}$ cm. **b** 100 cm. **c** $\frac{1}{1000}$ cm. **d** 1000 cm.

23 The quotient of the physical measures 14 ft. and .3 ft. should be rounded to the nearer ~~~. (321)

a thousandth **c** ten
b hundredth **d** one

24 In $\triangle ABC$, $\angle B^\circ = 90$. If $m(\overline{BC}) = 3$ and $m(\overline{AB}) = 12$, then $\tan \angle C =$ ~~~~. (332)

a 36 **b** 4 **c** .25 **d** 12^3

25 In triangle DEF, $\angle D^\circ = 90$. $m(\overline{EF}) = 13$. $m(\overline{ED}) = 12$. $m(\overline{DF}) = 5$. ~~~~ ~ 5/13. (332)

a $\sin \angle F/1$ **c** $\sin \angle E/1$

b $\cos \angle E/1$ **d** $\tan \angle F/1$

Test 165a

For each problem, first write a sentence that expresses a condition. Tabulate the solution set of the condition, if possible. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The universe for each variable is D.

26 In 1952, Mr. Meyer's salary was \$4500. In 1960, his salary was greater, but it was less than \$8200. What was the increase in Mr. Meyer's salary in the eight-year period? (294)

27 Together, Ann and Fred spent \$24.60 shopping. Ann spent twice as much as Fred. How much did each of them spend? (292)

28 Mr. Wells had fewer than the 48 concrete blocks he needed for a patio. After he bought a load of 32 blocks, he had more than 52. How many blocks did he have before he bought the load of 32 blocks? (295)

29 The measure of a side of a square is $1\frac{1}{3}$ ft. What is its area in square inches? (387)

30 The difference of two numbers is $^-4$. The minuend is the sum of $\frac{2}{3}$ and $\frac{1}{2}$. What are the numbers? (296)

31 A certain rate pair is equivalent to 27/9. The sum of the components is 16. What is the rate pair? (296)

32 The product of two numbers is 16. The first number is equal to the sum of $2\frac{3}{4}$ and $2\frac{7}{12}$. What are the numbers? (296)

33 The product of three numbers is $40\frac{1}{3}$. The first number is equal to the product of 3 and the second number. The second number is $22 \div 6$. What are the three numbers? (296)

Test 165b

What words or symbols best complete exercises 34 through 46?

34 If the sum of the measures in degrees of two angles is ~~~~, then the angles are supplementary. (26)

35 The vertices of two triangles can be put in one-to-one correspondence in ~~~~ ways. (38)

36 In a parallelogram, it is not always true that opposite ~~~~. (44)

a sides are congruent

b angles are supplementary

c sides are parallel

d angles are congruent

37 Each rational number that is greater than 1 is a solution of ~~~~. (64)

a $x + \frac{3}{5} > \frac{2}{5}$ **c** $x - \frac{2}{5} > \frac{3}{5}$

b $\frac{3}{5} - x > \frac{2}{5}$ **d** $\frac{2}{5} - x > \frac{3}{5}$

38 The ordered pair (4, 7) is an (x, y) that satisfies ~~~~. (77)

a $y > x \wedge x < y - 3$

b $x + y < 15 \wedge y = x$

c $x + 3 > y \vee x + y < 8$

d $x + y > 12 \vee y = x + 3$

39 Name the starting point and the terminal point of $\overrightarrow{SK}$. (154)

40 What is the non-zero component of the basic pair for $^+4.75$? (174)

41 The number ~~~~ is a perfect square. (258)

a 721 **b** 576 **c** 664 **d** 825

42 The real number ~~~~ satisfies $4x = 5(2x - 3)$. (278)

43 What unit of length was used to obtain a physical measure of 2.67 ft.? (311)

44 For any two triangles, if the corresponding angles are congruent, then the two triangles are ~~~~. (326)

45 In $y = 5x^2$, y varies ~~~~ as x^2 varies. (351)

46 If a jet plane flies at the rate of 575 mi. per hour, it will travel ~~~~ miles in 4 hours. (376)

Test 165c

What words or symbols best complete exercises 47 through 58?

47 Lines that are included in the same plane are ~~~~ lines. (24)

48 Tabulate the set of vertices of $\triangle$ LMN. (37)

49 In $\triangle$YCL, angle ~~~~ is opposite CY. (40)

50 In quadrilateral MGRS, side ~~~~ is opposite side GR. (42)

51 Make a Venn diagram to show that sets M and R have some common members but neither is a subset of the other. (63)

52 The solution set of $x \geq 4$ is the ~~~~ of the solution sets of $x > 4$ and $x = 4$. (71)

53 The reciprocal of $2\frac{3}{7}$ is ~~~~. (150)

54 Each point whose coördinate is a positive rational number is located ~~~~ of point 0. (166)

55 The square of $\frac{2}{7}$ is ~~~~. (253)

56 A ~~~~ is a unit of length that is the same as 10^{-2} metre. (304)

a kilometre **c** millimetre
b decametre **d** centimetre

57 The rate pair ~~~~ is a member of the same proportional relation as $\frac{3}{4}$. (325)

a $\frac{2}{6}/1\frac{1}{4}$ **b** $\frac{1}{2}/\frac{2}{3}$ **c** $\frac{1}{3}/\frac{4}{3}$ **d** $1\frac{1}{3}/2\frac{1}{4}$

58 The set of points that are associated with the elements of the solution set of a condition is the ~~~~ of the solution set of the condition. (348)

Test 165d

What words or symbols best complete the following exercises?

59 $\angle a$ and $\angle b$ are supplementary angles. What is the sum of their measures? (27)

60 Name the sides of $\triangle$JKL. (38)

61 The symbol ~~~~ is a name for the set of rational numbers of arithmetic. (64)

62 The solution set of ~~~~ contains 4.3. (89)

a $x \approx {}_{.1}4.2$ **c** $x \approx {}_{.5}5$
b $x \approx {}_{.2}4$ **d** $x \approx {}_{1}3$

63 A standard name of ~~~~ is 3.2. (153)

a $\{(x, y) \mid (x, y) \sim (5.4, 8.6)\}$
b $\{(x, y) \mid (x, y) \sim (4.1, .9)\}$
c $\{(x, y) \mid (x, y) \sim (2.1, 1.1)\}$
d $\{(x, y) \mid (x, y) \sim (1.2, 3)\}$

64 If one or both of the numbers used in forming the product of two rational numbers are replaced by equal rational numbers, the product is still the same rational number. This property is the ~~~~ property of multiplication of rational numbers. (185)

a commutative **c** transitive
b associative **d** well-defined

65 Point $(^{-}3, 5)$ is in the ~~~~ quadrant. (285)

a first **c** third
b second **d** fourth

66 What is the repetend of $52.7\overline{8584}\ldots$? (261)

67 Give a standard description of the solution set of $5 - x > {}^{-}1$. (281)

68 Tabulate the solution set of $(x + y - 2z = 0) \wedge (x - 3z = 2) \wedge (2z + 1 = 3)$. (290)

69 There are ~~~~ significant digits in the numeral .0005872. (316)

70 The cosine of an acute angle of a right triangle is first component of a rate pair that has a second component of 1 and that is equivalent to the rate pair representing the comparison of the measure of the side ~~~~ to the angle with the measure of the hypotenuse. (332)

71 Any two angles that have a common vertex and a common side and whose interiors are disjoint sets are ~~~~ angles. (28)

a adjacent **c** opposite
b vertical **d** supplementary

72 $\triangle DGK \cong \triangle MTR$. What side of $\triangle MTR$ corresponds to side KD of $\triangle DGK$? (39)

73 The ordered pair ~~~~ is the basic ordered pair that is equivalent to $(1\frac{1}{2}, 5\frac{3}{4})$. (165)

a $(1\frac{1}{2}, 0)$ **c** $(0, 4\frac{1}{4})$
b $(0, 7\frac{1}{4})$ **d** $(1, 5\frac{1}{4})$

74 The numeral ~~~~ is scientific notation for the quotient $32 \div 400{,}000$. (234)

Unit 14

Probability and statistics

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Events and their probabilities

In this unit you will consider events that are called chance events. When someone says, "It will probably rain at the faculty picnic," he is making a remark about a chance event. In this lesson you will learn what is meant by a "measure of chance" of an event.

A ? Suppose that you are looking for a particular book and there are two new copies of it on a shelf in the library. Does it seem more likely that you would choose one copy rather than the other?

B ? Suppose that there are five pieces of apple pie of the same size on the dessert counter in a cafeteria. Do you think that the chance of someone's choosing one of the five pieces of pie is the same for each piece?

C ? Suppose that a bag of candy contains twice as many pieces of lemon-flavored candy as of mint-flavored candy. The bag is shaken after the candy is put in it. If you take a piece of candy from the bag without looking, does it seem more likely that you will take a piece of lemon candy rather than a piece of mint candy?

Your answers to exercises A, B, and C are based upon past experiences and common sense. However, we could have asked you questions about chance events to which the answers would not be quite so obvious. Sometimes, when the answer to a question about a chance event is not obvious, you can decide upon the answer by making a careful study of an *experiment* (eks per′ə mənt). To explain

Without looking, Jim is to take a marble from a box that contains 1 blue marble and 1 white marble. What is the chance that he will take the blue marble?

D1

Hal and Tony are playing a game that requires them to use a spinner like the one shown in the sketch below. At each turn, one of the boys spins the pointer. What is the chance that the pointer will stop on the numeral 4?

D2

what we mean by an experiment, we will begin with some simple examples.

Read the problem in D1. Jim's taking a marble from the box is the experiment. Taking the blue marble is one possible *outcome* of the experiment.

D What is the other possible outcome of the experiment? Is each outcome a chance event?

E Do you think that the two outcomes of the experiment are equally likely to occur?

Since the two outcomes are equally likely, we say that each outcome has 1 chance in 2 of occurring.

F Think about the outcome of taking the blue marble from the box. How many chances in 2 does it have of occurring?

Now read the experiment described in D2.

G You are to assume that the pointer always stops on one of the numerals, but never on the marks between the numerals. You are also to assume that the regions containing the numerals are the same size. What are the 6 possible outcomes of the experiment? Is each of these outcomes a chance event?

H Assume that the 6 outcomes are equally likely to occur. How many chances in 6 does each numeral on the spinner have of occurring?

I Does the numeral 4 have 1 chance in 6 of occurring, more than 1 chance in 6, or less than 1 chance in 6?

In the exercises that follow, you will learn some of the language that the mathematician uses to describe chance events. You will use the experiments that have already been described.

Read the experiment in D1 again. You will remember that you decided there is 1 chance in 2 that Jim will take the blue marble. You can use this information to determine the *probability* of Jim's taking the blue marble. You can think of the probability of an outcome as a "measure of chance." You will use a condition to determine the probability.

A Read sentence A in D3. The rate pair 1/2 represents the rate of blue marbles taken per 2 chances. The rate pair $P/1$ represents the rate of blue marbles taken per 1 chance. What does condition A require about these two rate pairs? For what is P a variable?

B How can you obtain condition B from condition A? How can you obtain condition C from condition B? What number satisfies condition C? Condition A?

U = D.

A $1/2 \sim P/1$.

B $2P = 1$.

C $P = \frac{1}{2}$.

D3

"The probability of taking the blue marble is equal to one half."

P (blue) $= \frac{1}{2}$.

D4

This morning Mrs. Andrews absent-mindedly put some hard-boiled eggs along with raw eggs in the egg shelf of her refrigerator. There are 10 eggs in the shelf, and 3 of them are hard-boiled. What is the probability that Mrs. Andrews will pick up a hard-boiled egg when she takes an egg from the shelf?

D5

C From $1/2 \sim \frac{1}{2}/1$, you know that, because Jim has 1 chance in 2 of taking the blue marble, he has $\frac{1}{2}$ chance in 1 of taking it. If Jim takes only one marble, does it make sense to say that he will take $\frac{1}{2}$ of the blue marble?

Obviously, Jim cannot take $\frac{1}{2}$ of a blue marble. The number $\frac{1}{2}$ is the mathematical probability that, when Jim takes 1 marble from the box, the marble will be blue. D4 shows how to write and read a sentence about the probability that Jim will take the blue marble. Study D4. Notice that the word "blue" in parentheses is used to indicate the outcome of taking the blue marble.

In the experiment about the spinner with the numerals from 1 through 6, you know that the numeral 4 has 1 chance in 6 of occurring. Now you can determine the probability that the numeral 4 will occur. U = D.

D In the condition expressed at the right below, P is a variable for the probability that the numeral 4 will occur. What does 1/6 represent? To what does the second component of $P/1$ refer?

$1/6 \sim P/1.$

E Tabulate $\{P|1/6 \sim P/1\}$. Write a sentence that expresses the probability that the numeral 4 will occur. Why will you use the symbol P(4) in the sentence?

Read the experiment described in D5.

F Does Mrs. Andrews have 3 chances in 10 of picking up a hard-boiled egg? What rate pair can you use to represent this rate?

G The sentence at the right below expresses a condition for finding the probability that she will pick up a hard-boiled egg. For what is P a variable? To what does the second component of $P/1$ refer?

$3/10 \sim P/1.$

H Tabulate $\{P|3/10 \sim P/1\}$. Is the probability that Mrs. Andrews will pick up a hard-boiled egg equal to $\frac{3}{10}$? Write a sentence that expresses this probability. Use the symbol P(hard-boiled).

When you are determining the probability of a given outcome of an experiment, it is convenient to speak of this outcome as *favorable*. This does not mean that the outcome is "good" or "pleasant"; it only means that it is the outcome whose probability you are considering.

Universe for each variable = D.

D $f/t \sim P/1$.

E $Pt = f$.

F $P = \frac{f}{t}$.

D6

prob a bil i ty of a favorable outcome (prob'ə bil'ə tē). The quotient of the number of favorable outcomes and the number of possible outcomes of an experiment. If 7 of 12 possible outcomes are favorable, then P(favorable) = $\frac{7}{12}$.

I For each of the experiments described on pages 398 and 399, what is the favorable outcome?

J For each of the experiments that you have considered so far, the probability of the favorable outcome is equal to the quotient of what two numbers?

Now you can use a condition for equivalence for determining the probability of any given favorable outcome. Look at D6.

In the condition expressed by sentence D, we have used P as a variable for the probability of a favorable outcome. f is a variable for the number of favorable outcomes, and t is a variable for the total number of possible outcomes. All the possible outcomes are equally likely.

K From condition D, you know that the rate pair that represents the rate of favorable outcomes per 1 chance is equivalent to another rate pair. Name this rate pair.

L How can condition E be obtained from condition D? How can condition F be obtained from condition E?

From condition F, you know that the probability of a favorable outcome of an experiment is equal to the quotient of the number of favorable outcomes and the number of possible outcomes, when all the outcomes are equally likely.

M What is the probability of a favorable outcome if 3 of 5 possible outcomes are favorable? If 1 of 4 outcomes are favorable?

N If the number of favorable outcomes and the number of possible outcomes are the same, is the probability equal to 1?

O For any given number of possible outcomes, if the number of favorable outcomes is 0, what is the probability of a favorable outcome?

P Can you have more favorable outcomes than you have possible outcomes? Can the probability of an outcome ever be greater than 1? Explain your answers.

Q Can there ever be fewer than 0 favorable outcomes? Explain why the probability of a favorable outcome cannot be less than 0.

Now you know how to determine and express the probability of a favorable outcome of an experiment. You know that you determine the probability by finding the quotient of the number of favorable outcomes and the number of possible outcomes of an experiment.

On your own

For each of exercises 1 through 8, first name the number of possible outcomes and the number of favorable outcomes. Then give the answer to the exercise.

1 One day, the Emersons' small son removed the labels from 6 cans of vegetables that his mother had just brought from the grocery store. Mrs. Emerson had bought 2 cans of peas and 4 cans of green beans. All the cans were of the same size. If Mrs. Emerson opens one of the cans, what is the probability that she will open a can of peas?

2 What is the probability that Mrs. Emerson will open a can of green beans?

3 What is the probability that Mrs. Emerson will open a can of corn?

4 Mr. Miller, who is the coach of a boys' baseball team, has 4 complimentary tickets for a professional baseball game. He plans to have his 12 players draw lots to see who will get the 4 tickets. For each of the 12 members of the team, what is the probability that he will get one of the tickets?

5 Cal, Ernie, Ted, and Russ are 4 members of the baseball team mentioned in exercise 4. What is the probability that these 4 boys will get the 4 tickets?

THINK How many possible combinations of 4 boys are there among the 12 members of the team?

6 If there are 6 pieces of candy in a bag and all the pieces are chocolate, what is the probability that you will get a chocolate piece when you take one piece of candy?

7 The waitress at a lunch counter has put 7 cartons of coffee in a bag for Mr. Levy to take to his office. Four of the cartons have coffee with cream, and the other cartons contain black coffee. The waitress forgot to mark the cartons to indicate whether there is cream in the coffee or not. If Mr. Levy takes 1 carton of coffee from the bag, what is the probability that he will get black coffee?

8 In the experiment described in exercise 7, what is the probability that the first carton that Mr. Levy takes from the bag has cream in the coffee?

A probability is expressed in each of exercises 9 through 13. Describe an experiment for which the probability of a favorable outcome is equal to the probability expressed in each exercise.

9 $\frac{4}{9}$ **10** $\frac{99}{100}$ **11** $\frac{1}{15}$ **12** 1 **13** $\frac{2}{1000}$

Probabilities in a sample space

In the last lesson you learned how to determine the probability of a favorable outcome of an experiment. Now you will learn what is meant by a sample space of an experiment, and how a favorable outcome is related to a subset of a sample space.

Read the description of an experiment in D1. In the exercises that follow, you will think about the possible outcomes of this experiment.

A The experiment described in D1 involves drawing just 1 tag from each hat. What numbers are named on the blue tags? Think of each of these numbers as representing a possible outcome of drawing a tag from the hat that contains blue tags. What are the possible outcomes of drawing a tag from the hat that contains red tags?

Four blue tags are put into one hat, and 5 red tags are put into a second hat. The numbers 1, 2, 3, and 4 are named on the blue tags, with one numeral on each tag. The numbers 1, 2, 3, 4, and 5 are named on the red tags, with one numeral on each tag. Someone who has been blindfolded draws one tag from each hat. What are the possible outcomes of the experiment?

D1

B Can each outcome of drawing a blue tag be matched with each outcome, in turn, of drawing a red tag?

C Although you have not studied the fundamental counting property, try to determine the number of ways in which the outcomes of drawing a blue tag can be matched with the outcomes of drawing a red tag.

Instead of thinking of a drawing from each hat as a separate outcome of the experiment, you are to think of two drawings, one from each hat, as a single outcome. You can use an ordered pair of numbers to represent each of these possible outcomes. We will use the number named on the blue tag as the first component of the ordered pair, and the number named on the red tag as the second component.

D Is (2, 3) a possible outcome of the experiment? Is (1, 5) a possible outcome? Is (5, 5) a possible outcome? Is (2, 7) a possible outcome? Explain your answers.

E The set of all ordered pairs that represent the possible outcomes has how many members?

The elements of the set of all the possible outcomes of the experiment described in D1 are named in the chart in D2. The numerals in the top row, to the right of the letter r, are the numerals written on the red tags. The numerals in the column below the letter b are the numerals written on the blue tags.

F Find the numeral 2 in the first column. What ordered pairs are named at the right of the numeral 2? What is the first component of each of these ordered pairs? Does each of these ordered pairs represent an outcome in which the numeral 2 is written on the blue tag?

G Find the numeral 4 in the top row. What is the second component of each ordered pair that is named below this numeral? For each of these outcomes, what do you know about the numeral on the red tag?

Four blue tags are put into one hat, and 5 red tags are put into a second hat. The numbers 1, 2, 3, and 4 are named on the blue tags, with one numeral on each tag. The numbers 1, 2, 3, 4, and 5 are named on the red tags, with one numeral on each tag. Someone who has been blindfolded draws one tag from each hat. What are the possible outcomes of the experiment?

D1

b \ r	1	2	3	4	5
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)

D2

sam ple space of an experiment (sam′pel). A set whose elements are all possible outcomes of the experiment. The elements of the sample space are **sample points.**

The chart in D2 is one way of representing *a sample space of the experiment* described in D1. A sample space of an experiment is a set whose members are all the possible outcomes of the experiment. Each ordered pair named in the chart in D2 is called a *sample point.*

H Look again at D2. Which of the sample points named have the same first and second components? Which have 1 as a first component? Which have 5 as a second component?

I Tabulate the set of first components that you can obtain from the sample space represented in D2. Use the letter B to name this set.

J Is $B = \{1, 2, 3, 4\}$? Does set B contain each of the numbers that are named on the blue tags? Does set B contain any other numbers?

K Tabulate the set of second components that you can obtain from the sample space represented in D2. Use the letter R to name this set.

L Is $R = \{1, 2, 3, 4, 5\}$? Does set R contain each of the numbers that are named on the red tags? Does set R contain any other numbers?

M Is the sample space represented in D2 the same set as the Cartesian set $B \times R$?

N Is each outcome of the experiment that is described in D1 represented by an element of $B \times R$?

For convenience, we will say that each outcome of the experiment *is* an element of $B \times R$. $B \times R$ is the universe for the experiment described in D1.

Next you will learn how to develop sample spaces for other experiments. Read the description of an experiment in D3. Think of each pair of books that Martin might take from the library as representing a possible outcome of this experiment.

A Can each copy of the astronomy book be matched with each copy of the history book in turn?

B Can you use an ordered pair of numbers to represent each possible outcome of the experiment? Think of the numbers named on the astronomy books as the first components of the ordered pairs. Is (2, 4) a possible outcome? Is (6, 1) a possible outcome?

C Make a chart like the one in D2 to represent a sample space of the experiment involving

There are 6 copies of a particular astronomy book in the public library; these copies are separately labelled 1 through 6. There are 4 copies of a particular history book; these copies are separately labelled 1 through 4. Martin plans to take 1 copy of each kind of book from the library. What are the possible outcomes of the experiment?

D3

the library books. Is the sample space a set of ordered pairs? How many sample points are in the sample space?

D Tabulate the set of first components that you can obtain from your chart. Use the letter A to name this set.

E Tabulate the set of second components that you can obtain from your chart. Use the letter H to name this set.

F Name the Cartesian set that is the universe for the experiment described in D3. How do you know that the sample space represented in your chart is the same set as the Cartesian set you just named?

You can also represent sample spaces of experiments of the kind that you studied in lesson 178. One of the experiments in that lesson involved a spinner and the probability of the pointer's stopping on any one of the numerals on the card.

G A set of possible outcomes of the experiment involving the spinner is {1, 2, 3, 4, 5, 6}. Explain why the sample space of the experiment is not a Cartesian set.

In the exercises that follow, you will study subsets of sample spaces. You will work with one of the experiments described earlier in the lesson.

Read the problem in D4. Now you are interested in the probability of drawing a blue tag and a red tag so that 1 is the number named on the blue tag. Later, you will consider experiments in which you are concerned about the numbers named on both kinds of tag.

Earlier in the lesson you found that the Cartesian set $B \times R$ is a sample space of this experiment. B is the set of numbers named on the blue tags, and R is the set of numbers named on the red tags. You can again think of the sample space $B \times R$ as the universe for the experiment involving the tags. $B \times R$, represented in D2, is shown again on this page.

A Each element of $B \times R$ that has 1 as the first component is a favorable outcome of the experiment described in D4. Tabulate the set of favorable outcomes. Use the letter A to name this set.

Notice that the requirement for a favorable outcome is that the first component is 1. No requirement is made of the second component.

B Is A a subset of $B \times R$? Use b as a variable for the number named on the blue tag; use r as a variable for the number named on the red tag. Is $A = \{(b, r) | b = 1\}$? $B \times R$ is the universe for (b, r).

Set A is called an *event*. An event is a subset of a sample space.

C How do you know that each element of event A is a favorable outcome of the experiment described in D4?

D How many members does $B \times R$ have? Is this the number of possible outcomes of the experiment?

E How many members does event A have?

Next you will develop a formula for finding the probability of an event such as event A.

F From lesson 178, you know that the probability of a favorable outcome of an experiment is the quotient of what two numbers?

b \ r	1	2	3	4	5
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)

D2

Four blue tags are put into one hat, and 5 red tags are put into a second hat. The numbers 1, 2, 3, and 4 are named on the blue tags, with one numeral on each tag. The numbers 1, 2, 3, 4, and 5 are named on the red tags, with one numeral on each tag. Someone who has been blindfolded draws one tag from each hat. What is the probability that 1 is the number named on the blue tag?

D4

event. A subset of a sample space.

For a sample space whose elements are equally likely outcomes, the probability of any event is the quotient of the number of favorable outcomes that corresponds to the event and the number of possible outcomes of the experiment.

Now look at D5. $P(M) = \frac{m}{n}$ is a formula for finding the probability of an event that is a subset of a sample space all of whose elements are equally likely outcomes. $P(M)$ is a variable for the probability of the event; m is

a variable for the number of elements in the event; and n is a variable for the number of elements in the sample space. In the work that follows, unless you are told otherwise, you will consider only sample spaces whose elements are equally likely outcomes.

You can use the formula $P(M) = \frac{m}{n}$ to determine the probability of event A, which you are asked to find in the experiment described in D4.

G In $P(M) = \frac{m}{n}$, why should you replace $P(M)$ by P(A)? By what number should you replace m? By what number should you replace n? What statement do you obtain?

$P(A) = \frac{5}{20}$ gives the probability of event A. Now you can give the answer to the problem.

The probability that 1 is the number named on the blue tag is $\frac{1}{4}$.

In this lesson you learned that a sample space of an experiment is the set of all possible outcomes of the experiment. You also learned that an event is a subset of a sample space and that the probability of an event can be found by using the formula $P(M) = \frac{m}{n}$.

On your own

Use the experiments described in D6 in connection with exercises 1 through 10. You will also use $P(M) = \frac{m}{n}$. The universe for m is the set of non-negative integers. The universe for n is I_p.

1 Read the description of experiment A. Make a chart to represent a sample space of this experiment. The sample space should contain all the ordered pairs of numbers whose first components refer to the rows and whose second components refer to the columns.

2 R is the set of rows. C is the set of columns. $R \times C$ has how many elements?

Universe for m is the set of non-negative integers.
Universe for $n = I_p$.

$$P(M) = \frac{m}{n}.$$

D5

A The seats in a meeting room are identified according to row and column. The columns are named from left to right with the numerals from 1 through 7. The rows are named from front to back with the numerals from 1 through 6. It is equally likely that any one of the seats will be occupied.

B A box contains 5 objects: 3 sharpened pencils and 2 unsharpened pencils. The pencils are in no special order in the box. You are to take a pencil from the box without looking.

D6

3 Event A involves a person's sitting in a seat in a row and a column that have the same number. What is the probability of event A?

4 Event B involves a person's sitting in a row with a number greater than the number of the column. What is the probability of event B?

5 Event C is that the sum of the number of the row and the number of the column of the seat in which a person is sitting is greater than 12. Find the probability of event C.

6 Event D is that the sum of the number of the row and the number of the column of the seat in which a person is sitting is less than 3. What is the probability of event D?

7 Read the description of experiment B in D6. Call the sharpened pencils "s_1," "s_2," and "s_3," and call the unsharpened pencils "u_1" and

A The seats in a meeting room are identified according to row and column. The columns are named from left to right with the numerals from 1 through 7. The rows are named from front to back with the numerals from 1 through 6. It is equally likely that any one of the seats will be occupied.

B A box contains 5 objects: 3 sharpened pencils and 2 unsharpened pencils. The pencils are in no special order in the box. You are to take a pencil from the box without looking.

D6

"u_2." Each pencil that you might draw is an outcome of the experiment. Tabulate the sample space.

8 What is the probability of taking a sharpened pencil from the box?

9 What is the probability of taking an unsharpened pencil from the box?

10 What is the probability of taking a pen from the box?

Use $P(M) = \frac{m}{n}$ in connection with exercises 11 through 14.

11 The probability of event Q is $\frac{2}{3}$. The number of elements in event Q is 8. How many elements are there in the sample space for the experiment?

12 There is 1 element in event R. There are 25 elements in the sample space. What is the probability of event R?

13 The probability of event S is 1. The number of elements in the sample space is 4. How many elements are there in event S?

14 The probability of event T is $\frac{3}{8}$. The number of elements in the sample space is 40. How many elements are there in event T?

Mutually exclusive events

In the preceding lesson you learned that an event is a subset of a sample space of an experiment. In this lesson you will study the intersection of two events, the union of two events, and the probabilities associated with the intersection and the union.

You will recall that in the last lesson you studied an experiment that concerns the drawing of blue tags and red tags from two hats. One hat contains 4 blue tags with the numerals 1 through 4 on them. The other hat contains 5 red tags with the numerals 1 through 5 on them. Throughout this lesson, keep in mind that the experiment consists of drawing one tag from each hat.

The chart that represents a sample space of the experiment is repeated in D1.

A Each ordered pair expressed in the chart represents an outcome of the experiment. To what does the first component of each ordered pair refer? To what does the second component refer?

B Use S as a name of the sample space of the ? experiment. Give a standard description of sample space S.

In the exercises in this lesson, the universe for (b, r) is S.

C How many elements are in S? The experiment has how many possible outcomes? Give the probability of any one of these outcomes.

D Event A is $\{(b, r) | b = 2\}$. Tabulate A. Is A the set of all outcomes in which you draw a blue tag with the numeral 2?

b \ r	1	2	3	4	5
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)

D1

A $A = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5)\}$.

B $B = \{(1, 3), (2, 3), (3, 3), (4, 3)\}$.

C $A \cap B = \{(2, 3)\}$.

D $A \cup B = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (1, 3), (3, 3), (4, 3)\}$.

D2

E Event B is $\{(b, r) | r = 3\}$. Tabulate this event. Describe B in terms of the outcomes contained in B.

F Look at D2. Does sentence A express a true statement about event A? Does sentence B express a true statement about event B?

G How do you know that $\frac{5}{20}$, or $\frac{1}{4}$, is the probability of event A? How do you know that $\frac{1}{5}$ is the probability of event B?

You will recall that, to find the solution set of a compound condition that includes the connective "and," you use the idea of intersection of sets. You can again make use of the idea of intersection of sets to find the probability that event A *and* event B will occur. The probability that event A and event B will occur is the probability of $\{(b, r) | b = 2 \wedge r = 3\}$. Another way of saying this is: The probability that event A and event B will occur is the probability that you will draw a blue tag with the numeral 2 and a red tag with the numeral 3.

H Does the intersection of two sets contain those objects, and only those objects, that belong to both sets?

I Look again at D2. What members do set A and set B have in common? Does sentence C express a true statement about the intersection of set A and set B?

J Is $\{(2, 3)\}$ a subset of S? How do you know that $\frac{1}{20}$ is the probability of the event $\{(2, 3)\}$? That $\frac{1}{20}$ is the probability of $A \cap B$?

The probability of $A \cap B$ is the probability that event A and event B will occur. The probability that you will draw a blue tag with the numeral 2 and a red tag with the numeral 3 is $\frac{1}{20}$.

Next you will decide the probability that event A *or* event B will occur. This is the probability of $\{(b, r) | b = 2 \vee r = 3\}$. It is the probability that you will draw a blue tag with the numeral 2 or a red tag with the numeral 3. Remember that "or" in a compound condition means "one or the other or both." You know that you use the idea of union of sets to find the solution set of a compound condition that includes "or." You can also use the idea of union of sets to find the probability that event A or event B will occur.

K Is the union of two sets the set that contains all those objects, and only those objects, that belong to either set or to both sets?

L Look again at D2. Which elements are contained in both A and B? Which elements are contained only in A? Only in B? Does sentence D express a true statement about the union of A and B?

M $A \cup B$ has how many elements? Explain how you know that $\frac{2}{5}$ is the probability of $A \cup B$.

The probability of $A \cup B$ is the probability that event A or event B will occur. The probability that you will draw a blue tag with the numeral 2 or a red tag with the numeral 3 is $\frac{2}{5}$.

N You obtained $\frac{1}{4}$ for P(A) and $\frac{1}{5}$ for P(B). Is $\frac{1}{4} + \frac{1}{5} = \frac{9}{20}$? What is P(A) + P(B)?

O Did you obtain $\frac{1}{20}$ for $P(A \cap B)$? Is $\frac{9}{20} - \frac{1}{20} = \frac{2}{5}$? What is $P(A) + P(B) - P(A \cap B)$?

P What number did you obtain for $P(A \cup B)$? Is this the same number that you obtained for $P(A) + P(B) - P(A \cap B)$?

In exercises N, O, and P, you developed a property of the probability of the union of two events. The property is expressed below.

For each M *and* N, *if* M *and* N *are events in a finite sample space, then*

$P(M \cup N) = P(M) + P(N) - P(M \cap N)$.

You have found the probability of the union of two events when the events have some element in common. In the example that you have been studying, event A and event B have an element in common because there is one possible outcome in which you can draw both a blue tag with the numeral 2 and a red tag with the numeral 3. Now you will study the probability of the union of two events when the events have no element in common.

A Look again at the chart in D1. Event X is $\{(b, r) | b + r = 4\}$. Tabulate X. Describe event X in words.

B Event Y is $\{(b, r) | r = 4\}$. Tabulate Y. Describe event Y in words.

C Suppose that you draw a blue tag and a red tag and that the numerals on the tags express two numbers whose sum is 4. Can either of the tags have the numeral 4 on it? Explain your answer.

D Now look at D3. Do sentences E and F express true statements?

E Do X and Y have any members in common?

D1

b \ r	1	2	3	4	5
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)

A $A = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5)\}$.

B $B = \{(1, 3), (2, 3), (3, 3), (4, 3)\}$.

C $A \cap B = \{(2, 3)\}$.

D $A \cup B = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (1, 3), (3, 3), (4, 3)\}$.

D2

E $X = \{(3, 1), (2, 2), (1, 3)\}$.

F $Y = \{(1, 4), (2, 4), (3, 4), (4, 4)\}$.

D3

mu tu al ly ex clu sive events (mū′chü əl lē eks klü′siv). Events of a finite sample space that have no elements in common. Mutually exclusive events are disjoint sets.

Events X and Y are *mutually exclusive events* of sample space S. Such events have no elements in common. They are disjoint sets.

F Look again at D3. Why is $\frac{3}{20}$ the probability of event X? Why is $\frac{1}{5}$ the probability of event Y?

G How many members are in $X \cap Y$? What is the intersection of any two disjoint sets?

Why is 0 the probability of the intersection of event X and event Y?

H For X and Y, how do you know that $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$? Is $P(X \cup Y) = P(X) + P(Y) - 0$? Explain your answers.

I What is the probability of the union of event X and event Y? What is the probability that event X or event Y will occur?

J Are $P(X) + P(Y) - 0$ and $P(X) + P(Y)$ the same number? Is the probability that event X or event Y will occur equal to the sum of the probability of X and the probability of Y?

K Look again at the chart in D1. Event J is $\{(b, r) | b + r = 2\}$. Event K is $\{(b, r) | b + r = 9\}$. Tabulate J. Tabulate K.

L Are J and K disjoint sets? Are J and K mutually exclusive events? Explain your answers.

M What is the probability of event J? Of event K? What is the probability that event J and event K will occur? What is the probability that event J or event K will occur?

N Is the probability that event J or event K will occur the sum of the probability of event J and the probability of event K?

In exercises F through N, you have studied a property of the probability of mutually exclusive events. The property is expressed below.

For each M *and* N, *if* M *and* N *are mutually exclusive events of a finite sample space, then* $P(M \cup N) = P(M) + P(N)$.

Next you will use the idea of the complement of a set to study mutually exclusive events that are related in a special way.

A Look again at the chart in D1. The symbol $\{(b, r) | b < 4\}$ is a standard description of event D. Tabulate D.

B Now tabulate $\{(b, r) | b \geq 4\}$. Are any members of $\{(b, r) | b \geq 4\}$ also members of set D? Does $\{(b, r) | b \geq 4\}$ contain all the members of S that are not members of D?

C Is $\{(b, r) | b \geq 4\} = \{\overline{(b, r) | b < 4}\}$? How do you know that $\{(b, r) | b \geq 4\} = \overline{D}$?

Remember that a bar over a symbol for a standard description, or a bar over the letter that names a set, means the complement of that set.

REMINDER

The complement of a given subset of a universe is the set that contains all the members of the universe, and only those members, that do not belong to the given subset.

See lesson 112, page 78.

Since D and $\overline{D}$ are complements of each other, these two sets are *complementary events*. Any two subsets of a finite sample space, each of which is the complement of the other, are complementary events.

D You know that $\overline{D} = \{\overline{(b, r) | b < 4}\}$. Is $\overline{D} = \{(b, r) | \sim (b < 4)\}$?

E Is $\overline{D}$ the event that D does *not* occur?

F Are D and $\overline{D}$ mutually exclusive events? ? Explain your answer.

G How do you know that $D \cup \overline{D}$ is the same set as the sample space S?

H If you consider S as an event, how do you know that the number of favorable outcomes in the event is the same as the number of all possible outcomes?

I How do you know that 1 is the probability of event S?

complementary events. Two subsets of a finite sample space, each of which is the complement of the other.

J Look at D4. $P(D \cup \overline{D})$ represents the probability of $D \cup \overline{D}$. Does $P(D)$ represent the probability of D? Does $P(\overline{D})$ represent the probability of $\overline{D}$?

K How do you know that sentence G in D4 expresses a true statement?

L Remember that D and $\overline{D}$ are mutually exclusive events. What property of mutually exclusive events enables you to obtain statement H from statement G?

M How can you obtain statement I from statement H? Study sentence I. Is the probability of the complement of D the difference of 1 and the probability of D?

N Is the probability that D will not occur the difference of 1 and the probability that D will occur?

O Is the union of any two complementary sets the same as the universe? How does this explain why 1 is the probability that an event will occur or will not occur?

P Is the sum of the probabilities of any two complementary events equal to 1? If you know the probability of a given event, how can you find the probability that it will not occur?

In exercises F through P, you have developed a property of complementary events.

For each M *and* $\overline{M}$, *if* M *and* $\overline{M}$ *are two complementary events in a finite sample space, then* $P(\overline{M}) = 1 - P(M)$.

Q Suppose that E and $\overline{E}$ are two complementary events in S. Suppose also that E = S. What is the probability of E? What property tells you that the probability of $\overline{E}$ is $1 - 1$? If $\overline{E} = \{\ \}$, should 0 be the probability of $\overline{E}$?

In this lesson you learned that the probability that event A *and* event B will occur is the probability of $A \cap B$. You also learned that the probability that event A *or* event B will occur is the probability of $A \cup B$. Then you studied

G $P(D \cup \overline{D}) = 1$.
H $P(D) + P(\overline{D}) = 1$.
I $P(\overline{D}) = 1 - P(D)$.

D4

A There are 15 tags in a box. The numerals 1 through 15 are written on the tags, with one numeral on each tag. Someone is to draw a tag from the box without looking.

B At a party, four prizes are wrapped in exactly the same way. The packages are all the same size. Carol is to choose two of the prizes.

C A committee of three boys is to be chosen from among five volunteers. The volunteers are Carl, Jim, Ed, Bill, and Tom.

D5

mutually exclusive events and complementary events.

On your own

Use the experiments described in D5 in connection with exercises 1 through 12. You will also need to use the following formulas:

$P(M \cup N) = P(M) + P(N) - P(M \cap N)$.
$P(M \cup N) = P(M) + P(N)$.
$P(\overline{M}) = 1 - P(M)$.

1 Read the description of experiment A. Is set S a sample space of experiment A?

S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}.

2 Event A is the event in which the tag drawn has a numeral for a number that is divisible by 3. Event B is the event in which the tag drawn has a numeral for an even number. Are A and B mutually exclusive events? Explain your answer.

3 What is the probability that A or B will occur? That A and B will occur?

4 Event C is the event in which the tag drawn has a numeral for a prime greater than 2. Are event B (described in exercise 2) and event C mutually exclusive? Explain your answer.

5 What is the probability that B and C will occur? That B or C will occur?

6 Describe in words the event that B does *not* occur. Is this event the complement of B?

7 What is the probability of B? Of $\overline{B}$? What is the probability that B and $\overline{B}$ will occur? That B or $\overline{B}$ will occur?

8 Read the description of experiment B in D5. Tabulate a sample space of experiment B. Use G, H, I, and J to represent the prizes. The sample space should contain all the possible pairs of prizes that Carol could choose.

9 What is the probability that Carol will choose prize G or prize I? That she will choose prize G and prize I?

10 Read the description of experiment C. Tabulate a sample space of experiment C. Use C, J, E, B, and T to represent the volunteers. The sample space should contain all the possible committees of three that can be chosen.

11 What is the probability that Ed and Jim will be chosen? What is the probability that Ed or Jim will be chosen?

12 What is the probability that Tom will be chosen? That Tom will not be chosen?

13 A bag contains five objects: three green marbles and two red marbles. Call the green marbles "g_1," "g_2," and "g_3." Call the red marbles "r_1" and "r_2." Suppose that you draw a marble from the bag and put it back. Then you draw a second marble. What is the probability that the first marble will be red or the second marble will be red?

14 What is the probability that the first marble and the second marble will be green?

Independent events

In the previous lesson you learned that the probability that a first event and a second event will occur is the probability of the intersection of the two events. In this lesson you will learn more about the probability of the intersection of events that are called independent events.

The chart in D1 is a chart that you have used before. It represents a sample space of an experiment. You will recall that the experiment involves one hat containing 4 blue tags, with one of the numerals 1 through 4 written on each tag; and a second hat containing 5 red tags, with one of the numerals 1 through 5 written on each tag. The experiment consists of drawing one tag from each hat.

A Use S to name the sample space of the experiment. What does each member of S represent? Is $S = \{(b, r) \mid 1 \leqq b \leqq 4 \wedge 1 \leqq r \leqq 5\}$? In this lesson, the universe for (b, r) is S.

b \ r	1	2	3	4	5
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)

D1

B Describe event $\{(3, 2)\}$ in terms of blue tags and red tags. Describe event $\{(b, r) | b \leqq 3\}$ in terms of blue tags and red tags.

C Give a standard description of the event that you will draw a red tag with a numeral for a number greater than 5. Tabulate this event.

Now you will study two events in S to see how the probability of their intersection is related to their individual probabilities.

D Event K is the event that you will draw a blue tag with the numeral 2. Give a standard description of K.

E Tabulate K. What is the probability of K?

F Event L is the event that you will draw a red tag with the numeral 4. Give a standard description of L.

G Tabulate L. What is the probability of L?

H You know that $K \cap L$ is the event that you will draw a blue tag with the numeral 2 and a red tag with the numeral 4. Is the probability of this event the probability of $K \cap L$?

I Tabulate $K \cap L$. What is the probability that event K and event L will occur? How do you know?

J Now look at sentence A in D2. Is $\frac{1}{20}$ the probability of $K \cap L$? Is $\frac{1}{4}$ the probability of K? Is $\frac{1}{5}$ the probability of L? Does sentence A express a true statement?

K Is the probability of $K \cap L$ the product of the probability of K and the probability of L?

L Read sentence B in D2. P(K) represents the probability of K; P(L) represents the probability of L; and $P(K \cap L)$ represents the probability of $K \cap L$. Is $P(K \cap L) = P(K) \cdot P(L)$?

M Event A is the event that you will draw a blue tag with a numeral for a number that is less than or equal to 2. Give a standard description of A.

N Tabulate A. What is the probability of A?

O Event B is the event that you will draw a red tag with a numeral for a number that is greater than or equal to 4. Give a standard description of B.

P Tabulate B. How do you know that $\frac{2}{5}$ is the probability of B?

Q Tabulate $A \cap B$. How do you know that $\frac{2}{10}$ is the probability that A and B will occur?

R Now study sentence C in D3. Is the probability of $A \cap B$ the product of the probability of A and the probability of B?

S Read sentence D in D3. Is $P(A \cap B) = P(A) \cdot P(B)$?

In exercises D through S, you discovered that the probability of $K \cap L$ is the product of the probability of K and the probability of L. You also discovered that the probability of $A \cap B$ is the product of the probability of A and the probability of B. Events K and L are *independent events*. A and B are also independent events.

b \ r	1	2	3	4	5
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)

D1

A $\frac{1}{20} = \frac{1}{4} \cdot \frac{1}{5}$.

B $P(K \cap L) = P(K) \cdot P(L)$.

D2

C $\frac{2}{10} = \frac{1}{2} \cdot \frac{2}{5}$.

D $P(A \cap B) = P(A) \cdot P(B)$.

D3

in de pend ent events (in′di pen′dənt). Events that are related as follows: M and N are subsets of a finite sample space. If $P(M \cap N) = P(M) \cdot P(N)$, then M and N are independent events. If M and N are independent events, then $P(M \cap N) = P(M) \cdot P(N)$.

If the probability of the intersection of two events is the product of their individual probabilities, then the two events are independent. Also, if two events are independent, then the probability of their intersection is the product of their individual probabilities.

In discussions of probability, independent events are sometimes described as "events that have nothing to do with each other." This means that the occurrence of one of the events does not depend in any way on the occurrence of the other event. In the examples of independent events that you have studied so far, this is easy to see. For example, you would not expect that drawing a blue tag with the numeral 2 from one hat will affect whether or not you will draw a red tag with the numeral 4 from the other hat. However, it is not always easy to recognize independent events. We will use the definition of independent events to decide when two events are independent.

T Now you will study events R and T in sample space S. Event R is the event that you will draw a blue tag with a numeral for an even number. Tabulate R. T is the event that you will draw a red tag with a numeral for an odd number. Tabulate T.

U What is the probability of R? What is the probability of T?

V Suppose you know that R and T are independent events. How can you use the probability of R and the probability of T to find the probability of $R \cap T$?

W What is the probability of drawing a blue tag with a numeral for an even number and a red tag with a numeral for an odd number?

If you know that two events are independent, you can use the definition of independent events to help you find the probability of their intersection. You can also use the definition to decide when two events are independent.

A Look again at D1. Event G is the event that you will draw a red tag with a numeral for a number greater than or equal to 3. Tabulate G. Event H is the event that you will draw a blue tag and a red tag with numerals for two numbers whose sum is 5. Tabulate H.

B What is the probability of G? Of H?

C If G and H are independent events, what product should be the probability of $G \cap H$?

D Tabulate $G \cap H$. How many members are contained in $G \cap H$? What is the probability of $G \cap H$?

E Is $\frac{2}{20}$, or $\frac{1}{10}$, the product of the probability of G and the probability of H? Are G and H independent events? Explain your answer.

F Suppose that X and Y are two mutually exclusive events and that neither of them is the empty set. How do you know that $\{\ \}$ is the intersection of X and Y?

G How do you know that the probability of $X \cap Y$ is 0?

H Can the probability of X be 0? Can the probability of Y be 0? Explain your answers.

I How do you know that the product of the probability of X and the probability of Y cannot be 0? Explain why the probability of $X \cap Y$ is not the same as the product of the probability of X and the probability of Y.

J Are X and Y independent events?

In this lesson you learned that, if M and N are events in a finite sample space, and if the prob-

A The first digit in the numeral for a telephone number may be chosen from the numerals 1 through 9. The second digit may be any one of the numerals 0 through 9. You decide to guess the first two digits for a telephone number that you have forgotten.

B A five-story building has two elevators, e_1 and e_2. Each elevator runs independently of the other and stops at each floor. Assume that an elevator does not stop between floors. While waiting for an elevator, you decide to guess at which floor each elevator is at a given moment.

C Evelyn has one black scarf, two red scarfs, and four figured scarfs in a drawer. Each day, without looking, she takes a scarf from the drawer to wear for that day.

D4

ability of $M \cap N$ is the product of the probability of M and the probability of N, then M and N are independent events.

On your own

Use the experiments described in D4 in connection with exercises 1 through 13. You will need to use $P(M \cap N) = P(M) \cdot P(N)$ for some of the exercises.

1 Read the description of experiment A. Make a chart to represent a sample space that contains all the possible pairs of numbers that can be expressed by the first two digits of the numeral for the telephone number.

2 What is the probability that the first number is 3? That the second number is 5? Are these independent events? How do you know?

3 What is the probability that the sum of the two numbers is 13? That the second number is greater than 7? Are these independent events? How do you know?

4 What is the probability that the sum of the two numbers is 9? That the second number is 5? Are these independent events? Explain your answer.

5 What is the probability that the first number is 1 greater than the second number? That the second number is 0? Are these independent events? Explain your answer.

6 What is the probability that the sum of the numbers is 7? That the sum of the numbers is 10? How do you know, without determining the probability of their intersection, that these two events are not independent events?

7 Read the description of experiment B in D4. Is {1, 2, 3, 4, 5} the set of all possible positions for e_1? How do you know that $\frac{1}{5}$ is the probability that e_1 is at the third floor? Give the set of all possible positions for e_2. What is the probability that e_2 is at the third floor?

8 What is the probability that e_1 is at the third floor and e_2 is at the third floor? Assume that these are independent events.

9 What is the probability that e_1 is at the fifth floor and e_2 is at the first floor? Assume that these are independent events.

10 What is the probability that e_1 is above the third floor and e_2 is below the third floor? Assume that these are independent events.

11 Read the description of experiment C. What is the probability that Evelyn will wear a black scarf on Monday? That she will wear a red scarf on Tuesday?

12 What is the probability that Evelyn will wear a black scarf on Monday and a red scarf on Tuesday? These are independent events.

13 What is the probability that Evelyn will wear a figured scarf on Monday and will not wear a figured scarf on Tuesday? These are independent events.

KEEPING SKILFUL

For each of exercises 1 through 22, find the sum, difference, product, or quotient.

1 $18\frac{9}{16} + 8\frac{5}{8}$
2 $213.65 - 82.7$
3 $20\frac{1}{4} \div {}^{-}16\frac{1}{2}$
4 6.3×4.8
5 ${}^{-}125.7 + {}^{-}85.9$
6 $605\frac{2}{3} - {}^{-}84$
7 ${}^{-}105 \times 4\frac{2}{3}$
8 ${}^{-}3.87 \div {}^{-}.036$
9 ${}^{-}13\frac{1}{5} - {}^{-}29\frac{3}{4}$
10 $80.32 + {}^{-}1.4 + 8.6$
11 ${}^{-}9\frac{11}{16} \times {}^{-}18\frac{2}{5}$
12 $980.59 \div 39.7$
13 ${}^{-}1000 - 72.45$
14 $25.6 \times 11.3 + {}^{-}.9$
15 ${}^{-}1102\frac{5}{6} + 2099\frac{7}{10}$
16 $25 \div 10\frac{5}{9}$
17 $29.104 - 48.666$
18 $2\frac{3}{4} \times 18\frac{1}{2} \times 16$
19 $80.073 \div 65.1$
20 $74 + 5\frac{2}{3} + 8\frac{3}{4}$
21 $84.35 - {}^{-}61.8$
22 $38.2 \div .025$

For each of exercises 23 through 29, tabulate the solution set if possible. Otherwise, give a standard description of the solution set. The universe for each variable is D.

23 $y + .6 < 21.5 \wedge 12 + y > 18.5$.
24 $g - 8\frac{1}{5} > 12 \vee 6g = 121\frac{1}{5}$.
25 $3x - 8 = {}^{-}2 \wedge 2x > 8.5 - 9.3$.
26 $\sim (21.5t + {}^{-}10.5t > 2.75)$.
27 $10 - m < {}^{-}6 \wedge 11m \geqq 30\frac{1}{4}$.
28 $8(n - 8) = {}^{-}100 \vee \frac{1}{2}/\frac{1}{3} \sim 6\frac{1}{2}/n$.
29 ${}^{-}4(a + 3.1) < 25.6 \wedge {}^{-}15 - a > {}^{-}5$.

For each of exercises 30 through 37, give a standard name of the number named.

30 7! 32 9! 34 P(6, 6) 36 C(6, 2)
31 5! 33 11! 35 P(10, 5) 37 C(8, 5)

Give the complete factorization of each integer named in exercises 38 through 45.

38 72 40 320 42 441 44 1690
39 150 41 864 43 588 45 7000

For each integer named in exercises 46 through 53, find the number of divisors and the sum of the divisors.

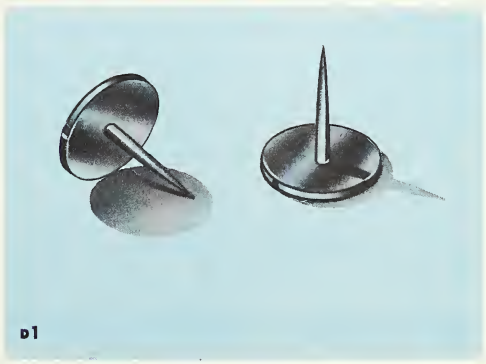
46 27 48 100 50 144 52 440
47 175 49 136 51 735 53 252

Empirical probability

So far in this unit you have considered probabilities of events in finite sample spaces. The sample spaces you studied were those in which each outcome of an experiment had the same probability. You could, therefore, determine the probability of any event that might occur in the experiment. In this lesson you are going to consider experiments in which you do not know the probability of each outcome. You will find approximations of the probabilities of events by making predictions based on observations. A probability determined in this way is called an *empirical* (em pir′ə kəl) probability.

A Suppose that you want to know the probability that a tack, when tossed on a level surface, will land point up. D1 shows the possible outcomes of tossing the tack. How many possible outcomes does this experiment have?

B ? Do you think that the outcomes of the experiment of tossing the tack are equally likely? Explain your answer.



D1

Number of toss	Result of toss	Number of toss	Result of toss
1	D	11	D
2	D	12	D
3	U	13	D
4	D	14	U
5	D	15	D
6	D	16	D
7	U	17	D
8	U	18	D
9	D	19	D
10	D	20	D

D2

relative frequency of an event (frē′kwən sē). If an event E occurs x times in N trials, then $\frac{x}{N}$ is the relative frequency of E. The universe for x is the set of non-negative integers. The universe for N is I_p.

To find a good approximation of the probability that the tack will land point up, you could toss the tack many times and collect data concerning the outcomes. You could then use the data to estimate the probability that the tack will land point up in one toss.

C Suppose that you toss a tack and collect the data recorded in D2. U is the event that the tack lands point up. D is the event that the tack lands point down. Are U and D complementary events? How did the tack land on the first toss? On the fifth toss? On the fourteenth toss? On the last toss?

D How many times did U occur in the first five tosses? In the first ten tosses?

Since you tossed the tack 20 times, you can say that you collected data based on 20 *trials*.

E How many times did U occur in 20 trials? Is $\frac{4}{20}$ the quotient of the number of times that U occurred and the number of trials?

The number $\frac{4}{20}$ is the *relative frequency* of U in 20 trials. The relative frequency of an event in a given number of trials is the quotient of the number of times the event occurs and the number of trials.

F Suppose that, instead of 20 trials, you complete 100 trials of the experiment. If the tack lands point up in 27 of the 100 trials, what is the relative frequency of U?

G Suppose that the relative frequency of U, based on 1000 trials, is $\frac{251}{1000}$. How many times did U occur in 1000 trials?

You can determine an approximation of the probability of an event by finding its relative frequency in a great number of trials.

H Suppose that, in 100,000 trials, the tack lands point up 25,002 times. Based on this information, what is an approximation of the probability that the tack will land point up if it is tossed only once? Is .25 the probability of U to the nearer hundredth?

I ? What is an approximation of the probability that the tack will land point up twice in succession? You can think of this as the probability of the intersection of two events. Assume that the events are independent.

J ? If .25 is an approximation of the probability of U, the event should occur .25 times for one toss of the tack. How can you determine the number of times that U will occur in the next 100 tosses?

Empirical probability is used in business and science for making many kinds of predictions. Look at D3. The data recorded in the display are derived from a *mortality* (môr tal′ə tē) *table* such as insurance companies use when they decide upon life insurance rates. The table is called a mortality table because it can be used to obtain the mortality (death rate) of persons in different age groups.

Age in years	Persons living
1	1,000,000
10	971,804
20	951,483
30	924,609
40	883,342
50	810,900
60	677,771
70	454,548
80	181,765

D3

A The data given in D3 are based on a sample consisting of 1,000,000 persons who were living at the age of 1 year. What number is expressed directly to the right of the numeral for age 10? This number tells you that, of the original 1,000,000 persons, 971,804 were still living at the age of 10 years.

B How many of the 1,000,000 persons were still living at age 20? At age 60? At age 80?

C Suppose that you want to know the probability that a person who is 1 year old now will reach the age of 80 years. Why can you use $\frac{181{,}765}{1{,}000{,}000}$ as an approximation of the probability of this event? What is the probability of this event expressed to the nearer hundredth?

D How many of the original 1,000,000 persons died before they reached the age of 60? What is the probability (to the nearer hundredth) that a person who is 1 year old now will die before he is 60 years old?

E How many persons were living at age 40? Of these, how many were living at age 50?

F Can you use $\frac{810{,}900}{883{,}342}$ as an approximation of the probability that a person who is 40 years old now will be alive at age 50?

G Mr. Colfax is 40 years old now. He wants to buy a life insurance policy that will insure him for the next 20 years. What is the probability that he will die during that time?

In this lesson you studied experiments for which you could not tabulate a sample space whose elements were equally likely outcomes. You learned that, to determine the probability of an event in such a sample space, you need to collect data from which you can determine the relative frequency of an event.

On your own

For each of exercises 1, 2, and 6, first name the number of trials and the number of times the event has occurred. Then give the answer. For some of the exercises, you will need to use probability formulas.

1 For five football seasons, Kevin has recorded his predictions of the results in the games played in a certain conference. He has correctly predicted the results in 75 out of a total of 135 games. What is the probability that his next prediction will be correct?

2 A manufacturer tested one lot of staplers by examining a sample consisting of 1000 of the staplers. He found that 2 staplers were defective. What is the probability that any one stapler picked from this lot will be defective? If a store receives a shipment of 1500 staplers from this lot, how many of the staplers can be expected to be defective?

3 A baseball player's batting average is the quotient of the number of hits he has made and the number of his official times at bat. Why can you think of his batting average as the probability that he will obtain a hit in one time at bat?

4 A player has a batting average of .320. What is the probability that he will get a hit the next time he is at bat?

5 What is the probability that the player described in exercise 4 will get two hits for his

next two times at bat? Assume that these are two independent events and that the batting average does not change significantly for one time at bat.

6 Winslow High School has graduated 12,500 students in the past 10 years. Of these students, 975 graduated with honors. What is the probability that a student at Winslow High School will graduate with honors?

7 According to a mortality table, the probability that Mr. Kramer will live to be 90 yr. old is .01. The probability that his wife will live to be 90 is .008. What is the probability that both Mr. Kramer and his wife will live to be 90 yr. old? Assume that these are two independent events.

8 What is the probability that Mr. Kramer or his wife will live to be 90 yr. old?

APPLYING MATHEMATICS

For each problem first select the appropriate formula. Next use the information given in the problem to make replacements and write a sentence that expresses these replacements. Then give the answer to the problem.

1 There are 9 books on a bookshelf. Tom's mother put a five-dollar bill in each of 2 of these books. If Tom takes a book from the shelf, what is the probability that he will take a book with a five-dollar bill in it?

2 At a bake sale, there are 3 apple pies, 1 peach pie, 1 blueberry pie, 3 cherry pies, and 2 mince pies. All the pies are wrapped so that they look alike. Margaret is going to select 1 pie. What is the probability that Margaret will select an apple pie?

3 What is the probability that Margaret will select a mince pie?

4 What is the probability that Margaret will select a lemon pie?

5 Goodwin's Grocery is going to give each customer a package of flower seeds and a package of vegetable seeds. Petunia seeds are one of the 5 kinds of flower seeds. Carrot seeds are one of the 6 kinds of vegetable seeds. What is the probability that a customer will be given a package of petunia seeds?

6 What is the probability that a customer will be given a package of carrot seeds?

7 What is the probability that a customer will be given a package of petunia seeds and a package of carrot seeds? Assume that these are two independent events.

8 What is the probability that a customer will be given a package of petunia seeds or a package of carrot seeds?

9 Mr. Moretti has correctly predicted the weather 42 out of 56 times. What is the probability that his next weather prediction will be correct?

10 A box contains 12 red pencils, 5 blue pencils, and 8 green pencils. Jack is going to draw a pencil from the box. What is the probability that Jack will draw a blue pencil?

11 What is the probability that Jack will draw a red pencil or a green pencil?

12 Jack is going to draw a pencil from the box, put it back into the box, and draw again. What is the probability that he will draw a green pencil from the box both times? Assume that these are two independent events.

13 For the last year, George has kept a record of the number of times he has had strikes in bowling. In the 400 times that George has rolled the bowling ball, he has had 65 strikes. What is the probability that George will have a strike the next time he rolls the ball?

14 What is the probability that George will have a strike the next two times he rolls a bowling ball? Assume that these are two independent events.

SPECIAL CHALLENGE

Read the problem in D1. You can determine the probability of the event that there are at least 2 students in the class who have the same birthday by first determining the probability that no 2 students have the same birthday.

A Suppose that B is the event that there are at least 2 students who have the same birthday. What is event $\overline{B}$? How do you know that $P(B) = 1 - P(\overline{B})$?

B How many possibilities are there for each student's birthday in a 365-day year?

Suppose that the birthday of one student is on a given day of a given month. If a second student does not have the same birthday, then his birthday must be on one day of the remaining 364 days.

C If you consider only 2 students, explain why the probability that they do not have the same birthday is $\frac{364}{365}$.

D Next consider a third student along with the first 2. If the birthday of the third student is to be different from the birthdays of the first 2, then how many possible days are there for his birthday?

E Is the probability that the third student does not have the same birthday as either of the first 2 equal to $\frac{363}{365}$?

We will consider these events (the event that the first 2 students do not have the same birthday and the event that the third student does not have the same birthday as either of the first 2) as independent events.

F Why should you find the product of $\frac{364}{365}$ and $\frac{363}{365}$ to find the probability that all 3 students do not have the same birthday?

G What is the probability that the birthday of a fourth student is not the same as the birthdays of the first 3 students? Is the probability that all 4 students do not have the same birthday equal to the product $\frac{364}{365} \times \frac{363}{365} \times \frac{362}{365}$?

H What product is equal to the probability that the first 6 students do not have the same birthday?

I What product is equal to the probability that the first 10 students do not have the same birthday?

The sentence in D2 expresses a statement concerning the probability that no 2 of the 30 students have the same birthday.

J Explain how the product expressed in D2 was obtained.

If you were to compute, you would find that the probability of event $\overline{B}$ is approximately equal to .294. You can use the probability of $\overline{B}$ to determine the probability of B.

K How do you obtain $P(B) = 1 - .294$ from $P(B) = 1 - P(\overline{B})$? What is the probability that at least 2 of the 30 students have the same birthday?

You may be surprised to find that the probability that at least 2 of the 30 students have the same birthday is so great. In fact, for any number of persons greater than 22, the probability that at least 2 of these persons have the same birthday is greater than $\frac{1}{2}$.

There are 30 students in Mr. Jansen's eighth-grade class. What is the probability that at least 2 of the 30 students were born on the same day of the same month? The probability should be determined on the basis of a 365-day year.

D1

$$P(\overline{B}) = \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \ldots \times \frac{336}{365}.$$

D2

CHECKING UP

If you have trouble with this test, you can find help in lessons 178 through 182.

Test 166

1 Sam, Harry, Leo, and Kathy are candidates for class president. The election of each candidate is equally likely. What is the probability that Harry will not be elected?

2 Events D and E are independent events. The probability of D is $\frac{3}{4}$. The probability of E is $\frac{1}{4}$. What is the probability of D $\cap$ E?

3 The probability of event T is $\frac{3}{20}$. The number of elements in the sample space is 40, and the elements are equally likely outcomes. How many elements are there in T?

Use the experiment described in the display for exercises 4 through 8.

4 Tabulate a sample space of the experiment. Use w as a variable for the number named on the white card. Use y as a variable for the number named on the yellow card. Name the sample space W $\times$ Y.

5 Event A is $\{(w, y) | w = 1\}$. What is the probability of A?

6 Event B is $\{(w, y) | y > 3\}$. What is the probability of B?

7 What is the probability of A $\cap$ B? Are A and B independent events?

8 What is the probability of A $\cup$ B? Are A and B mutually exclusive events?

There are 3 white cards in one box. The numbers 1, 2, and 3 are named separately on these cards. There are 6 yellow cards in another box, with the numbers 1 through 6 named separately on these cards. Someone is to take a card from each box without looking.

Organization and presentation of data

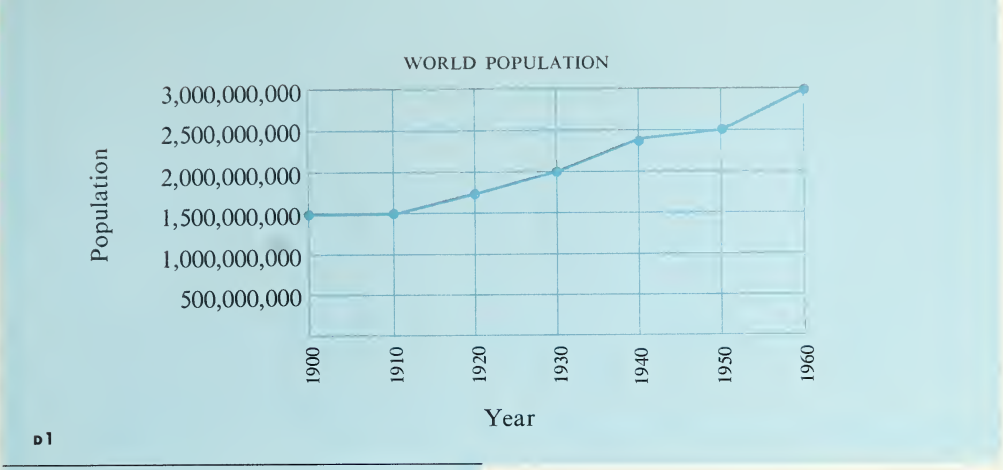
In lesson 182 you learned that you can use collected data to make reasonable estimates about the probability of an event. We refer to such data as *statistical* (stə tis′tə kəl) *data*. Statistical data often are easier to use when you have arranged them in a chart or a graph. In this lesson you will first study some of the kinds of graphs that commonly appear in newspapers, magazines, and books. As you do this, you will learn some useful ways in which to organize statistical data.

A The graph in D1 is a graph of data concerning the world's population. The numerals along the bottom of the graph represent certain years covered by the data in the graph. What is the first year covered? What is the last year? How many ten-year periods are covered?

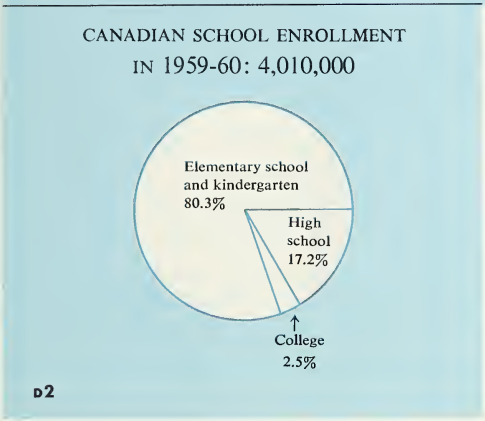
B The numerals at the left of the graph express the world's population. Do you think that the number expressed by each of these numerals is an approximation? Explain your answer.

Notice that the graph contains dots for each of the years 1900, 1910, 1920, 1930, 1940, 1950, and 1960. The location of the dot for 1900 shows that the world's population was about 1,500,000,000 in 1900.

C What was the world's approximate population in 1910? What was the world's approximate population in each of the other years named in the graph?



D1



D2

The dots were used to make the picture of the curve in the graph in D1. Graphs like this are usually called *line graphs*.

D From D1, does the world's population appear to be greater in 1910 than in 1900? Does it appear to be about the same in both years?

E Of all the ten-year periods covered by the graph, which of the periods represents the greatest change in the world's population? Which represents the least change?

F Between 1910 and 1930, the world's population increased by approximately how much?

G Between 1930 and 1950, what was the approximate increase in the world's population?

H How does the approximate increase from 1910 to 1930 compare with the approximate increase from 1930 to 1950?

I Based upon the data presented in the graph in D1, which of the following statements seems more reasonable? "The world's population will probably be greater in 1970 than in 1960." "The world's population will probably be less in 1970 than in 1960."

Now look at the graph in D2. Graphs like this are often called *circle graphs*.

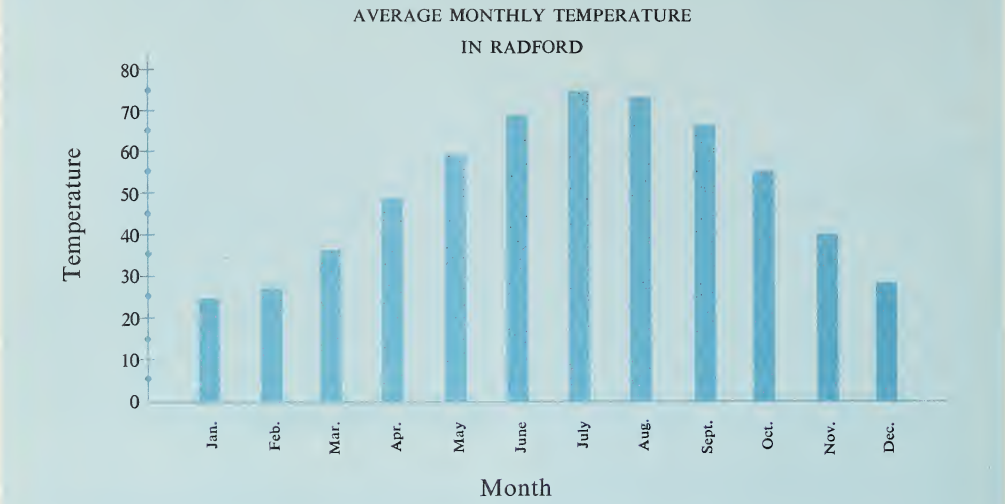
J The picture of the circle and its interior represents the total school enrollment of 4,010,000 persons in 1959-60. What per cent of the total school enrollment is represented by the picture of the circle and its interior?

K Each of the per cents expressed in the graph in D2 is a per cent of what number?

L Look at the portion of the graph that represents the elementary school and kindergarten enrollment. Does this portion correspond to the greatest per cent?

M Does the portion of the graph that represents the college enrollment correspond to the least per cent?

N What per cent of the total enrollment is high school enrollment?



D3

Look at the graph in D3. Graphs like this are called *bar graphs*. This graph shows the average temperature in Radford for each month of the year. Notice the abbreviations for the names of the months at the bottom of the graph. The numerals at the left of the graph express temperatures in degrees Fahrenheit.

O Look at the bar above the name of the first month. Does the top of this bar appear to be aligned with the dot that is halfway between the numerals 20 and 30 on the left? This means that the average temperature in Radford in January is about 25° F.

P What is the approximate average temperature in Radford in February? In May? In September?

Q In which month is the average temperature highest? In which month is the average temperature lowest?

Mr. Monroe had some copies of 5 new phonograph records to sell in his record shop. During the first week he sold 25 copies of record A, 32 copies of record B, 18 copies of record C, 68 copies of record D, and 46 copies of record E.

D4

Record	Number of copies sold
A	25
B	32
C	18
D	68
E	46

D5

Next you will see how you can arrange statistical data so that the data are useful for making predictions about probabilities. First you will use the data given in D4 to learn how to organize data. There are various ways of organizing these data. One way is to make a chart like the one in D5. Study D5.

A Does the number named for each record in D5 correspond to the data given in D4?

B Mr. Monroe sold the most copies of which record? He sold the fewest copies of which record?

Notice that it is easier to compare the sales of the 5 different records by using the chart in D5 than by reading the data as it is expressed in D4. The chart in D5 is called a *frequency array* (ə rā′). A frequency array is an arrangement or organization of data according to the frequency of the events or measures included in the data.

C One way that Mr. Monroe might use the data expressed in the frequency array in D5 is to determine the number of copies of each record he should order in the future. Based upon the frequency array, should he order more copies of record A or of record B? Should he order more copies of record B or of record E? Should he order more copies of record D or of record E?

Now read the experiment described in D6.

The data given in D6 have been organized as a frequency array in D7. Study D7.

D In the frequency array, which numeral expresses the total number of red disks drawn by all four students? How was this number obtained?

E How was each of the other numbers named in the frequency array determined?

F Each of the four students drew 50 disks. How many disks did all four students draw? What is the sum of the numbers expressed in the frequency array?

G Have all of the drawings been included in the frequency array?

H From the information given in the frequency array, can you conclude that there were probably more white disks than disks of any other color in the bag?

Four members of the eighth-grade mathematics class at Evans School took part in the following experiment: The students were given a paper bag that contained some red, white, blue, and green disks. Each of the four students drew 1 disk from the bag. He then observed the color of the disk drawn, replaced the disk, and shook the bag. Each student made a total of 50 such drawings and kept a record of the color of each disk drawn.

Marge recorded 12 red, 13 blue, 16 white, and 9 green.

Terry recorded 17 red, 8 blue, 23 white, and 2 green.

Allen recorded 18 red, 9 blue, 16 white, and 7 green.

Sally recorded 17 red, 8 blue, 20 white, and 5 green.

D6

Disk	Number of times drawn
Red	64
Blue	38
White	75
Green	23

D7

I From the information given in the frequency array, what can you conclude about the number of green disks in the bag?

J Suppose that each of the four students were to draw 100 more disks in the same way. How can you use the frequency array in D7 to make predictions about the probabilities associated with this experiment?

You can also use a frequency array to make a graph. The frequency array in D8 shows the results of a telephone survey to determine the number of persons who were watching four different television programs on a certain evening. The column with the heading "Frequency" refers to the number of viewers of each program. Study D8.

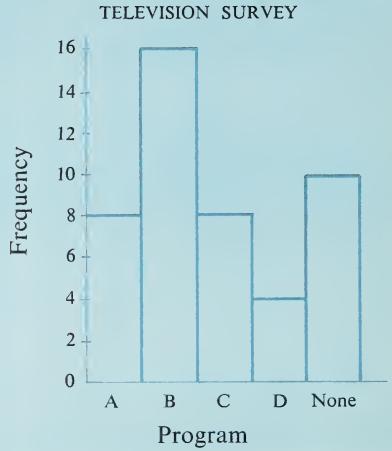
Program	Frequency
A	8
B	16
C	8
D	4
None	10

D8

K How many persons in all are included in the data in the frequency array?

L Among the persons who were called, which program was being watched by the most viewers? By the fewest viewers?

Now look at the graph in D9. This graph is a picture of the data presented in the frequency array in D8.



D9

M The graph in D9 is called a *histogram* (his′tə gram′). How does a histogram differ from a bar graph?

N Suppose that the numerals at the left of the histogram were omitted. Would you still be able to tell which program had the most viewers? Would you still be able to tell which program had the fewest viewers? Would you still be able to tell that programs A and C had the same number of viewers?

O Suppose that an advertiser wants to advertise the fact that program B was the most popular program. Give some reasons why he might use a histogram rather than a frequency array in an advertisement.

You learned in this lesson that statistical data can be organized and presented in many different ways. You also learned that the way in which you organize statistical data is often determined by the use you want to make of the data.

On your own

Use the graphs in D10 in connection with exercises 1 through 9.

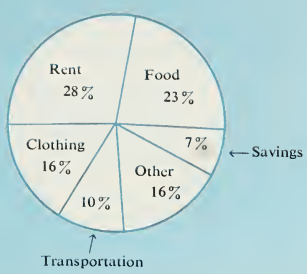
1 The data presented in the circle graph in D10 concern the amount of income that the Perry family spends on various items. What per cent of the Perrys' income is represented by the picture of the circle and its interior?

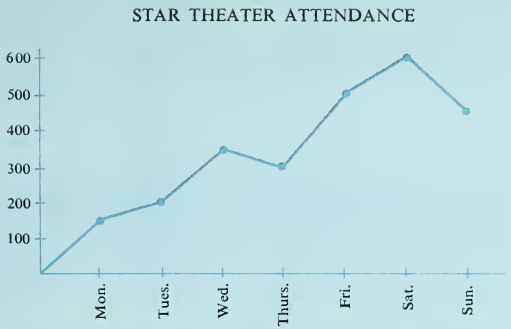
2 Do the Perrys spend a greater per cent of their income for food or for rent?

3 For which do the Perrys spend more, clothing or transportation?

4 What are some expenses that might be included under the heading "Other"?

5 Suppose that the Perrys' monthly income is $500. Use the per cents expressed in the circle graph to determine the amount they spend for each of the items named.





D10

6 Now look at the line graph in D10. What kind of data is presented in this graph?

7 The abbreviations at the bottom of the graph name the days of the week. The numerals at the left express the number of persons attending the theater. On which day did the greatest number of persons attend the theater? On which day did the least number of persons attend?

8 On which day did more persons attend the theater, on Wednesday or on Thursday? On Monday or on Tuesday?

9 Suppose that there is a popcorn concession in the lobby of the theater. How might the owner of the concession use the data presented in this graph to determine the amount of popcorn that he will need for each evening of the week?

In the following exercises, you will use statistical data to make frequency arrays and graphs.

10 30 of the eighth-grade students at Hooker School bring lunches, 75 buy lunch in the school cafeteria, and 15 go home for lunch. Use this information to make a frequency array.

11 Use the frequency array you made for exercise 10 to make a histogram.

12 As part of a probability experiment in his mathematics class, Clarence asked 75 persons to name a number from 1 through 9. He found that 4 persons named 1; 6 persons named 2; 15 persons named 3; 7 persons named 4; 12 persons named 5; 5 persons named 6; 13 persons named 7; 7 persons named 8; and 6 persons named 9. Use this information to make a frequency array.

13 Use the frequency array you made for exercise 12 to make a histogram.

KEEPING SKILFUL

For each condition expressed below, tabulate the solution set. The universe for (x, y) is $D \times D$.

1 $x = {}^{-}5y \wedge \frac{3}{5}y = 12\frac{3}{4}$.

2 $x - {}^{-}8.6 = 20.4 \wedge y = 2.5 + x$.

3 $y = \frac{4}{5}x \wedge 2x + y = 56$.

4 $x + y = {}^{-}8.36 \wedge 12y = {}^{-}.72$.

5 $y = 1 + x \wedge 10x = 3(x + 4\frac{2}{3})$.

6 $7y + 3 = x \wedge x + 20y = 84$.

7 $y = .2x \wedge 2(x + 4) = 32.6$.

8 $x = y \wedge 4x + 10 = 12y + 58$.

9 $x - 16.5 = 12.25 \wedge x + y = 4$.

10 $x = y - 2 \wedge x + 4y = 683$.

11 $y = x - 5 \wedge 6x + 3\frac{1}{3} = 4y$.

The arithmetic mean

In lesson 183 you learned that a frequency array is a useful way of organizing and presenting statistical data. In this lesson you will see how the arithmetic mean provides useful information about statistical data.

A Read the problem in D1. On which day did the greatest number of students attend school? On which day did the least number of students attend school? What is the absolute difference of these two numbers?

59, the absolute difference of 544 and 485, is the *range* of the set of data given in D1. The range of a set of data is the absolute difference of the greatest and least measures included in the data. We often use the word "measure" to refer to the numbers expressed in statistical data, even though these numbers may have been obtained by counting rather than by measurement.

B In lesson 174 you learned how to use the formula $a = \frac{t}{n}$ for finding averages. In $a = \frac{t}{n}$, for what is a a variable? For what is t a variable? For what is n a variable?

You can use the average formula to help you develop a condition for the problem in D1. The universe for each variable is D.

C What is the replacement for n in $a = \frac{t}{n}$? How can you find the replacement for t? The sentence in blue below expresses a condition for the problem in D1.

$$a = \frac{t}{5} \wedge t = 485 + 538 + 544 + 535 + 492.$$

During one week at the McKay School, 485 students attended school on Monday, 538 attended on Tuesday, 544 attended on Wednesday, 535 attended on Thursday, and 492 attended on Friday. What was the average daily attendance at McKay School for the week?

D1

Jim made the following scores on 6 science tests: 87, 92, 78, 90, 86, 94. What was the mean of Jim's scores?

D2

range (rānj). For any set of data, the absolute difference of the greatest and least measures.

arithmetic mean. For any set of data, the quotient of the sum of the measures and the number of measures.

D What number satisfies the simple condition $t = 485 + 538 + 544 + 535 + 492$?

E What number satisfies $a = \frac{2594}{5}$?

F Give the answer to the problem in D1.

As you learned in *Book 1*, another name for the average is the *arithmetic mean*. The arithmetic mean of a set of data is the quotient of the sum of the measures and the number of measures. Often we will use just the word "mean" to refer to the arithmetic mean.

For a set of statistical data, the mean of the measures is one good way of describing and

comparing measures. The mean is one indication of the "central" measure with which each measure in a set of data can be compared.

G Now read the problem in D2. What was Jim's highest score? What was his lowest score? What was the range of Jim's scores?

Instead of using a compound condition for the problem in D2, we will develop a new formula for finding the arithmetic mean. In this formula, we will use a variable for each measure in a set of measures. We will use x_1 as a variable for the first measure, x_2 as a variable for the second measure, x_3 as a variable for the third measure, and so on. We will use x_n as a variable for the nth, or last, measure.

H You read the symbol "x_1" as "x one." How do you read x_2? x_{100}? x_n?

You can use the symbol in D3 to express the sum of the elements of $\{x_1, x_2, x_3, \ldots, x_n\}$.

I Suppose that there are 4 measures in a set of data. Does $x_1 + x_2 + x_3 + x_4$ represent the sum of these measures? How can you represent the sum of the measures if there are 12 measures in a set of data?

In expressing the formula for the arithmetic mean, we will use the symbol below at the right to name the variable for the mean. The symbol is read "x bar."

$\bar{x}$

J Now look at D4. You know that sentence A expresses a formula for finding the average. Is $\bar{x}$ a variable for the same number as a?

K For what is t a variable in the formula for the average? Does $x_1 + x_2 + x_3 + \ldots + x_n$ represent the same number as t?

L Explain how condition B, expressed in D4, was obtained from condition A.

The condition $\bar{x} = \dfrac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$ is a formula for finding the arithmetic mean of the measures in a set of data.

Universe for x = D.
Universe for $n = I_p$.
$x_1 + x_2 + x_3 + \ldots + x_n$

D3

Universe for x = D.
Universe for $n = I_p$.

A $a = \dfrac{t}{n}$.

B $\bar{x} = \dfrac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$.

D4

The Midvale Senior High School play was presented on 3 different evenings. For the 3 performances of the play, 405 persons attended on the first evening, 461 on the second evening, and 490 on the third evening. What was the mean number of persons attending the play per evening?

D5

M Read the problem in D2 again. The sentence ? in blue below expresses the condition for this problem. How was this condition obtained from the formula for finding the mean?

$$\bar{x} = \frac{87 + 92 + 78 + 90 + 86 + 94}{6}.$$

N How do you obtain $\bar{x} = \dfrac{527}{6}$ from the condition expressed in exercise M? What number satisfies $\bar{x} = \dfrac{527}{6}$? Is $87\frac{5}{6}$ the mean score that Jim made on the six tests?

Next you will study a property of the arithmetic mean. To illustrate this property, we will use the problem in D5.

A Use the formula for finding the mean to find the answer to the problem in D5.

B Now subtract the mean from the number of persons attending the play on each evening. Is $405 - 452$, or $^{-}47$, the difference of the attendance on the first evening and the mean? What is the difference of the attendance on the second evening and the mean? On the third evening and the mean?

C What is the sum of the differences that you named for exercise B?

In exercises B and C, you studied an example of a useful property of the mean. The property is expressed below.

For any set of data, the sum of the numbers that are obtained by finding the difference of each measure in the set and the arithmetic mean is equal to 0.

Find the mean of each set of measures expressed in exercises D and E. Then verify that the sum of the differences of the measures and the mean is equal to 0.

D $5\frac{2}{3}, 3\frac{3}{4}, 8\frac{1}{2}, 7\frac{2}{3}, 6\frac{1}{2}$

E 610, 552, 484, 729, 692, 477

In the following exercises, you will study another way of describing the measures in a set of data. The data presented in the frequency array in D6 concern the test scores of students. Notice that the number of students with the lowest score is named first, and the number with the highest score is named last.

F The scores of how many students are included in the data? Explain why the middle score would be the score of the twelfth student.

G Explain why we say that the score of the twelfth student is 85.

85 is the *median* score of the measures expressed in D6. If the members of a set of measures are arranged in order from the least to the greatest or from the greatest to the least, then the median is the measure in the middle.

The Midvale Senior High School play was presented on 3 different evenings. For the 3 performances of the play, 405 persons attended on the first evening, 461 on the second evening, and 490 on the third evening. What was the mean number of persons attending the play per evening?

D5

Score	Frequency
75	2
78	1
83	6
85	4
89	5
93	4
97	1
Total	23

D6

Mrs. Henderson has 32 students in her class. She found that 4 of her students have no brothers or sisters; 7 have 1 brother or sister; 13 have 2 brothers or sisters; 2 have 3 brothers or sisters; 5 have 4 brothers or sisters; and 1 has 6 brothers or sisters.

D7

H You know that, to find the mean of several measures, you must first find the sum of the measures. Does the data given in D6 include 23 measures? What does $2 \cdot 75$ represent?

I Is $(2 \cdot 75) + (1 \cdot 78) + (6 \cdot 83)$ the sum of the first 9 measures? Find the sum of the 23 measures.

J What is the mean of the scores expressed in D6?

K What is the range of the set of measures expressed below?

5.3, 4.1, 7.6, 6.7, 5.3, 8.2, 7.3

L Find the mean of the measures expressed in exercise K. Find the median.

The mean and the median represent two different ways of describing a set of data in terms of the "central" measure of the set.

In this lesson you studied a formula for finding the arithmetic mean of a set of measures. You also learned that the range is the absolute difference of the greatest and least measures and that the median is the middle measure.

On your own

Use the data in D7 in connection with exercises 1 through 4.

1 What is the range of the measures expressed in D7?

2 How many measures in all are expressed in D7?

3 What is the mean number of brothers or sisters of the students in Mrs. Henderson's class?

4 What is the median of the measures expressed in D7?

THINK Why should you use the sixteenth and seventeenth students to determine the median?

A set of measures is expressed in each of exercises 5 through 10. For each exercise, determine the range, the mean, and the median.

5 \$5000, \$2000, \$3500, \$5400, \$3500

6 4.6, 2.8, 5.2, 3.3, 4.6, 7.9, 4.6, 2.8, 8.4, 5.2

7 $7\frac{1}{2}$, $8\frac{1}{3}$, $8\frac{1}{3}$, $5\frac{3}{8}$, $7\frac{1}{2}$, $8\frac{1}{3}$, $8\frac{1}{3}$, $5\frac{3}{8}$, 6, $7\frac{2}{3}$

8 16, 25, 32, 28, 18, 25, 19, 32

9 .02, .01, .024, .03, .026, .022, .031

10 175, 176, 169, 178, 182, 195, 171, 176, 176, 182, 174

APPLYING MATHEMATICS

For each problem, first write a sentence that expresses a condition. If possible, tabulate the solution set of the condition. Otherwise, give a standard description of the solution set. Then give the answer to the problem. The universe for each variable is D.

1 Roger has \$130.95 in his savings account, \$16.42 in a bank at home, and \$1.90 in his wallet. How much money does Roger have altogether?

2 Last year Mr. Martin spent \$145.35 for auto repairs, and Mr. Nelson spent \$116.28. The amount that Mr. Martin spent for repairs is how many times the amount that Mr. Nelson spent?

3 Ken worked 10 hr. at Hayes' Service Station on Friday and Saturday. If he had worked $\frac{3}{4}$ hr. more on Saturday, he would have worked 6 hr. that day. How many hours did Ken work on Friday and how many hours did he work on Saturday?

4 Last year 875 students were enrolled in Carlson High School. .8% of these students had perfect attendance. How many students at Carlson High School had perfect attendance last year?

5 Mrs. Wilson had $2\frac{1}{2}$ lb. of shelled hickory nuts. After she had used some of these nuts in a hickory nut cake, she still had more than $2\frac{1}{8}$ lb. of nuts left. How many pounds of hickory nuts did Mrs. Wilson use in the cake?

6 During March, Mr. Craft sold 152 men's jackets. Altogether, during March and April, he sold 321 men's jackets. How many men's jackets did Mr. Craft sell during April?

7 At 6:00 P.M. the temperature in Toronto was warmer than 10 degrees below zero. By 10:00 P.M. the temperature had dropped 4 degrees and was colder than 6 degrees below

zero. What was the temperature in Toronto at 6:00 P.M.?

8 Beth had more than 8 oz. of mints. She then bought a $10\frac{1}{2}$-oz. package of mints, but still had less than 24 oz. How many ounces of mints did Beth have before she bought the $10\frac{1}{2}$-oz. package?

9 The Harris Paper Company received an order for 3300 envelopes. The envelopes are available in packages of 100 and packages of 275. How many packages of envelopes, each containing the same number of envelopes, should the company send to the customer?

10 Last week Sally earned $12.80. Sally earned $\frac{4}{5}$ as much as Betty. Together, Betty and Alice earned $25.40. How much money did Betty earn and how much money did Alice earn?

11 Mrs. Andrews bought 3 qt. of floor wax. After she had used some, but not all, of this wax, she had at most $1\frac{1}{8}$ qt. of wax left. How many quarts of wax had Mrs. Andrews used?

12 A school bought 15 tennis rackets priced at $9.95 each and 18 tennis balls priced at 3 for $2.25. How much did the school spend in all for the tennis rackets and the tennis balls?

13 Mrs. Robin bought hamburger, bacon, and pork chops. The total weight of the meat was $9\frac{3}{4}$ lb. The hamburger weighed 2 times as much as the bacon. The total weight of the bacon and pork chops was $5\frac{3}{4}$ lb. How many pounds of each kind of meat did Mrs. Robin buy?

14 From what numbers can you subtract $16\frac{8}{9}$ so that each difference is at least $20\frac{1}{3}$?

15 A first number is 3 times a second number. The difference of the second number and 4 is 2.5. A third number is the sum of the first number and 9.5. What are the three numbers?

16 The difference of a first number and a second number is not greater than $^{-}6$. The sum of the second number and 2 is 10. What are the two numbers?

CHECKING UP

The small numerals within parentheses tell what pages to turn to for help if you need it.

Test 167

For exercises 1 through 10, decide what word or words best complete each exercise.

1 Each element of a sample space is a ~~~. (402)

2 The probability of $C \cap D$ is the probability that event C ~~~ event D will occur. (407)

3 If C and D are ~~~ events, then the probability that C or D will occur is equal to the sum of the probability of C and the probability of D. (409)

4 A probability that is determined by making a prediction based upon observations is an ~~~ probability. (415)

5 $\bar{x} = \dfrac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$ is a formula for determining the ~~~. (427)

6 If the number of trials is great, the relative frequency of an event can be used as an approximation of the ~~~ of the event. (416)

7 In a set of data, the absolute difference of the least measure and the greatest measure is the ~~~ of the measures. (426)

8 The ~~~ of an event in a given number of trials is the quotient of the number of times the event has occurred and the number of trials. (404)

9 If M and N are subsets of a finite sample space, and if $P(M \cap N) = P(M) \cdot P(N)$, then M and N are ~~~ events. (413)

10 The sum of the probabilities of two complementary events is ~~~. (410)

Test 168

Use the experiment described in D1 in connection with exercises 11 through 15.

11 Make a chart to represent a sample space of the experiment described. The sample space

Code numbers are often assigned to merchandise. Suppose that the first digit of a code is one of the numerals from 0 through 4 and the last digit is one of the numerals from 1 through 4.

D1

A scientist weighed some mice that he planned to use in an experiment. 3 of the mice weighed 22.8 g. each, 2 weighed 23.0 g. each, 4 weighed 23.2 g. each, 2 weighed 22.5 g. each, 1 weighed 24.1 g., and 1 weighed 23.5 g.

D2

should include all possible pairs of numbers expressed by pairs of first and last digits. (402)

12 What is the probability that the first and last numbers are the same? (413)

13 What is the probability that the first number is 3? That the last number is 2? Are these independent events? (413)

14 What is the probability that the first number is 1? That the sum of the two numbers is 6? Are these mutually exclusive events? (409)

15 What is the probability that the first number is greater than 2? That the sum of the two numbers is less than 4? Are these independent events? (413)

Use the data expressed in D2 in connection with exercises 16 through 20.

16 What is the range of the data given? (426)

17 Make a frequency array of the data. (423)

18 What is the arithmetic mean of the measures expressed? (427)

19 What is the median of the measures expressed? (428)

20 Make a histogram of the data given. (424)

21 The probability of T is $\frac{1}{5}$. What is the probability of event $\overline{T}$? (410)

22 Events P and Q are complementary events. What is the probability that event P or event Q will occur? (410)

23 Two baseball teams played a 4-game series. The attendance at the first game was 25,750; at the second game, 14,600; at the third game, 18,025; and at the fourth game, 23,450. Make a line graph to represent the attendance for the 4-game series. (421)

24 The data in a mortality table are based upon a sample of 100,000 persons who were living at age 1. Of the persons in the sample, 93,482 persons were still living at age 30. Based upon the data in this table, find the probability that someone who is now 1 year old will live to be 30 years old. Find the probability to the nearer thousandth. (417)

CHECKING UP

The small numerals within parentheses tell what pages to turn to for help if you need it.

Test 169

Seven subsets of the set of real numbers are listed below. Use the list for exercises 1 through 6. In each of these exercises, a real number is named. From the list, choose every set of which the number is an element.

The set of rational numbers (R)
The set of positive rational numbers (R_p)
The set of negative rational numbers (R_n)
The set of integers (I)
The set of positive integers (I_p)
The set of negative integers (I_n)
The set of irrational numbers ($\overline{R}$)

1 42 (262)
2 $^{-}6$ (262)
3 .101001 . . . (262)
4 $4\frac{3}{275}$ (262)
5 $^{-}\sqrt{67}$ (262)
6 $^{-}77.311\ldots$ (262)

Test 170

Write "T" for each sentence below that expresses a true statement. Write "F" for each sentence that expresses a false statement.

7 If point D is the midpoint of $\overline{AK}$, then $\overline{AD} \cong \overline{DK}$. (126)

8 34^{-2} is a positive integral power. (224)

9 $y = \frac{4}{x}$ is a condition for direct variation. (384, 355)

10 The quotient of $^{-}63$ and $^{-}7$ is $^{+}9$. (211)

11 The opposite sides of a rhombus are parallel. (124)

12 For any two complementary events, one event is a subset of the other event. (409)

Test 171

For each of the following exercises, decide what symbol, word, or words best complete the exercise.

13 The least solution of $x \approx_{\frac{1}{8}} 14$ is ~~~~. U = D. (90)

14 If the intersection of ℓ_4 and $\odot$P contains exactly 2 points, then ℓ_4 is a ~~~~ of $\odot$P. (115)

15 $23^6 \div 23^{-5} =$ ~~~~. (231)

16 The numeral 56000 $\pm$ 50 has ~~~~ significant digits. (316)

Make a graph of each solution set described in the following exercises.

17 $\{x \mid 3\frac{2}{3} \leqq x\}$ U = D. (283)

18 $\{x \mid \sim (x - \frac{1}{4} \geqq 5\frac{1}{4})\}$ U = D. (283)

19 $\{(x, y) \mid x = 3y + 2\}$ U = D $\times$ D. (287)

20 $\{(x, y) \mid y > 6x\}$ U = D $\times$ D. (288)

21 $\{(x, y) \mid y = 3x \wedge x + y = 7\}$ U = D $\times$ D. (289)

Test 172a

For each problem, write a sentence that expresses a condition. Tabulate the solution set of the condition and give the answer to the problem. The universe for each variable is D.

22 A certain company gives 1 hour of paid vacation for each $17\frac{1}{3}$ hours that an employee works. The paid vacation time is what per cent of the hours worked? (Hundredths) (373)

23 On December 31, Mr. Humphrey had \$473.50 in his savings account at a bank. The following day, the bank added interest for the year to his account at the rate of 4% per year. How much money should he have had in his account after the interest was added? (380)

24 Mr. Zaret fills the radiator of his car with antifreeze at the rate of 2 quarts of antifreeze to 5 quarts of water. The capacity of the radiator is 16 quarts. How much antifreeze and how much water should he use? (296)

25 The quotient of what number and $^{-}3$ is equal to $7\frac{1}{4}$? (295)

26 A first number is equal to the product of $^{-}.87$ and a second number. The second number is the difference of 4.1 and 6.9. What are the numbers? (296)

27 The sum of three numbers is $^{-}13\frac{1}{2}$. The second number is 3 times the first number. The third number is $\frac{1}{2}$ times the first number. What are the numbers? (296)

Test 172b

What words or symbols best complete exercises 28 through 35?

28 Two angles are ~~~~ angles if the sum of their measures is 90. (273)

29 In which quadrants is the locus of the solution set of $y = \frac{-6}{2}$? (301)

30 What y satisfies $y = {}^{-}2x^2$ if x is replaced by $\frac{-2}{3}$? (303)

31 What is the area of a triangle whose altitude is 3 in. and whose base is 7 in.? (329)

32 An ~~~~ is a subset of a sample space. (456)

33 The union of two ~~~~ events is the entire sample space. 461)

34 Events R and S are independent. If P(R) = $\frac{2}{3}$ and P(S) = $\frac{1}{4}$, then P(R $\cap$ S) = ~~~~. (464)

35 The arithmetic mean of the measures 26, 32, 46, 35, and 41 is ~~~~. (478)

Symbols and how to read them

These displays show you how to read the mathematical symbols used in this book. They are arranged in the order in which they were introduced. Whenever, in your work, you come to a symbol that you have forgotten how to read, look for one like it among those on the pages in this section. If you need more information about the symbol, turn to the page of the book whose numeral is given above the symbol. There you will find a complete explanation of the meaning of the symbol.

book 1 page 11

"The set whose members are

$\{ \text{Michigan, Huron, Superior, Erie, Ontario} \}$

Michigan, Huron, Superior, Erie, Ontario"

book 1 page 16

"x plus six equals

$x + 6 = 9.$

nine."

book 1 page 19

"K equals the set whose members are

$K = \{ 5, 6, 7, 8, 9, 10 \}.$

five, six, seven, eight, nine, ten."

book 1 page 20

"x plus two is greater than

$x + 2 > 6.$

six."

book 1 page 20

"Two times n is less than

$2n < 9.$

nine."

book 1 page 22

"The set whose members are

$\{ 0, 1, 2, \ldots, 1000 \}$

the natural numbers from 0 through 1000"

book 1 page 23

"The set whose members are

$\{ 0, 1, 2, \ldots \}$

the natural numbers"

book 1 page 27

"The set whose members are all x that satisfy the condition that x is less than six"

$\{ x \mid x < 6 \}$

book 1 page 33

$\overleftrightarrow{RS}$

"Line RS"

book 1 page 38

$\overline{XZ}$

"Segment XZ"

book 1 page 39

$\overline{AB} \cong \overline{CD}.$

"Segment AB is congruent to segment CD."

book 1 page 83

"The intersection of

$\{1, 2, 3, 4\} \cap \{2, 4, 6\}$

the set whose members are one, two, three, four and the set whose members are two, four, six"

book 1 page 83

"The intersection of C and D equals

$C \cap D = \{2, 4\}.$

the set whose members are two, four."

book 1 page 87

"The union of

$\{5, 10\} \cup \{6, 8, 10\}$

the set whose members are five, ten and the set whose members are six, eight, ten"

book 1 page 87

"The union of C and D equals

$C \cup D = \{5, 6, 8, 10\}.$

the set whose members are five, six, eight, ten."

book 1 page 91

"x plus three is greater than seven and

$x + 3 > 7 \wedge 6 + x < 14.$

six plus x is less than fourteen."

book 1 page 91

"The intersection of

$\{x|x+3>7\} \cap \{x|6+x<14\}$

the solution set of $x+3>7$ and

the solution set of $6+x<14$"

book 1 page 104

"Ray $\overrightarrow{AB}$

AB"

book 1 page 105

"Angle $\angle ABC$

ABC"

book 1 page 109

"Triangle $\triangle ABC$

ABC"

book 1 page 123

"The ordered pair (Paris, Spain)

whose first component is Paris and

whose second component is Spain"

book 1 page 124

"The ordered pair (1, 2)

one, two"

book 1 page 127

"A $A \times B$ cross B"

book 1 page 128

"The ordered pair (a, b)

a, b"

book 1 page 138

"The set whose members are $\{ (x, y) \mid x>y \}$

all ordered pairs x, y

that satisfy the condition that

x is greater than y"

book 1 page 139

"Point point (1, 4)

1, 4"

book 1 page 169

"The ordered pair two, seven

is equivalent to $(2, 7) \sim (4, 14)$.

the ordered pair four, fourteen."

book 1 page 174

"One to fifteen

is equivalent to $1/15 \sim 2/30$.

two to thirty."

book 1 page 174

"The set whose members are $\{ 4/7, \ldots \}$

the rate pair 4/7 and all rate pairs equivalent to 4/7"

book 1 page 175

"The set whose members are $\{ (x, y) \mid x/y \sim 4/7 \}$

the ordered pairs x, y

that satisfy the condition that

x to y

is equivalent to

four to seven"

book 1 page 187

"Eighty-eight 88 %

per cent"

book 1 page 216

"one zero three three, 1033_{four}

base four"

book 1 page 227

"The second power 10^2

of ten"

book 1 page 238

256, 193, 462, 109

"256 billion 193 million 462 thousand 109"

book 1 page 249

"The natural number 0 equals the number of members in the empty set."

$0 = n\{\}$.

book 1 page 257

"The number of members in A"

$n(A)$

book 1 page 257

"The number of members in the union of A and B"

$n(A \cup B)$

book 1 page 262

"The number of members in D cross E"

$n(D \times E)$

book 1 page 311

"The fraction $\frac{a}{b}$ over b"

book 1 page 311

"The fraction one third is equivalent to the fraction five fifteenths."

$\frac{1}{3} \sim \frac{5}{15}$.

book 1 page 314

"The set whose members are $\{\frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \ldots\}$ $\frac{1}{5}$ and all fractions equivalent to $\frac{1}{5}$"

book 1 page 371

"972 and 564 thousandths"

972.564

book 1 page 372

"57 hundredths"

$.57$

book 1 page 385

"Point six, three repetend six, three"

$.\overline{63}$

book 1 page 385

"Point one, six repetend six"

$.1\overline{6}$

book 2 page 5

"The measure of segment AC is equal to one and one half."

$m(\overline{AC}) = 1\frac{1}{2}$.

book 2 page 21

"The measure in degrees of angle ABC is equal to 40."

$\angle ABC^\circ = 40$.

book 2 page 67

"Eleven fourths is less than y is less than seventeen fourths."

$\frac{11}{4} < y < \frac{17}{4}$.

book 2 page 69

"s plus two is less than seven or s plus four is greater than six."

$s + 2 < 7 \vee s + 4 > 6$.

book 2 page 70

"m is greater than or equal to four and one half."

$m \geqq 4\frac{1}{2}$.

book 2 page 71

"x is less than or equal to three and twenty-five hundredths."

$x \leqq 3.25$.

book 2 page 78

"The complement of

$\overline{\{1, 4, 5\}} = \{2, 3\}.$

the set whose members are one, four, five

is equal to

the set whose members are two, three."

book 2 page 79

"The complement of M

$\overline{M} = \{2, 3\}.$

is equal to

the set whose members are two, three."

book 2 page 80

"It is not the case that

$\sim (x > 6).$

x is greater than six."

book 2 page 87

"Seventeen

is a member of

$17 \in \{a | a > 8\frac{3}{5}\}.$

the solution set of $a > 8\frac{3}{5}$."

book 2 page 88

"x

is approximately equal to

$x \approx 7.$

seven."

book 2 page 90

"x

is approximately equal to

$x \approx_1 7.$

seven

with a tolerance of one."

book 2 page 90

$U = Z.$

"x

is approximately equal to

$x \approx_t k.$

k

with a tolerance of t."

book 2 page 92

"The ordered triple

$(4, 2, 6)$

four, two, six"

book 2 page 93

"The set whose members are

$\{(x, y, z) \mid x \in Q \wedge y \in V \wedge z \in W\}$

all ordered triples x, y, z

that satisfy the condition that

x is a member of Q

and y is a member of V

and z is a member of W"

book 2 page 114

"Circle

$\odot S$

S"

book 2 page 118

"Arc

$\overset{\frown}{EHF}$

EHF"

book 2 page 119

"The measure in degrees of

$\text{minor } \overset{\frown}{PR}\,^\circ = 70.$

minor arc PR is equal to 70."

book 2 page 122

"Segment XY

$\overline{XY} \parallel \overline{WZ}.$

is parallel to

segment WZ."

book 2 page 123

"Segment EF

$\overline{EF} \perp \overline{FG}.$

is perpendicular to

segment FG."

book 2 page 154

"The directed segment

$\overrightarrow{RS}$

RS"

book 2 page 168

"The set whose members are

$\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, \ldots\}$

the negative integers,

zero,

and the positive integers"

book 2 page 205

"The additive inverse of

$^{-}(+3)$

positive three"

book 2 page 256

"The principal square root of

$\sqrt{a}$

a"

book 2 page 313

"A length of 150 ft.

$150 \text{ ft.} \pm 5$

with a greatest possible error of 5 ft."

book 2 page 326

"Triangle ABC

$\triangle ABC \sim \triangle DEF.$

is similar to

triangle DEF."

book 2 page 334

"The tangent of 30

$\tan 30 \approx .5774.$

is approximately equal to

.5774."

book 2 page 399

"The probability

$P(\text{blue}) = \frac{1}{2}.$

of taking the blue marble

is equal to

one half."

The displays below and on the next page explain the symbols used in this book for the various sets of numbers and their subsets.

$\mathbb{N}$ This symbol names the set of natural numbers. The symbol is read "the set of natural numbers." You learned about this set in lesson 6, *Book 1*, pages 22, 23, and 24.

$\mathbb{C}$ This symbol names the set of counting numbers. The symbol is read "the set of counting numbers." You learned about this set in lesson 20, *Book 1*, page 73.

$\mathbb{R}_a$ This symbol names the set of rational numbers of arithmetic. The symbol is read "the set of rational numbers of arithmetic." You learned about this set in lesson 73, *Book 1*, page 318.

$\mathbb{Z}$ This symbol names the set of non-zero rational numbers of arithmetic. The symbol is read "the set of non-zero rational numbers of arithmetic." You learned about this set in lesson 85, *Book 1*, page 367.

$\mathbb{R}$ This symbol names the set of rational numbers. The symbol is read "the set of rational numbers." You learned about this set in lesson 128, *Book 2*, page 163.

$\mathbb{R}_p$ This symbol names the set of positive rational numbers. The symbol is read "the set of positive rational numbers." You learned about this set in lesson 127, *Book 2*, pages 157 and 158.

$\mathbb{R}_n$ This symbol names the set of negative rational numbers. The symbol is read "the set of negative rational numbers." You learned about this set in lesson 128, *Book 2*, pages 161 to 163.

$\overline{\mathbb{R}}_n$ This symbol names the set of non-negative rational numbers. The symbol is read "the set of non-negative rational numbers." You learned about this set in lesson 134, *Book 2*, page 187.

$\mathbb{Z}_r$ This symbol names the set of non-zero rational numbers. The symbol is read "the set of non-zero rational numbers." You learned about this set in lesson 140, *Book 2*, page 211.

$\mathbb{I}$ This symbol names the set of integers. The symbol is read "the set of integers." You learned about this set in lesson 129, *Book 2*, page 168.

$\mathbb{I}_p$ This symbol names the set of positive integers. The symbol is read "the set of positive integers." You learned about this set in lesson 129, *Book 2*, page 168.

$\mathbb{I}_n$ This symbol names the set of negative integers. The symbol is read "the set of negative integers." You learned about this set in lesson 129, *Book 2*, page 168.

$\mathbb{D}$ This symbol names the set of real numbers. The symbol is read "the set of real numbers." You learned about this set in lesson 151, *Book 2*, page 263.

$\overline{\mathbb{R}}$ This symbol names the set of irrational numbers. The symbol is read "the set of irrational numbers." You learned about this set in lesson 151, *Book 2*, page 262.

BUSINESS

Cost (373)

$c = np.$

c—total cost
n—number of items
p—price

Interest (381)

$i = prt.$

i—interest
p—principal
r—rate
t—time

Amount (381)

$a = p + i.$

a—amount
p—principal
i—interest

Selling price (381)

$s = c + o + p.$

s—selling price
c—cost
o—overhead
p—profit

SCIENCE

Temperature (384)

$C = \frac{5}{9}(F - 32).$

$F = \frac{9}{5}C + 32.$

C—number of degrees centigrade
F—number of degrees Fahrenheit

Work (382)

$w = fd.$

w—work
f—force
d—distance

Power (382)

$p = \frac{w}{t}.$

p—power
w—work
t—time

Electricity (383)

$I = \frac{E}{R}.$

$W = EI.$

I—number of amperes of current
E—number of volts of pressure
R—number of ohms of resistance
W—number of watts of power

GEOMETRY

Trapezoid

Area (389)

$A = \frac{1}{2}a(b_1 + b_2).$

A—area
a—measure of an altitude
b_1, b_2—measures of the bases

Parallelogram

Area (387)

$A = ab.$

A—area
a—measure of an altitude
b—measure of the base

Rectangle

Perimeter (385)

$p = 2l + 2w.$

p—perimeter
l—length
w—width

Area (385)

$A = lw.$

A—area
l—length
w—width

Square

Perimeter (386)

$p = 4s.$

p—perimeter
s—measure of a side

Area (329)

$A = s^2.$

A—area
s—measure of a side

Triangle

Perimeter (386)

$p = a + b + c.$

p—perimeter
a, b, c—measures of the sides

Area (387)

$A = \frac{1}{2}ab.$

A—area
a—measure of an altitude
b—measure of the base

Equilateral triangle

Perimeter (328)

$p = 3s.$

p—perimeter
s—measure of a side

Trigonometry

$\tan t = \frac{y}{x}.$ (375)

$\sin t = \frac{y}{r}.$ (375)

$\cos t = \frac{x}{r}.$ (375)

t—measure of an acute angle of a right triangle
x—measure of side adjacent to acute angle
y—measure of side opposite acute angle
r—measure of hypotenuse of right triangle

MISCELLANEOUS

Distance (372)

$d = rt.$

d—distance
r—rate
t—time

Percentage (373)

$p = br.$

p—percentage
b—base
r—rate

Average (374)

$a = \frac{t}{n}.$

a—average
t—total
n—number of items

Conversion (374)

$l = kg.$

l—lesser unit of measure
k—constant
g—greater unit of measure

Probability of an event (405)

$P(M) = \frac{m}{n}.$

$P(M)$—probability of the event
m—number of elements in the event
n—number of elements in the sample space

Probability of the union of two events (408)

$P(M \cup N) = P(M) + P(N) - P(M \cap N).$

$P(M \cup N)$—probability of the union of events M and N
$P(M)$—probability of event M
$P(N)$—probability of event N
$P(M \cap N)$—probability of the intersection of events M and N

Probability of the union of two mutually exclusive events (409)

$P(M \cup N) = P(M) + P(N).$

$P(M \cup N)$—probability of the union of events M and N
$P(M)$—probability of event M
$P(N)$—probability of event N

Probability of the complement of an event (410)

$P(\overline{M}) = 1 - P(M).$

$P(\overline{M})$—probability of the complement of event M
$P(M)$—probability of event M

Probability of the intersection of two independent events (413)

$P(M \cap N) = P(M) \cdot P(N)$.

$P(M \cap N)$ — probability of the intersection of events M and N

$P(M)$ — probability of event M

$P(N)$ — probability of event N

Arithmetic mean (427)

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}.$$

$\bar{x}$ — arithmetic mean

$x_1 + x_2 + x_3 + \ldots + x_n$ — sum of n measures

n — number of measures

liquid measures*

2 tablespoonfuls (tbs.)	1 fl. ounce (fl. oz.)
8 fluid ounces (fl. oz.)	1 cup (c.)
2½ cups (c.)	1 pint (pt.)
2 pints (pt.)	1 quart (qt.)
4 quarts (qt.)	1 gallon (gal.)
8 pints	1 gallon

*Canada uses imperial liquid measures.
Canada: 1 pt. = 20 fl. oz.
United States: 1 pt. = 16 fl. oz.

dry measures

2 dry pints	1 dry quart
8 dry quarts	1 peck (pk.)
4 pecks (pk.)	1 bushel (bu.)
32 dry quarts	1 bushel

calendar measures

7 days (da.)	1 week (wk.)
52 weeks (wk.)	1 year (yr.)
12 months (mo.)	1 year
365 days (da.)	1 year
366 days	1 leap year

clock measures

60 seconds (sec.)	1 minute (min.)
60 minutes (min.)	1 hour (hr.)
24 hours (hr.)	1 day (da.)

length

12 inches (in.)	1 foot (ft.)
36 inches	1 yard (yd.)
3 feet (ft.)	1 yard
16½ feet	1 rod (rd.)
5280 feet	1 mile (mi.)
1760 yards (yd.)	1 mile
320 rods (rd.)	1 mile

weight

16 ounces (oz.)	1 pound (lb.)
100 pounds (lb.)	1 hundredweight (cwt.)
2000 pounds	1 ton (T.)

square measures

144 square inches (sq. in.)	1 square foot (sq. ft.)
9 square feet (sq. ft.)	1 square yard (sq. yd.)
160 square rods (sq. rd.)	1 acre (A.)
43560 square feet	1 acre
640 acres (A.)	1 square mile

cubic measures

1728 cubic inches (cu. in.)	1 cubic foot
27 cubic feet	1 cubic yard
277 cubic inches	1 gallon
2150 cubic inches	1 bushel

metric measures

1000 millimetres (mm.)	1 metre (m.)
100 centimetres (cm.)	1 metre
10 decimetres (dm.)	1 metre
10 metres (m.)	1 decametre (dkm.)
100 metres	1 hectometre (hm.)
1000 metres	1 kilometre (km.)
1000 milligrams (mg.)	1 gram (g.)
1000 grams (g.)	1 kilogram (kg.)
1000 millilitres (ml.)	1 litre (l.)
1000 litres (l.)	1 kilolitre (kl.)
1000 cubic centimetres	1 litre

English—metric

.039 inch	1 millimetre
.394 inch	1 centimetre
1.094 yards	1 metre
.621 mile	1 kilometre
.035 ounce	1 gram
2.2 pounds	1 kilogram
.88 quart	1 litre

25.4 millimetres	1 inch
2.54 centimetres	1 inch
.914 metre	1 yard
1.6 kilometres	1 mile
28.35 grams	1 ounce
.45 kilogram	1 pound
1.136 litres	1 quart

Responses to ? exercises

Responses to ? exercises are keyed to your book by page number and exercise letter. For example, the response for exercise I on page 5 is labeled 5 I
When an exercise includes two or more questions, the responses to these questions are separated by an asterisk (*).

5 I To find the measure of a segment, you subtract the lesser coördinate from the greater coördinate; therefore, $m(\overline{AC})$ is the difference of the same two numbers as $m(\overline{CA})$.

7 M $4\frac{2}{8}$, or $4\frac{1}{4}$ * $4\frac{1}{4}$

7 N Subdivide the segment whose measure is .1 that contains B into 10 congruent segments. The measure of each of these segments will be .01. If B is still not an endpoint of one of these segments, you can subdivide again. You continue to subdivide until B is an endpoint of one of the segments.

8 Q The measure of a segment depends upon the segment you use as the unit segment. For exercises O and P, different unit segments were used.

10 H $1\frac{3}{16}$ * Since the left end of the wire is above the mark for 1, and the right end is above the mark for $2\frac{3}{16}$, you know that the measure of the wire is $2\frac{3}{16} - 1$, or $1\frac{3}{16}$.

14 F $.394/1 \sim 1/y$. * One inch is the same length as approximately 2.54 centimeters.

19 N 20 * 20 * $\angle DTF$ is congruent to $\angle FTH$.

19 T The two unit angles are not congruent. * If these unit angles were congruent, then ray MW, represented in D2 would be associated with the same number as ray TG, represented in D3.

20 J When the unit angle is the degree, the union of the ray associated with 180 and the ray associated with 0 is a line, not an angle. Therefore, the measure of an angle in degrees must be less than 180. * The measure in degrees of an angle must be greater than 0, because, if the difference of the numbers associated with 2 rays is 0, then the 2 rays are the same ray, and a single ray is not an angle.

27 U 360 * The four angles make up two pairs of supplementary angles. Since the sum of the measures in degrees of a pair of supplementary angles is 180, the sum of the measures of the two pairs of supplementary angles must be 360.

29 P Yes. The interiors of the two angles that form a linear pair are disjoint, and the angles have a common side. * No. The sides of the adjacent angles that are not common do not necessarily form a line.

29 Q Yes. If two angles form a linear pair, then they are supplementary angles. * No. The only requirement for supplementary angles is that the sum of their measures be 180.

29 R No. The sum of the measures of a pair of adjacent angles need not be 180. * No. The only requirement for supplementary angles is that the sum of their measures be 180.

29 C Yes. Since the transversal contains two points in the plane that includes the two coplanar lines, the transversal is included in the plane.

31 E Opposite angles are congruent; therefore, $\angle q \cong \angle r$ and $\angle u \cong \angle v$. Since $\angle q \cong \angle u$, you know that $\angle r \cong \angle v$. * Yes

31 F Yes * Yes * $\angle q^\circ + \angle p^\circ = 180$, and $\angle u^\circ + \angle t^\circ = 180$. Since $\angle q^\circ = \angle u^\circ$, then $\angle p^\circ = \angle t^\circ$. This means that $\angle p \cong \angle t$. * Yes

31 H $\angle s$, $\angle t$, and $\angle w$ 75 * $\angle r$, $\angle u$, and $\angle v$

31 M $\angle g$ is a supplement

of $\angle h$ and $\angle m$ is a supplement of $\angle n$. Therefore, $\angle g \cong \angle m$. $\angle k$ is a supplement of $\angle h$ and $\angle q$ is a supplement of $\angle n$. Therefore, $\angle k \cong \angle q$. $\angle h$ and $\angle j$ are opposite angles. $\angle n$ and $\angle p$ are opposite angles. Therefore, $\angle j \cong \angle p$.

32 P No * No * For example, consider the *true* statement, "If an animal is a dog, then it has four legs." The *false* converse is: "If an animal has four legs, then it is a dog." Now consider the *true* statement, "If two natural numbers are not equal, then one of the numbers is less than the other." The *true* converse is: "If one natural number is less than another, then the two numbers are not equal."

34 H Yes * If two lines are parallel, then each pair of corresponding angles is congruent. Congruent angles, like $\angle d$ and $\angle c$, have the same measure.

34 I Yes * The interiors of $\angle e$ and $\angle a$ are both subsets of the same half-plane bounded by ℓ_5, and a side of $\angle a$ is a proper subset of a side of $\angle e$. * Yes. Since ℓ_4 and ℓ_7 are parallel lines, the corresponding angles a and e are congruent. Congruent angles have the same measure.

34 L Yes * The three numbers in the sum $\angle a° + \angle b° + \angle c°$ are the same as the three numbers in the sum $\angle d° + \angle e° + \angle f°$.

35 W No * If a triangle had only one acute angle, then each of the other angles would be right or obtuse. Each of these angles would, therefore, have a measure of 90 or greater. Then the sum of the measures of the three angles would be greater than 180.

39 D Sides PQ and ST, sides QR and TU, sides RP and US * Yes * The corresponding sides determined by a congruence correspondence are congruent.

39 G No * It is possible that only one of the six correspondences is a congruence correspondence. This is true because the sides and angles are mapped onto each other in a different way for each correspondence between the vertices. For certain correspondences, the corresponding sides and angles may not be congruent.

40 K If the corresponding sides of two triangles are congruent, then the two triangles are congruent. * This converse is true.

40 M If the corresponding angles of two triangles are congruent, then the two triangles are congruent. * This converse is false. Your sketch should show two triangles whose corresponding angles have the same measure, but whose sides do not.

41 F Yes. A segment is congruent to itself. * Yes * Yes. If the corresponding sides of two triangles are congruent, then the two triangles are congruent.

41 H Yes * For example, if $\triangle ABC$ is equilateral, then $\angle A \cong \angle B$ and $\angle B \cong \angle C$. This is true because $\angle A$ and $\angle B$ are the angles opposite congruent sides and $\angle B$, and $\angle C$ are the angles opposite congruent sides. $\angle A \cong \angle C$ because these two angles are opposite congruent sides.

42 J 2 * B and D

43 M Yes * $\angle JKL$ and $\angle LIJ$ are corresponding angles. Since the triangles are congruent, the corresponding angles are congruent.

44 B Yes * A linear pair of angles are supplementary angles, and the sum of their measures is 180. This means that $\angle RNQ° = 90$ because $\angle QNO° = 90$. Therefore, $\angle RNQ$ is a right angle.

44 C Yes * If two lines are parallel, then the angles of each pair of corresponding angles formed by the two lines and a transversal are congruent. Since $\angle RNQ$ is a right angle and has a measure of 90, $\angle NOP$ is also a right angle and has a measure of 90.

44 D The opposite angles of a parallelogram are congruent. * The opposite angles of a parallelogram are congruent.

52 F $\overline{UR}$ is not perpendicular to the lines that include a pair of opposite sides. * $\overline{RS}$ is not perpendicular to the lines that include a pair of opposite sides.

52 J First, you can show that the altitudes form right angles with the lines that include the opposite sides because an altitude is perpendicular to the lines that include the opposite sides. The altitudes are parallel because corresponding angles are congruent. Therefore, the points of intersection of each pair of altitudes and the lines that include the opposite sides determine a rectangle. Since each pair of opposite sides of a rectangle are congruent, the altitudes have the same measure.

53 K Each side of a rectangle is perpendicular to a pair of opposite sides and has its endpoints in the lines that include these opposite sides. * Since any side of a rectangle can be thought of as an altitude, any side can be thought of as the base. If a certain side is thought of as the altitude, the adjacent sides are bases.

53 C Since the area of parallelogram ABCD is the sum of two areas that are equal to the two areas whose sum is the area of rectangle EFCD, the area of parallelogram ABCD is the same as the area of rectangle EFCD.

54 M You can use the same condition for equivalence to find the area of the parallelogram as the condition you would use to find the area of a rectangle whose base and altitude have the same measure as the base and altitude of the rectangle.

54 G The two triangles are congruent; therefore they have the same area.

63 E 5 and 7. If an object belongs to two sets, it is a common member of the sets. * The portion of the diagram that has both vertical and horizontal shading

64 G No. A geometric figure cannot belong to both the set of triangles and the set of rectangles. * Yes. Sets that have no common members are disjoint sets.

64 H Yes. The intersection of two sets contains all those members, and only those members, that belong to both sets. Since A and B are disjoint, the intersection of A and B contains no members and is the empty set. * { }. The symbol { } names the empty set.

65 F Yes. There are infinitely many rational numbers that are greater than $\frac{7}{6}$. * To tabulate a set, you must be able to name the least member of the set and the next two or three members. You cannot name the least member of $\{x|x > \frac{7}{6}\}$.

65 J Yes * It is an infinite set.

65 F Each member of $\{x|x < \frac{5}{4} \wedge x < \frac{2}{3}\}$ satisfies not only $\frac{1}{8} + x < \frac{11}{8}$ but also $\frac{4}{3} - x > \frac{2}{3}$.

66 H Yes * $\{x|x < \frac{5}{4} \wedge x < \frac{2}{3}\}$ contains the common members of $\{x|x < \frac{5}{4}\}$ and $\{x|x < \frac{2}{3}\}$. Similarly, the intersection of $\{x|x < \frac{5}{4}\}$ and $\{x|x < \frac{2}{3}\}$ contains the common members of these two sets.

66 I Yes * There are no members of either set that are not also contained in the other set.

66 M $\{x|x < \frac{5}{4} \wedge x < \frac{2}{3}\}$ is the same set as $\{x|x < \frac{2}{3}\}$; therefore, the graphs of these two sets are the same.

67 R Yes * For each replacement of y, $y > \frac{11}{4}$ also means $\frac{11}{4} < y$.

67 Z Yes * The density property of the set of rational numbers of arithmetic is that, between any two given members of R_a, there is another member of R_a. This means that no matter how many rational numbers you name that are between $\frac{11}{4}$ and $\frac{17}{4}$, you cannot name all of them. Therefore, $\{y|\frac{11}{4} < y < \frac{17}{4}\}$ is an infinite set.

70 G Yes * The union of the solution set of $s + 2 < 7$ and the solution set of $s + 4 > 6$ contains every member of each solution set. The solution set of $s + 2 < 7 \vee s + 4 > 6$ also contains every member of each of these solution sets.

70 M Yes * Both $\{m|m = 4\frac{1}{2}\} \cup \{m|m > 4\frac{1}{2}\}$ and $\{m|m = 4\frac{1}{2} \vee m > 4\frac{1}{2}\}$ contain those numbers, and only those numbers, that satisfy either or both of the conditions $m = 4\frac{1}{2}$ and $m > 4\frac{1}{2}$.

71 Q $\{x|x < 3.25\}$

71 S Yes. $x \leqq 3.25$ is the same condition as $x < 3.25 \vee x = 3.25$. * The union. The solution set of $x \leqq 3.25$ is the same set as the union of the solution sets of $x < 3.25$ and $x = 3.25$.

72 B Yes * Each dot in the graph of $\{s|s + 2 < 7\}$ is also in the graph of $\{s|s + 2 < 7 \vee s + 4 > 6\}$.

74 C Two ordered pairs are the same only if they have the same first and second components. The first component of (2, 3) is not the same as the first component of (3, 2). The second component (2, 3) is not the same as the second component of (3, 2). Therefore, (2, 3) is not the same ordered pair as (3, 2).

74 F No. There is no natural number that can be added to 8 to obtain a sum of 7. * If x in $x + y = 7$ is replaced by a number greater than 7, then there is no replacement for y that satisfies the condition.

74 H Yes * 8 * Yes * No * {(0, 7), (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (7, 0)}

75 S Yes. Each encircled dot represents a point associated with an ordered pair of natural numbers that satisfies $y > x + 1$. * The encircled dots are only a few of the dots that are in the graph of $\{(x, y)|y > x + 1\}$.

78 D All the members of the universe that are not in subset A are in subset B. These are the only members in subset B.

79 K Yes. 5 belongs to the universe but is not a member of set D. * Yes. Each natural number greater than 4 is a member of the universe but is not a member of set D. * 4 is a member of D.

79 M Yes. {5, 6, 7, . . .} contains all those members of N, and only those members, that are not in $\{x|x < 5\}$.

79 R The universe * The empty set

79 S Set K

79 U The universe * { }

80 I Yes * Both the solution set of $\sim(x > 6)$ and the complement of the solution set of $x > 6$ contain those members of the universe, and only those members, that do not satisfy $x > 6$.

81 F Yes. $\{m|\sim(m = 2)\}$ is the same set as $\{m|m \neq 2\}$. Therefore, their graphs contain the same dots.

81 K The solution set of $b + 1 < 4$ is the same as the solution set of $b < 3$. Therefore, the solution set of $\sim(b + 1 < 4)$ is the same as the solution set of $\sim(b < 3)$. * The members of N that are not less than 3 are either equal to 3 or greater than 3. * The solution set of $\sim(b + 1 < 4)$, is the same set as the complement of the solution set of $b < 3$. Each of the sets, $\{b|\sim(b + 1 < 4)\}$,

$\{b|\sim(b<3)\}$, $\{b|b\geqq 3\}$, and $\{b|b<3\}$, contains every natural number either greater than or equal to 3.

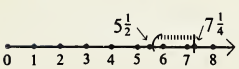
83 D No. Since $1\frac{1}{2}$ is not less than $1\frac{1}{2}$, $1\frac{1}{2}$ does not satisfy $1\frac{1}{2}<x$. * No. Since $4\frac{2}{3}$ is not less than $4\frac{2}{3}$, $4\frac{2}{3}$ does not satisfy $x<4\frac{2}{3}$. * Yes. Each rational number of arithmetic that is greater than $1\frac{1}{2}$ and less than $4\frac{2}{3}$ satisfies both $1\frac{1}{2}<x$ and $x<4\frac{2}{3}$.

83 G $6\frac{1}{2}<x<2$. * No. $6\frac{1}{2}$ is not less than $6\frac{1}{2}$. * No. 2 is not less than 2. * No. There is no member of R_a that is both greater than $6\frac{1}{2}$ and less than 2, so there is no member of R_a that satisfies $6\frac{1}{2}<x<2$.

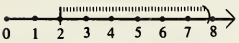
84 O Yes. Since $4\frac{1}{3}$ is equal to $4\frac{1}{3}$, $4\frac{1}{3}$ satisfies $4\frac{1}{3}\leqq x$; and since $4\frac{1}{3}$ is less than $6\frac{2}{5}$, $4\frac{1}{3}$ satisfies $x\leqq 6\frac{2}{5}$. * Yes. Since $6\frac{2}{5}$ is greater than $4\frac{1}{3}$, $6\frac{2}{5}$ satisfies $4\frac{1}{3}\leqq x$; and since $6\frac{2}{5}$ is equal to $6\frac{2}{5}$, $6\frac{2}{5}$ satisfies $x\leqq 6\frac{2}{5}$.

84 U Yes. a satisfies both $a\leqq x$ and $x\leqq b$. b satisfies both $a\leqq x$ and $x\leqq b$. * The set of rational numbers of arithmetic that contains a and b and all the rational numbers between a and b

86 G $\{x|5\frac{1}{2}<x\leqq 7\frac{1}{4}\}$



$\{x|2\leqq x<8\}$



88 I Yes. The absolute difference of 7 and 6.99 is .01, which is less than 1. * Yes. The absolute difference of 7 and 7.01 is .01, which is less than 1.

89 A Subtract .5 from 7. * 6.5 * Yes

89 B .5 * 7.5 * .5

89 E Yes. The absolute difference of 7 and 6.9 is .1, which is less than .5. * Yes. The absolute difference of 7 and 7 is 0, which is less than .5. * Yes. The absolute difference of 7 and 7.35 is .35, which is less than .5.

89 H Yes. $\{x|6.5\leqq x\leqq 7.5\}$ contains 6.5 and 7.5 and all the rational numbers between 6.5 and 7.5.

90 K Subtract .1 from 7. * Add .1 to 7. * 6.9 * 7.1

90 Q To find the least number that is a solution of $x\approx_t k$, the tolerance t is subtracted from the given number k. * $k+t$

90 S $x\approx_{.2} 17$.

93 H $\{(4, 2, 3), (4, 2, 6), (4, 3, 3), (4, 3, 6), (4, 5, 3), (4, 5, 6), (7, 2, 3), (7, 2, 6), (7, 3, 3), (7, 3, 6), (7, 5, 3), (7, 5, 6)\}$

93 J Find the product of the number of elements in Q, the number of elements in V, and the number of elements in W. The product you obtain is the number of ordered triples you can form from the elements of these three sets.

94 K 0, 1, 2, 3 * 3 is the only solution of $y=3$, and the replacement for y is the second component of each ordered triple.

97 D Each member of Z that satisfies $x+27<40\frac{3}{8}$ must be less than $13\frac{3}{8}$. $\{x|x<13\frac{3}{8}\}$ contains each member of Z that is less than $13\frac{3}{8}$, so the symbol $\{x|x<13\frac{3}{8}\}$ is a standard description of the solution set of $x+27<40\frac{3}{8}$. $\{x|x+27<40\frac{3}{8}\}=\{x|x<13\frac{3}{8}\}$.

97 E No. Since 0 is not a member of the universe, it cannot be a member of the solution set. * You cannot name the least member of the solution set of $x+27<40\frac{3}{8}$.

97 D You can use $n-5.25=20.00$ to help you find the members of R_a that satisfy $n-5.25>20.00$. Since 25.25 satisfies $n-5.25=20.00$, each number that satisfies $n-5.25>20.00$ must be greater than 25.25. Therefore, $\{n|n>25.25\}$ is the solution set of $n-5.25>20.00$.

98 G It is an open interval in R_a because the limits are not members of the interval. * 25.25 and 30.00. An open interval is the set of all members of R_a between two given numbers. The two given numbers are the limits of the interval. In this case, all the members of R_a between 25.25 and 30.00 are elements of $\{n|25.25<n<30.00\}$.

98 I Yes. 25.26 is the least number in hundredths that is greater than 25.25. * 29.99

100 S Neither 1 nor 2 satisfies $x\geqq 3$. * If a number greater than 6 is used as a replacement for x in $x+y=7$, there is no member of C that satisfies the condition. *

x	3	4	5	6
y	4	3	2	1

101 E The first component of each ordered pair that satisfies $x+y=5\wedge\sim(x=3)$ cannot be 3. * Five

102 C The condition is a condition for equivalence; this means that you must find a replacement for x so that the rate pair $x/60$ is equivalent to the rate pair 16/9.

103 M Yes. A per cent is a rate pair that represents a comparison and that has a second component of 100. * The increase in enrollment for each 100 students enrolled in summer classes last summer

104 I $168\frac{2}{3}$ * The problem asks you to find the average length per portage. Since x, not y, is the variable for the average length, $168\frac{2}{3}$ is used to get the answer to the problem. 506 refers to the total length of the three portages.

105 P Yes. A proportional relation is a set of ordered pairs in which each member is equivalent to each of the other members. $\{5/1, \ldots\}$ contains each ordered pair that is equivalent to 5/1. * An infinite set

106 F Yes * Mr. Owen has a choice of two payments. He can pay either $\frac{1}{12}$ of the total amount or he can

pay $\frac{1}{3}$ of the total amount. Since the number that satisfies $\frac{1}{12}/1 \sim x/126$ and also the number that satisfies $\frac{1}{3}/1 \sim x/126$ can be used to get an answer to the problem, "or" is used in the condition.

108 H $22\frac{1}{2} - x = 15$ includes only one variable. Each of the other conditions includes at least two variables.

108 J The first component of each solution is $7\frac{1}{2}$. * The solutions of the compound condition must satisfy all three simple conditions. Only those ordered triples whose first components are $7\frac{1}{2}$ satisfy $22\frac{1}{2} - x = 15$.

109 O {($7\frac{1}{2}$, 5, 10)}

109 L $1.50/1 \sim x/3$. * $1.50/1 \sim x/3$ includes only one variable. Each of the other conditions includes at least two variables.

109 Q {(4.50, 9.00, 13.50)}

110 C $r - s = 2$. * The problem states that the difference of the first number and the second number is 2. This means that when the second number (s) is subtracted from the first number (r), the difference is 2.

110 J Three * {(2, 0, 0), (2, 0, 1), (2, 0, 2)}

113 E Points contained in the same line are collinear points. Since points E, F, and G are contained in ℓ_1, they are collinear.

113 G Point H * Every point is a member of space; the set of all points is called space.

113 H Three noncollinear points determine a plane. The names of any three noncollinear points can be used to name the plane that they determine. Points S, T, and N are noncollinear because they are not contained in the same line. * Points V, F, and R are noncollinear points. * Points V, S, and T are collinear points. Collinear points do not determine a plane because any line is included in infinitely many planes.

114 O A radius of a circle is a segment determined by the centre of the circle and a point in the circle. Point S is the centre of $\odot$S and point T is in $\odot$S. * Segments SF, SV, and SG are also radii of $\odot$S.

115 B ℓ_2 and $\odot$K have points D and C as common members. Geometric figures that have common members are said to meet, or to intersect. * {D, C}

115 C A segment whose endpoints are in a circle is a chord of that circle. C and D are in $\odot$K. * $\overline{EF}$ is a chord of $\odot$K.

115 F A chord that contains the centre of a circle is a diameter of the circle. * Each point in chord EF is also in ℓ_3.

116 D $\overline{EF}$ is a chord of $\odot$D that contains the centre of the circle. * The union of rays DE and DF is a line, and therefore not an angle.

117 F Points A, B, and C * Points G, H, I, N, and R * Points A, B, C, G, H, I, N, and R

117 G Points B, C, G, and H

117 H Points J, K, L, S, and V * Points A, B, C, J, K, L, S, and V

117 I Points B, C, J, and V

117 J Two. A line separates a plane that includes the line into two half-planes. * Line EF. A line is the boundary of each half-plane into which it separates the plane. The boundary is not included in either half-plane.

117 K The points in the same half-plane as L are G, H, N, and T. * Points B, D, E, and F * Points B, D, E, F, G, H, L, N, and T

118 L Points E, H, N, and F

118 O Yes * Line KJ includes a diameter of $\odot$G. $\overset{\frown}{KIJ}$ is the subset of $\odot$G that is included in the union of $\overrightarrow{KJ}$ and one of the half-planes bounded by $\overleftrightarrow{KJ}$. * $\angle HGJ$ is a central angle of $\odot$G. $\overset{\frown}{HDJ}$ is the subset of $\odot$G that is included in the union of $\angle HGJ$ and the interior of $\angle HGJ$. * $\angle IGJ$ is a central angle of $\odot$G. $\overset{\frown}{IHJ}$ is the subset of $\odot$G that is included in the union of $\angle IGJ$ and the exterior of $\angle IGJ$.

119 A 60 * 120 * Find the difference of 120 and 60. $120 - 60 = 60$, so the measure of $\angle LCQ$ is 60.

119 B 30 * 90 * An angle whose measure in degrees is 90 is a right angle.

120 K 90 * Each angle formed by two intersecting perpendicular lines is a right angle. A right angle has a measure in degrees of 90.

120 L 180. Semicircle BCD is the union of the two adjacent arcs, minor $\overset{\frown}{BC}$ and minor $\overset{\frown}{CD}$. Since each of these arcs has a measure in degrees of 90, the measure of semicircle BCD in degrees is equal to 90 + 90, or 180. * 180. $\overset{\frown}{CDE}°$ = minor $\overset{\frown}{CD}°$ + minor $\overset{\frown}{DE}°$.

121 O Yes * {N, P} * $\odot$M

122 C Polygon PQRS. A quadrilateral is a simple polygon that has four sides. * Triangle ABC. A triangle is the union of three segments determined by any three noncollinear points.

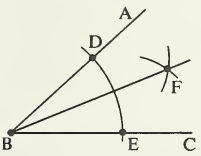
122 E Yes. Parallel segments are segments that are included in parallel lines. * { }. The intersection of two parallel lines is the empty set.

123 K Opposite sides of a parallelogram are congruent. If two segments are congruent, then their measures are equal. * Opposite angles of a parallelogram are congruent. If two angles are congruent, then their measures are equal.

123 M No * A parallelogram has two pairs of parallel sides. A trapezoid has only one pair of parallel sides.

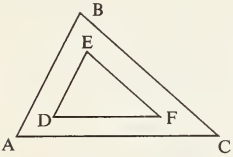
123 B A rectangle is a parallelogram in which each angle is a right angle. * Two lines that intersect to form a right angle are perpendicular. Two segments are perpendicular if they intersect and if they are subsets of lines that are perpendicular.

124 F A parallelogram in which each angle is a right angle is a rectangle. * Each side of IJKL is congruent to each of the other sides.
125 E 45 * 45. The angles opposite the congruent sides of an isosceles triangle are congruent. * 180. For any triangle, 180 is the sum of the measures of the angles in degrees. * 90. $\angle A^\circ + \angle C^\circ = 90$, and $90 + \angle B^\circ = 180$. Therefore, $\angle B^\circ = 90$.
126 I In an equilateral triangle, any two angles are opposite congruent sides. Therefore, each of the angles is congruent to each of the other angles. Since 180 is the sum of the measures of the three angles in degrees, each of the angles must have a measure in degrees of 60.
126 L $\triangle GHI$. A scalene triangle is a triangle with no two sides congruent. * $\triangle DEF$. An equilateral triangle is a triangle with three sides congruent. * $\triangle DEF$ and $\triangle ABC$. An isosceles triangle is a triangle with two sides congruent.
126 N $\triangle DEF$. An acute triangle is a triangle that has three acute angles. * $\triangle ABC$. A right triangle is a triangle with one right angle. * $\triangle GHI$. An obtuse triangle is a triangle with one obtuse angle. * $\triangle DEF$
126 B An altitude of a triangle is a segment determined by a vertex of the triangle and a point in the line that includes the side opposite the vertex. The altitude is perpendicular to the line that includes the side opposite the vertex. $\overline{JM}$ is determined by vertex J and a point in ℓ_1 that includes side KL, and $\overline{JM}$ is perpendicular to ℓ_1. * Side KL. When $\overline{JM}$ is used as an altitude, the side opposite vertex J is the base.
130 B {N} * Three noncollinear points determine a plane. Since points M, N, and S are noncollinear, these points determine plane MNS. If a line contains two points of a plane, the line is included in the plane. Since line MN contains points M and N, line MN is included in plane MNS. Since line RS contains points S and N, line RS is included in plane MNS. * Examples of names of the plane that include $\overleftrightarrow{MN}$ and $\overleftrightarrow{RS}$ are plane MNR, plane MNS, and plane MRS.
130 F A transversal is a line that intersects one of two coplanar lines in a point and also intersects the other line in a different point. Since $\overleftrightarrow{MN}$ intersects $\overleftrightarrow{TU}$ in point T and intersects $\overleftrightarrow{RS}$ in point N, $\overleftrightarrow{MN}$ is a transversal of $\overleftrightarrow{TU}$ and $\overleftrightarrow{RS}$.
133 F Use your ruler to find the measure of the segments determined by the midpoint and each of the endpoints of $\overline{EF}$. The two measures that you find should be approximately equal. Use your protractor to measure the four angles formed by line EF and the perpendicular bisector of $\overline{EF}$. The measures that you find should be approximately equal to 90.
134 K In the drawing below, certain dots are labelled to help explain the construction.

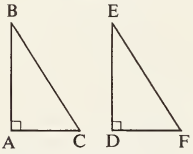


With the compasses set to a convenient radius and the tip on dot B, arc DE is constructed. With the compasses set to a convenient radius and the tip first on dot E and then on dot D, arcs are constructed that intersect in point F. A straightedge is used to construct $\overleftrightarrow{BF}$, which bisects $\angle ABC$.
135 F Yes. The corresponding sides and the corresponding angles determined by the correspondence represented by diagram D are congruent. Therefore, diagram D represents a congruence correspondence. * If there is a congruence correspondence between the vertices of two triangles, the triangles are congruent.
136 H None * For each of the correspondences represented by the four diagrams, all the pairs of corresponding sides and angles are not congruent.
137 H Yes * In the correspondence between the vertices of $\triangle JLK$ and $\triangle FGH$, the three pairs of corresponding sides are congruent, and the three pairs of corresponding angles are congruent. Therefore, the correspondence is a congruence correspondence, and $\triangle JLK$ is congruent to $\triangle FGH$.
138 G For any two triangles, if the corresponding sides are congruent, then the two triangles are congruent. (s-s-s property of congruence for triangles)
138 G Yes. Sides ST and SU and $\angle S$ of $\triangle STU$ are congruent to the corresponding sides and angle of $\triangle VWX$. * The s-a-s property of congruence for triangles
139 I 180 is the sum of the measures of the angles of a triangle in degrees. You know that $\angle A^\circ + \angle B^\circ = \angle D^\circ + \angle E^\circ$. You also know that $\angle C^\circ = 180 - (\angle A^\circ + \angle B^\circ)$ and that $\angle F^\circ = 180 - (\angle D^\circ + \angle E^\circ)$. $180 - (\angle A^\circ + \angle B^\circ) = 180 - (\angle D^\circ + \angle E^\circ)$. Therefore, $\angle C^\circ = \angle F^\circ$. This means that $\angle C \cong \angle F$.
139 J The a-s-a property of congruence for triangles * Because $\angle H \cong \angle K$ and $\angle I \cong \angle L$, you know that $\angle G \cong \angle J$. Therefore, in $\triangle GHI$, two angles ($\angle G$ and $\angle H$) and the side that is a subset of each of these angles ($\overline{GH}$) are congruent to the corresponding parts in $\triangle JKL$ ($\angle J$, $\angle K$, and $\overline{JK}$).
139 L Yes. The idea expressed by the "if" clause

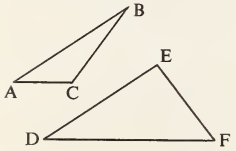
and the idea expressed by the "then" clause in "If two triangles are congruent, then the corresponding angles of the two triangles are congruent" have been interchanged. * No * See the sketch below. $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$. However, $\triangle ABC$ is not congruent to $\triangle DEF$.



139 MYes. Since all right angles have a measure in degrees of 90, all right angles are congruent. Therefore, you can use the s-a-s property of congruence for triangles. * See the sketch below. $\angle A$ and $\angle D$ are right angles. $\overline{AB} \cong \overline{DE}$. $\overline{AC} \cong \overline{DF}$. So, $\triangle ABC \cong \triangle DEF$.



139 NNo * See the sketch below. $\angle A \cong \angle D$. $\overline{AB} \cong \overline{DE}$. $\overline{BC} \cong \overline{EF}$. $\triangle ABC$ is not congruent, in this case, to $\triangle DEF$.



141 AYes. Rectangle ABCD is a simple closed curve. A closed region is the union of a simple closed curve and its interior. * Yes. A closed region that is the union of a rectangle and its interior is called a rectangular region.

141 HYes * Because the measures of the sides are equal, each of the sides is congruent to each of the other sides. A square is a rectangle in which each side is congruent to each of the other sides. * $4.6/1 \sim x/4.6$.

142 A$\overline{FI}$ and $\overline{GJ}$. An altitude of a parallelogram is a segment that is perpendicular to both of the lines that include a pair of opposite sides of the parallelogram. The endpoints of the altitude are in these lines. * $\overline{FG}$

143 N$\{(1\frac{1}{2}, 6)\}$ * The area of the piece of plywood is 6 sq. ft.

143 BA parallelogram and either of its diagonals form two congruent triangles. * Yes. If two triangles are congruent, then they have the same area.

144 M$\{(\frac{15}{32}, 4\frac{7}{32})\}$ * The area of the window is $4\frac{7}{32}$ sq. ft.

148 DThere is no natural number that you can add to 28 to obtain 13.

148 EYes * N is closed under multiplication.

148 FNo. $a - b$ is an element of N only if a is greater than b. * No. If you can find at least one example for which it is not true that $a - b \in$ N, then it is not true that, for each a and b, $a - b \in$ N. Such an example was given in exercise D. $13 - 28$ is not a member of N.

148 HThere is no natural number by which 10 can be multiplied to obtain 18.

148 I No. $a \div b$ is an element of N only if a is a multiple of b. * No. If you can find at least one example for which it is not true that $a \div b \in$ N, then it is not true that, for each a and b, $a \div b \in$ N. Such an example was given in exercise H. $18 \div 10$ is not a member of N.

149 B$\frac{3}{5} \sim \frac{9}{15}$ because $3 \times 15 = 9 \times 5$. * $\frac{6}{10} \sim \frac{9}{15}$ because $6 \times 15 = 9 \times 10$.

149 CThe only common factor of 3 and 5 is 1. In other words, 3 and 5 are relatively prime. * No * Both 1 and 2 are common factors of 6 and 10. Both 1 and 3 are common factors of 9 and 15.

149 DA rational number of arithmetic is a set of equivalent fractions. A rational number of arithmetic contains a basic fraction and all fractions equivalent to the basic fraction.

150 KYes * From the closure property of multiplication of natural numbers, $ac \in$ N, and, since neither b nor d is 0, $bd \in$ C. Therefore, $\frac{ac}{bd} \in R_a$.

150 NYes * From the closure property of multiplication of natural numbers, $ad \in$ N and $bc \in$ N. Since bd is not 0, $bd \in$ C. From the closure property of addition of natural numbers, $ad + bc \in$ N. Therefore, $\frac{ad+bc}{bd} \in R_a$.

151 TThere is no rational number of arithmetic that you can add to $\frac{7}{8}$ to get $\frac{3}{8}$.

152 GIf the universe for x, y, and z is N, then for each x, y, and z, $x + y \in$ N, $xy \in$ N, and true statements are obtained from the other conditions expressed in D5.

152 HIf the universe for x, y, and z is R_a, then for each x, y, and z, $x + y \in R_a$, $xy \in R_a$, and true statements are obtained from the other conditions expressed in D5.

154 L No. The starting point of $\overrightarrow{GW}$ is a different point from the starting point of $\overrightarrow{WG}$. The terminal points are also different. * $\overrightarrow{GW}$ and $\overrightarrow{WG}$ are the same set of points. * $\overrightarrow{GW}$ and $\overrightarrow{WG}$ are directed in opposite directions.

154 OThe coördinate of A is 0. The coördinate of B is $\frac{1}{2}$. The absolute difference of 0 and $\frac{1}{2}$ is $\frac{1}{2} - 0$, or $\frac{1}{2}$. * $\frac{5}{2} - 2$, or $\frac{1}{2}$ * Yes. The measure of each of these directed segments is $\frac{1}{2}$. * Yes. Both $\overrightarrow{AB}$ and $\overrightarrow{EF}$ are directed to the right.

155 QTo the right. Since $\frac{51}{2} > 25$, terminal point V is to the right of starting point W. * $\frac{51}{2} - 25$, or $\frac{1}{2}$. $m(\overrightarrow{WV})$ is the absolute difference of 25 and $\frac{51}{2}$, which are the coördinates of the endpoints of $\overrightarrow{WV}$.

156 GThey are equivalent directed segments and, therefore, have the same measure and the same direction.

156 H They are not equivalent ordered pairs for directed segments.

156 L Yes. $(\frac{1}{2}, 0) \sim (\frac{1}{2}, 0)$ because $\frac{1}{2} + 0 = \frac{1}{2} + 0$. * Yes. $\frac{2}{4}$ is the same member of R_a as $\frac{1}{2}$. * No. When you replace y in $(x, y) \sim (\frac{1}{2}, 0)$ by 0, you obtain $(x, 0) \sim (\frac{1}{2}, 0)$. If $(x, 0) \sim (\frac{1}{2}, 0)$, then $x + 0 = \frac{1}{2} + 0$. Since $\frac{1}{2}$ is the only member of R_a that satisfies $x + 0 = \frac{1}{2} + 0$, $\frac{1}{2}$ is the only member of R_a that satisfies $(x, 0) \sim (\frac{1}{2}, 0)$. Of course, $\frac{1}{2}$ can be named by many other fraction numerals, such as $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}$, etc. But each of these numerals names the same member of R_a. Each of the ordered pairs, $(\frac{2}{4}, 0)$, $(\frac{3}{6}, 0)$, and $(\frac{4}{8}, 0)$, is the same as $(\frac{1}{2}, 0)$.

157 S $(\frac{2}{3}, 0)$ * Any member of a positive rational number can be used to indicate the number. Since $\frac{5}{3} + 0 = \frac{2}{3} + 1$, $(\frac{5}{3}, 1) \sim (\frac{2}{3}, 0)$. Therefore, $(\frac{5}{3}, 1)$ is a member of the same positive rational number as $(\frac{2}{3}, 0)$ and can be used to indicate $^+\frac{2}{3}$.

157 F Replace b by 0, c by 3, and d by 2.

158 M An infinite set * Yes. If a and b are different members of Z, then $(a, 0)$ and $(b, 0)$ indicate different positive rational numbers.

158 N An infinite set. Since Z is an infinite set, the set of ordered pairs whose first components are members of Z and whose second components are 0 is also an infinite set.

160 E Both $\overrightarrow{QP}$ and $\overrightarrow{UT}$ have a measure of $\frac{3}{5}$ and are directed to the left.

160 J $(\frac{7}{5}, 2)$ * Since $\frac{7}{5} + \frac{3}{5} = 0 + 2$ is a true statement, $(\frac{7}{5}, 2) \sim (0, \frac{3}{5})$.

161 L $(1, \frac{3}{2})$ * Since $1 + \frac{3}{5}$ does not equal $0 + \frac{3}{2}$, $(1, \frac{3}{2})$ is not equivalent to $(0, \frac{3}{5})$.

161 P One * When you replace x in $(x, y) \sim (0, \frac{3}{5})$ by 0, you obtain $(0, y) \sim (0, \frac{3}{5})$. If $(0, y) \sim (0, \frac{3}{5})$, then $0 + \frac{3}{5} = 0 + y$. Since $\frac{3}{5}$ is the only member of R_a that satisfies $0 + \frac{3}{5} = 0 + y$, $\frac{3}{5}$ is the only member of R_a that satisfies $(0, y) \sim (0, \frac{3}{5})$. Therefore, the only member of $R_a \times R_a$ that has a first component of 0 and that satisfies $(x, y) \sim (0, \frac{3}{5})$ is $(0, \frac{3}{5})$.

162 D (1, 3) determines a directed segment that is directed to the left, not to the right. * Yes. Each member of the infinite set of ordered pairs that are equivalent to (1, 3) indicates the same number as (1, 3).

162 N An infinite set * Yes. If a and b are two different members of Z, then $(0, a)$ and $(0, b)$ indicate different negative rational numbers.

164 K No * Yes. A point is associated with each member of R and R is an infinite set.

164 L The arrow at the right tells you that the points associated with positive rational numbers go on and on. The arrow at the left tells you that the points associated with negative rational numbers go on and on.

164 R The directed segments associated with $^-3$ are directed to the left. The directed segments associated with $^+3$ are directed to the right.

166 F Point 0 is to the left of each point associated with a positive rational number. * Each point associated with a negative rational number is to the left of point 0.

166 J R_a is closed under the operation of addition. * Yes. 0 is less than $\frac{9}{8}$.

167 C Yes. $0 + \frac{1}{2}$, or $\frac{1}{2}$, is less than $0 + \frac{2}{3}$, or $\frac{2}{3}$. * The sum of the first component of $(0, \frac{2}{3})$ and the second component of $(0, \frac{1}{2})$ is less than the sum of the first component of $(0, \frac{1}{2})$ and the second component of $(0, \frac{2}{3})$. Therefore, the rational number $(0, \frac{2}{3})$, or $^-\frac{2}{3}$, is less than the rational number $(0, \frac{1}{2})$, or $^-\frac{1}{2}$.

168 Q The universe for a and b is Z. b, or $(0, b)$, is any negative rational number. a, or $(a, 0)$, is any positive rational number. Since, for each a and b, $0 + 0$ is less than $a + b$, $(0, b)$, or b, is less than $(a, 0)$, or a.

171 I $^+12$. A trip of $^+5$ takes you from point 0 to point $^+5$. From point $^+5$, a trip of $^+7$ takes you to point $^+12$. A single trip of $^+12$ also takes you from point 0 to point $^+12$. * $^+22$. A trip of $^+13$ takes you from point 0 to point $^+13$. From point $^+13$, a trip of $^+9$ takes you to point $^+22$. A single trip of $^+22$ also takes you from point 0 to point $^+22$. * $^+\frac{9}{4}$. A trip of $^+\frac{3}{2}$ takes you from point 0 to point $^+\frac{3}{2}$. From point $^+\frac{3}{2}$, a trip of $^+\frac{3}{4}$ takes you to point $^+\frac{9}{4}$. A single trip of $^+\frac{9}{4}$ also takes you from point 0 to point $^+\frac{9}{4}$.

171 J If you start at point 0 and take any two successive trips to the right, you will always end up at a point to the right of point 0.

172 N $^-11$. A trip of $^-5$ takes you from point 0 to point $^-5$. From point $^-5$, a trip of $^-6$ takes you to point $^-11$. A single trip of $^-11$ also takes you from point 0 to point $^-11$. * $^-\frac{7}{9}$. A trip of $^-\frac{4}{9}$ takes you from point 0 to point $^-\frac{4}{9}$. From point $^-\frac{4}{9}$, a trip of $^-\frac{1}{3}$ takes you to point $^-\frac{7}{9}$. A single trip of $^-\frac{7}{9}$ also takes you from point 0 to point $^-\frac{7}{9}$. * $^-20$. A trip of $^-13$ takes you from point 0 to point $^-13$. From point $^-13$, a trip of $^-7$ takes you to $^-20$. A single trip of $^-20$ also takes you from point 0 to point $^-20$.

172 O If you start at point 0 and take any two successive trips to the left, you will always end up at a point to the left of point 0.

172 E $^-3$. A trip of $^-5$ takes you from point 0 to point $^-5$. From point $^-5$, a trip of $^+2$ takes you to point $^-3$. A single trip of $^-3$ also takes you from point 0 to point $^-3$. * $^-8$. A trip of $^-14$ takes you from point 0 to point $^-14$. From point $^-14$, a trip of $^+6$ takes you to point $^-8$. A single trip of $^-8$ also takes you from point 0 to

Responses to ? page 172

point $^-8$. * $^-\frac{4}{2}$. A trip of $^-\frac{5}{2}$ takes you from point 0 to point $^-\frac{5}{2}$. From point $^-\frac{5}{2}$, a trip of $^+\frac{1}{2}$ takes you to point $^-\frac{4}{2}$. A single trip of $^-\frac{4}{2}$ also takes you from point 0 to point $^-\frac{4}{2}$.

173 J $^+6$. A trip of $^+9$ takes you from point 0 to point $^+9$. From point $^+9$, a trip of $^-3$ takes you to point $^+6$. A single trip of $^+6$ also takes you from point 0 to point $^+6$. * $^+6$. A trip of $^+16$ takes you from point 0 to point $^+16$. From point $^+16$, a trip of $^-10$ takes you to point $^+6$. A single trip of $^+6$ also takes you from point 0 to point $^+6$. * $^+\frac{4}{4}$. A trip of $^+\frac{7}{4}$ takes you from point 0 to point $^+\frac{7}{4}$. From point $^+\frac{7}{4}$, a trip of $^-\frac{3}{4}$ takes you to point $^+\frac{4}{4}$. A single trip of $^+\frac{4}{4}$ also takes you from point 0 to point $^+\frac{4}{4}$.

173 K If the trip represented by the positive number is "longer" than the trip represented by the negative number, then the sum of the two numbers should be positive. If the trip represented by the negative number is "longer" than the trip represented by the positive number, then the sum should be negative. In other words, if the positive number is greater than the negative number, then the sum should be positive; and if the negative number is greater than the positive number, then the sum should be negative.

173 L When the trip to the right is the same number of units as the trip to the left * When the directed segments determined by the two numbers have the same measure but opposite directions

173 M Yes. A trip to the left of a certain number of units, followed by a trip to the right of the same number of units, takes you to point 0. * A trip of $^-3$ takes you from point 0 to point $^-3$. From point $^-3$, a trip of $^+3$ takes you back to point 0.

173 O You do not think of 0 as representing either a trip to the right or left. For this reason, when you find the sum of 0 and any given rational number, you start at 0 and take the trip represented by the given number. Therefore, the sum of 0 and any given rational number should be the given number.

174 E Yes. $^+\frac{3}{4} = (\frac{3}{4}, 0)$, and $^+\frac{1}{2} = (\frac{1}{2}, 0)$. * $\frac{5}{4}$ * $^+\frac{5}{4}$. If you are finding the sum of two positive rational numbers, the component you obtain is the first component of a basic pair, and the sum is positive.

174 F Yes. $^-2 = (0, 2)$, and $^-\frac{2}{3} = (0, \frac{2}{3})$. $2 + \frac{2}{3} = \frac{8}{3}$. * Second component * $(0, \frac{8}{3})$, or $^-\frac{8}{3}$.

175 M The non-zero component of the basic pair for $^-\frac{4}{3}$ is $\frac{4}{3}$. The non-zero component of the basic pair for $^+\frac{4}{3}$ is $\frac{4}{3}$. Find the difference of these non-zero components. Since this difference is 0, the basic pair for the sum of $^-\frac{4}{3}$ and $^+\frac{4}{3}$ is (0, 0). * 0

176 C $^+5$ * Yes. (11, 6) = $^+5$, (7, 5) = $^+2$, and (4, 1) = $^+3$.

176 I From your work with trips, you know that the sum of $^-4$ and $^+1$ should be $^-3$. (7, 10) = $^-3$.

176 J From your work with trips, you know that the sum of (8, 4), or $^+4$, and (3, 4), or $^-1$, should be $^+3$. The sum of the first components of (8, 4) and (3, 4) is 11, and the sum of the second components is 8. You can form the ordered pair (11, 8) from these sums. (11, 8) = $^+3$.

177 A Closure property of addition

177 B From the definition of the sum of two rational numbers, if (a, b) and (c, d) are any two rational numbers, $(a, b) + (c, d) = (a + c, b + d)$. From the closure property of addition of rational numbers of arithmetic, $(a + c, b + d)$ is an ordered pair of rational numbers of arithmetic. Therefore, $(a + c, b + d)$ is a rational number and the sum of any two rational numbers is always a rational number.

177 C Yes. From your response to exercise B, you know that if $x \in R$ and $y \in R$, then $x + y \in R$.

178 B Yes. The definition of the sum of two rational numbers * Yes. $0 + 0 = 0$. * A positive rational number. Since the universe is Z, the first component of the basic pair $(m + n, 0)$ is non-zero.

178 E Yes. The definition of the sum of two rational numbers * Yes. $0 + 0 = 0$. * Negative. The second component of the basic pair $(0, m + n)$ is non-zero.

178 H Yes. The definition of the sum of two rational numbers * Yes. $m + 0 = m$, and $0 + n = n$. * Negative. If $m < n$, then the second component of (m, n) is greater than the first component.

178 J No. If the non-zero component of the basic pair for the negative rational number is greater than the non-zero component of the basic pair for the positive rational number, then the sum of the two numbers is negative. * No. If the non-zero component of the basic pair for the positive rational number is greater than the non-zero component of the basic pair for the negative rational number, then the sum is positive. * If the non-zero component of the basic pair for the positive number is the same as the non-zero component of the basic pair for the negative number, then the sum is zero.

178 K Yes. From the definition of the sum of two rational numbers, $(0, 0) + (c, d) = (0 + c, 0 + d)$ and $(0 + c, 0 + d) = (c, d)$. * The given rational number

180 H Statement A tells us that $^+3 + ^-3$ is the same rational number as 0. Since we want multiplication of rational numbers to be well defined, we should

be able to replace 0 in the product $^+2(0)$ by $^+3 + {}^-3$ without affecting the product. Statement B tells us that the product $^+2(0)$ is 0. If we replace 0 in $^+2(0)$ by $^+3 + {}^-3$, the product $^+2(^+3 + {}^-3)$ should also be 0.

180 K The transitive property of equality

180 M $^-6$. If multiplication of rational numbers has the commutative property, then the product of $^-3$ and $^+2$ should be the same as the product of $^+2$ and $^-3$.

180 Q The well-defined property of multiplication of rational numbers * The rational number 0 in the product $^-3(0)$ was replaced by the equal rational number, $^+2 + {}^-2$.

183 L Yes. To simplify an expression like $3 \cdot 0 + 1 \cdot \frac{2}{3}$, you always perform the operation of multiplication before you perform the operation of addition. Since $3 \cdot 0 = 0$ and $1 \cdot \frac{2}{3} = \frac{2}{3}$, $3 \cdot 0 + 1 \cdot \frac{2}{3} = 0 + \frac{2}{3}$. * Yes. Since $3 \cdot \frac{2}{3} = 2$ and $1 \cdot 0 = 0$, $3 \cdot \frac{2}{3} + 1 \cdot 0 = 2 + 0$.

184 A From the closure property of multiplication of rational numbers of arithmetic, $ac \in R_a$, $bd \in R_a$, $ad \in R_a$, and $bc \in R_a$. From the closure property of addition of rational numbers of arithmetic, $ac + bd \in R_a$ and $ad + bc \in R_a$. Therefore, $(ac + bd, ad + bc)$ is an ordered pair of rational numbers of arithmetic.

184 B From the definition of the product of two rational numbers, $(a, b) \times (c, d) = (ac + bd, ad + bc)$. From your response to exercise A, $(ac + bd, ad + bc)$ is an ordered pair of rational numbers of arithmetic. Therefore, $(ac + bd, ad + bc) \in R$, and the product of any two rational numbers is always a rational number.

184 C Yes. The operation of multiplication of rational numbers is a binary operation because it is defined for any two members of R.

185 I 0 * 0

187 F Since 0 is a rational number but is not a negative number, 0 is a non-negative rational number. * A proper subset of a set is a subset that does not contain all the elements of the set. Each element of $\{0\} \cup R_p$ is also an element of R, and $\{0\} \cup R_p$ does not contain all the elements of R.

187 H Each member of R_a is mapped onto just one member of $\overline{R}_n$, and each member of $\overline{R}_n$ is mapped onto just one member of R_a. * Sets that can be put in one-to-one correspondence are equivalent sets. R_a and $\overline{R}_n$ can be put in one-to-one correspondence.

188 R By the definition of the sum of two rational numbers, $(a, 0) + (b, 0) = (a + b, 0)$, or $^+a + {}^+b = {}^+(a + b)$. Since $a + b$ is the mate of $^+(a + b)$, $a + b$ is also the mate of $^+a + {}^+b$.

188 V By the definition of the product of two rational numbers, $(a, 0) \cdot (b, 0) = (a \cdot b, 0)$, or $^+a \cdot {}^+b = {}^+(a \cdot b)$. Since $a \cdot b$ is the mate of $^+(a \cdot b)$, $a \cdot b$ is also the mate of $^+a \cdot {}^+b$.

190 C The order in which you add $\frac{5}{6}$ and $\frac{1}{4}$ does not affect the sum.

190 G Yes * The definition of the sum of two rational numbers * $(c + a, d + b)$

191 H Yes * Yes * The commutative property of addition * Yes

191 J The transitive property of equality. Since true statements are obtained from conditions H and J for each a, b, c, and d, a true statement is also obtained from condition K.

192 F $(ac + bd, ad + bc)$ * $(ca + db, cb + da)$ * Because of the definition of the product of two rational numbers

192 G A: $(a, b) \cdot (c, d) = (ac + bd, ad + bc)$. Definition of the product of two rational numbers. B: $(c, d) \cdot (a, b) = (ca + db, cb + da)$. Definition of the product of two rational numbers. C: $(ac + bd, ad + bc) = (ca + db, da + cb)$. Commutative property of multiplication of rational numbers of arithmetic. D: $(a, b) \cdot (c, d) = (ca + db, da + cb)$. Conditions A and C and the transitive property of equality. E: $(ca + db, da + cb) = (ca + db, cb + da)$. Commutative property of addition of rational numbers of arithmetic. F: $(a, b) \cdot (c, d) = (ca + db, cb + da)$. Conditions D

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and E and the transitive property of equality. G: $(a, b) \cdot (c, d) = (c, d) \cdot (a, b)$. Conditions B and F and the transitive property of equality

194 I From conditions E and G and the transitive property of equality, you can obtain the condition $[(a, b) + (c, d)] + (e, f) = \big(a + (c + e), b + (d + f)\big)$. From the condition just expressed and condition F, you can again use the transitive property of equality to obtain condition H.

198 D One. Any replacement for which you obtain a false statement shows that it is not the case that a true statement is obtained for each replacement of the variables.

198 I Yes. The set of rational numbers are closed, commutative and associative under both addition and multiplication, and multiplication distributes over addition.

198 J Yes. Each member of R_p is also a member of R. * Yes. R_p is closed, commutative, and associative under both addition and multiplication, and multiplication distributes over addition.

198 K Yes. Each member of R_n is also a member of R. * No. R_n is not closed under multiplication. For example, the product of $^-4$ and $^-2$ is $^+8$, and $^+8$ is a member of R_p, not a member of R_n.

202 D Yes * The definition of the sum of two rational numbers

202 E The given rational number of arithmetic. 0 is the identity element for addition of rational numbers of arithmetic. * Yes. $0 + c = c$ and $0 + d = d$.

202 F Yes * Using the transitive property of equality, you can obtain condition G from conditions E and F.

202 C Yes * The definition of the product of two rational numbers. $1 \cdot c + 0 \cdot d$ is the same as $c + 0$, and $1 \cdot d + 0 \cdot c$ is the same as $d + 0$.

202 D $c + 0 = c$ and $d + 0 = d$ because of the identity-element property of addition.

202 E Yes * Using the transitive property of equality, you can obtain condition N from conditions L and M.

203 O Yes * The definition of the product of two rational numbers. For each c and d, $0 \cdot c + 0 \cdot d$ is the same as $0 + 0$, and $0 \cdot d + 0 \cdot c$ is the same as $0 + 0$.

203 P $0 + 0 = 0$ because of the identity-element property of addition.

203 Q Yes * Using the transitive property of equality, you can obtain condition T from conditions R and S.

203 R Yes * Commutative property of multiplication. $0 \cdot x = x \cdot 0$.

205 S Yes * For each rational number $(a, 0)$, there is a rational number $(0, a)$, and for each rational number $(0, a)$ there is a rational number $(a, 0)$.

206 B From the way in which subtraction is related to addition, you know that $^{-}\frac{2}{3} - {}^{+}\frac{1}{2} = x$ also means $^{-}\frac{2}{3} = {}^{+}\frac{1}{2} + x$.

206 D The well-defined property of addition of rational numbers. $^{-}({}^{+}\frac{1}{2})$ can be replaced by $^{-}\frac{1}{2}$ since $^{-}\frac{1}{2} = {}^{-}({}^{+}\frac{1}{2})$.

206 E The associative property of addition of rational numbers

206 F The definition of additive inverse. $^{-}\frac{1}{2} + {}^{+}\frac{1}{2} = 0$.

206 G The identity-element property of addition. $0 + x = x$.

206 H The commutative property of addition. $^{-}\frac{1}{2} + {}^{-}\frac{2}{3} = {}^{-}\frac{2}{3} + {}^{-}\frac{1}{2}$.

207 J $^{+}3 - {}^{-}\frac{7}{8} = x$. $^{+}3 = {}^{-}\frac{7}{8} + x$. $^{-}({}^{-}\frac{7}{8}) + {}^{+}3 = {}^{-}({}^{-}\frac{7}{8}) + ({}^{-}\frac{7}{8} + x)$. $^{+}\frac{7}{8} + {}^{+}3 = {}^{+}\frac{7}{8} + ({}^{-}\frac{7}{8} + x)$. $^{+}\frac{7}{8} + {}^{+}3 = ({}^{+}\frac{7}{8} + {}^{-}\frac{7}{8}) + x$. $^{+}\frac{7}{8} + {}^{+}3 = 0 + x$. $^{+}\frac{7}{8} + {}^{+}3 = x$. $^{+}3 + {}^{+}\frac{7}{8} = x$. $^{+}3 - {}^{-}\frac{7}{8} = {}^{+}3 + {}^{+}\frac{7}{8}$.

207 K $x - y = z$. $x = y + z$. Subtraction related to addition. $^{-}y + x = {}^{-}y + (y + z)$. Well-defined property of addition. $^{-}y + x = ({}^{-}y + y) + z$. Associative property of addition. $^{-}y + x = 0 + z$. Definition of additive inverse. $^{-}y + x = z$. Identity-element property of addition. $x + {}^{-}y = z$. Commutative property of addition. $x - y = x + {}^{-}y$. Transitive property of equality

207 P No. For example, $3 - 5$, $100 - 200$, and $4 - 40$ are not members of N. * No. For example, $\frac{3}{5} - \frac{4}{5}$, $7.6 - 18$, and $2\frac{1}{2} - 21\frac{3}{4}$ are not members of R_a.

207 Q The difference of two given rational numbers is equal to the sum of two rational numbers. Since R is closed under addition and since every rational number has an additive inverse, R must also be closed under subtraction. For each x and y, $x - y = x + {}^{-}y$. Since $x + {}^{-}y \in R$, so is $x - y \in R$.

207 R No * $^{+}3 - {}^{+}5 = {}^{-}2$, but $^{+}5 - {}^{+}3 = {}^{+}2$. $^{+}\frac{1}{2} - {}^{-}\frac{1}{3} = {}^{+}\frac{5}{6}$, but $^{-}\frac{1}{3} - {}^{+}\frac{1}{2} = {}^{-}\frac{5}{6}$. $^{-}\frac{13}{2} - {}^{+}\frac{3}{4} = {}^{-}\frac{29}{4}$, but $^{+}\frac{3}{4} - {}^{-}\frac{13}{2} = {}^{+}\frac{29}{4}$.

207 S $({}^{+}3 - {}^{-}\frac{7}{8}) - {}^{-}5 = {}^{+}\frac{31}{8} - {}^{-}5$. $^{+}\frac{31}{8} - {}^{-}5 = {}^{+}\frac{71}{8}$. $^{+}3 - ({}^{-}\frac{7}{8} - {}^{-}5) = {}^{+}3 - {}^{+}\frac{33}{8}$. $^{+}3 - {}^{+}\frac{33}{8} = {}^{-}\frac{9}{8}$. Therefore, $({}^{+}3 - {}^{-}\frac{7}{8}) - {}^{-}5$ does not equal $^{+}3 - ({}^{-}\frac{7}{8} - {}^{-}5)$. Thus, subtraction of rational numbers does not have the associative property.

208 G You know that their product must be $^{+}1$, and the product of two positive numbers is positive. * The product of two negative numbers is positive.

208 I There is no rational number by which you can multiply 0 to obtain $^{+}1$. * No. Because 0 does not have a reciprocal, 0 cannot be the reciprocal of any rational number.

209 A Z * Yes. By the definition of the product of two rational numbers, $(a, 0) \times (\frac{1}{a}, 0) = (a \cdot \frac{1}{a} + 0, 0 + 0)$, and $(a \cdot \frac{1}{a} + 0, 0 + 0) = (a \cdot \frac{1}{a}, 0)$. * Yes. $a \cdot \frac{1}{a} = 1$. * Yes. By the transitive property of equality, if $(a, 0) \times (\frac{1}{a}, 0) = (a \cdot \frac{1}{a}, 0)$ and $(a \cdot \frac{1}{a}, 0) = (1, 0)$, then $(a, 0) \times (\frac{1}{a}, 0) = (1, 0)$.

209 G Yes * The universe for a and b is Z. For each $(a, 0)$, there is an $(\frac{1}{a}, 0)$. For each $(0, b)$, there is an $(0, \frac{1}{b})$.

210 L From the associative property of multiplication, $(y \cdot {}^{-}\frac{2}{5})({}^{-}\frac{5}{2})$ is the same as $y({}^{-}\frac{2}{5} \cdot {}^{-}\frac{5}{2})$.

210 M Because $^{-}\frac{5}{2}$ is the reciprocal of $^{-}\frac{2}{5}$, $({}^{-}\frac{2}{5} \cdot {}^{-}\frac{5}{2}) = 1$.

210 N From the identity-element property of multiplication, $y(1) = y$.

210 P $^{-}\frac{10}{7} \div {}^{-}\frac{15}{14} = x$. $^{-}\frac{10}{7} = x \cdot {}^{-}\frac{15}{14}$. $({}^{-}\frac{10}{7})({}^{-}\frac{14}{15}) = (x \cdot {}^{-}\frac{15}{14})({}^{-}\frac{14}{15})$. $({}^{-}\frac{10}{7})({}^{-}\frac{14}{15}) = x({}^{-}\frac{15}{14} \cdot {}^{-}\frac{14}{15})$. $({}^{-}\frac{10}{7})({}^{-}\frac{14}{15}) = x(1)$. $({}^{-}\frac{10}{7})({}^{-}\frac{14}{15}) = x$. $^{-}\frac{10}{7} \div {}^{-}\frac{15}{14} = ({}^{-}\frac{10}{7})({}^{-}\frac{14}{15})$. * $\frac{4}{3}$

211 F From the quotient property, $x \div y = x \cdot (1/y)$. Since each non-zero rational number has a reciprocal and since $y \in Z_r$, you know that $(1/y) \in Z_r$. From the closure property of multiplication of rational numbers, $x \cdot (1/y) \in R$. Therefore, $x \div y \in R$.

211 G The quotient of a given number and 0 would

be the product of the given number and the reciprocal of 0. Since 0 does not have a reciprocal, there is no product of a given number and the reciprocal of 0. Therefore, there is no quotient of a given number and 0.

211 I No * Examples are: $^{+}4 \div {}^{+}2 = {}^{+}2$, but $^{+}2 \div {}^{+}4 = {}^{+}\frac{1}{2}$. $^{+}4 \div {}^{-}\frac{2}{9} = {}^{-}6$, but $^{-}\frac{2}{9} \div {}^{+}4 = {}^{-}\frac{1}{6}$. $^{-}\frac{3}{5} \div {}^{-}\frac{2}{15} = {}^{+}\frac{9}{2}$, but $^{-}\frac{2}{15} \div {}^{-}\frac{3}{5} = {}^{+}\frac{2}{9}$.

211 J No * $(^{-}\frac{5}{8} \div {}^{+}\frac{15}{4}) \div {}^{-}\frac{7}{12} = {}^{-}\frac{1}{6} \div {}^{-}\frac{7}{12}$. $^{-}\frac{1}{6} \div {}^{-}\frac{7}{12} = {}^{+}\frac{2}{7}$. $^{-}\frac{5}{8} \div (^{+}\frac{15}{4} \div {}^{-}\frac{7}{12}) = {}^{-}\frac{5}{8} \div {}^{-}\frac{45}{7}$. $^{-}\frac{5}{8} \div {}^{-}\frac{45}{7} = {}^{+}\frac{7}{72}$. Therefore, $(^{-}\frac{5}{8} \div {}^{+}\frac{15}{4}) \div {}^{-}\frac{7}{12}$ does not equal $^{-}\frac{5}{8} \div (^{+}\frac{15}{4} \div {}^{-}\frac{7}{12})$, and division of rational numbers does not have the associative property.

213 G The basic pair for $^{-}2\frac{1}{5}$ is $(0, 2\frac{1}{5})$. $(0, 2\frac{1}{5}) = (0, 2.2)$. Therefore, the decimal $^{-}2.2$ names $^{-}2\frac{1}{5}$, and statement D is true. The basic pair for $^{-}7\frac{4}{25}$ is $(0, 7\frac{4}{25})$. $(0, 7\frac{4}{25}) = (0, 7.16)$. Therefore, the decimal $^{-}7.16$ names $^{-}7\frac{4}{25}$, and statement E is true. The basic pair for $^{+}\frac{6}{10000}$ is $(\frac{6}{10000}, 0)$. $(\frac{6}{10000}, 0) = (.0006, 0)$. Therefore, the decimal $^{+}.0006$ names $^{+}\frac{6}{10000}$, and statement F is true.

215 J Multiply $^{+}1.1$ by $^{+}10$, or 10. * Multiply $^{-}.165$ by 10. * Yes

215 P Yes * .038. Since the last remainder is less than $\frac{1}{2}$ times the divisor, the quotient should be rounded to the next lesser thousandth.

219 C The only factors of 2 are 2 and 1. The only factors of 5 are 5 and 1. * $2^1 = 2$, and $5^1 = 5$. * Yes. The numeral $2^1 \cdot 5^1$ expresses 10 as the product of prime numbers.

219 H Multiply both the numerator and the denominator of the fraction $\frac{3}{8}$ by 125. * .375

219 I Divide both the numerator and the denominator of the fraction $\frac{27}{90}$ by 9. * .3

220 Q Yes * When you find the quotient of a given number and 1, you obtain the given number as the quotient and a remainder of 0.

220 R $\frac{3}{24}$ is not a basic fraction. $\frac{3}{24} = \frac{1}{8} \cdot \frac{1}{8}$ is a basic fraction, and each prime factor of 8 is also a prime factor of 10.

220 X $^{-}.5$ * $^{-}\frac{1}{2}$ * Yes * No matter how many digits for 0 you write after the digit 5 in the decimal $^{-}.5$, you still have a decimal for $^{-}\frac{1}{2}$. Therefore, you can think of 0 as the repetend of the decimal for $^{-}\frac{1}{2}$.

220 A 1 * They are the same.

220 F 2 * 6

221 O The non-zero component of the basic pair for every rational number is indicated by a fraction. To find a decimal for the fraction, you can think of the fraction as the quotient of its numerator and denominator. In the division process, the number of possible remainders cannot be greater than the denominator. Therefore, at some point in the division process, the remainders will begin to repeat, and there will be a repetend in the decimal for the quotient. At most, the number of digits in the repetend will be 1 less than the denominator. Since the non-zero component of the basic pair for the rational number can be named by a repeating decimal, the rational number can be named by a repeating decimal.

223 A If a decimal expresses a rational number and the decimal does not have a symbol for either "positive" or "negative," then you know that the decimal expresses a positive rational number.

223 G 3 * 10 * The third power * Yes. $10^3 = 10 \times 10 \times 10$, and $10 \times 10 \times 10 = 1000$.

224 C The associative property of multiplication

225 D $10 \times 10 \times 10$, or 1000 * Yes

226 M $45{,}000 = 4.5 \times 10^4$, and $5100 = 5.1 \times 10^3$.

226 N Yes * The commutative and associative properties of multiplication. $(4.5 \times 10^4) \times (5.1 \times 10^3) = 4.5 \times (10^4 \times 5.1) \times 10^3$. $4.5 \times (10^4 \times 5.1) \times 10^3 = 4.5 \times (5.1 \times 10^4) \times 10^3$. $4.5 \times (5.1 \times 10^4) \times 10^3 = (4.5 \times 5.1) \times (10^4 \times 10^3)$.

226 O $4.5 \times 5.1 = 22.95$, and $10^4 \times 10^3 = 10^7$.

228 C You must know the meaning of the symbol 10^0.

228 F If a number is used as a factor, it must be used as a factor at least once.

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For this reason, we define 10^0 without using the idea of a factor. However, we can use positive integral powers to help define 10^0.

228 G Since $0 + 3 = 3$, 10^{0+3} is the same as 10^3, and statement A is the same statement as statement B.

228 I The number 1 is the only identity element for multiplication of rational numbers. Therefore, the only number by which $(^{-}\frac{1}{2})^4$ can be multiplied to obtain $(^{-}\frac{1}{2})^4$ is 1. * Since $0 + 4 = 4$, $(^{-}\frac{1}{2})^{0+4}$ is the same as $(^{-}\frac{1}{2})^4$, and statement C is the same statement as statement D.

228 J Yes * x^n is the same as x^{0+n} because $n = 0 + n$.

229 L Yes * The only number that can be multiplied by itself to give that number is 1. For each replacement of x in $x^0 \cdot x^0 = x^0$, the true statement $1 \cdot 1 = 1$ is obtained.

229 G If the product of two numbers is 1, each of the numbers is the reciprocal of the other.

229 H Since $(^{-}\frac{3}{4})^{-5+5} = (^{-}\frac{3}{4})^0$, and $(^{-}\frac{3}{4})^0 = 1$, statement H tells you that the product of $(^{-}\frac{3}{4})^{-5}$ and $(^{-}\frac{3}{4})^5$ is 1. Therefore, $(^{-}\frac{3}{4})^{-5}$ is the reciprocal of $(^{-}\frac{3}{4})^5$. The symbol $1/(^{-}\frac{3}{4})^5$ also names the reciprocal of $(^{-}\frac{3}{4})^5$. $(^{-}\frac{3}{4})^{-5} \cdot (^{-}\frac{3}{4})^5 = (^{-}\frac{3}{4})^{-5+5}$. $(^{-}\frac{3}{4})^{-5} \cdot (^{-}\frac{3}{4})^5 = (^{-}\frac{3}{4})^0$. $(^{-}\frac{3}{4})^{-5} \cdot (^{-}\frac{3}{4})^5 = 1$.

229 I x^{-n}, or $1/x^n$ * The universe for n is I_p, so n is a positive integer. The inverse of a positive integer is a negative integer.

229 J If $x^{-n} \cdot x^n = x^{-n+n}$, then $x^{-n+n} = x^0$, and $x^0 = 1$. Therefore, $x^{-n} \cdot x^n = 1$, and x^{-n} must be the reciprocal of x^n. The reciprocal of x^n is $1/x^n$.

230 K $1/x^n$ and x^{-n} are the same. * x^{-n} and x^n are reciprocals of each other.

230 L If you replace x in x^{-n} by 0, you obtain 0^{-n}, which would be equal to $1/0^n$. For each replacement of n from I_p, $0^n = 0$, and $1/0^n = 1/0$. But 0 does not have a reciprocal, and 1/0 is not defined.

230 F Statement M:
$10^{-3} \cdot 10^{-5} = 1/10^3 \cdot 1/10^5$.
$10^{-3} \cdot 10^{-5} = 1/(10^3 \cdot 10^5)$.
$10^{-3} \cdot 10^{-5} = 1/10^{3+5}$.
$10^{-3} \cdot 10^{-5} = 1/10^8$.
$10^{-3} \cdot 10^{-5} = 10^{-8}$.
Statement N:
$10^9 \cdot 10^{-2} = 10^9 \cdot 1/10^2$.
$10^9 \cdot 10^{-2} = 10^9/10^2$.
$10^9 \cdot 10^{-2} = 10^7/1$.
$10^9 \cdot 10^{-2} = 10^7$.

231 B $10^{-6}/10^{-2}$ means $10^{-6} \div 10^{-2}$. From the definition of negative integral powers, $10^{-6} = 1/10^6$ and $10^{-2} = 1/10^2$. This means that $10^{-6} \div 10^{-2} = 1/10^6 \div 1/10^2$. Using the transitive property of equality, you can obtain statement P from $10^{-6}/10^{-2} = 10^{-6} \div 10^{-2}$ and from $10^{-6} \div 10^{-2} = 1/10^6 \div 1/10^2$.

231 C Yes * 10^2 is the reciprocal of $1/10^2$. Using the quotient property of rational numbers, you know that $1/10^6 \div 1/10^2 = (1/10^6) \cdot (10^2)$. Using the transitive property of equality, you can obtain statement Q from statement P, and $(1/10^6) \div (1/10^2) = (1/10^6) \cdot (10^2)$.

231 D $(1/10^6) \cdot (10^2)$ is the same as $10^2/10^6$.

231 J Yes * Since $10^0 = 1$, $10^0/10^{-4} = 1/10^{-4}$, and $1/10^{-4} = 10^4$.

233 H Form the product of two factors. Use a decimal for the rational number to name one factor and the numeral 10^0 to name the other factor.

233 J Yes. $.625 \times 10 = 6.25$, and $10^1 = 10$. * Yes. $1 \leqq 6.25 < 10$. * $.625 = 6.25 \div 10$; so $6.25 \div 10 = 6.25 \div 10^1$.

233 O $10^3 * 10^3$

233 Q $.0007 \times 10000 = 7$.
$.0007 \times 10^4 = 7$.
$.0007 = 7 \div 10^4$.
$.0007 = 7 \times 1/10^4$.
$.0007 = 7 \times 10^{-4}$.

234 C The associative and commutative properties of multiplication and the transitive property of equality * $(3 \times 10^6) \times (2.6 \times 10^4) = 3 \times (10^6 \times 2.6) \times 10^4$. Associative property of multiplication. $(3 \times 10^6) \times (2.6 \times 10^4) = 3 \times (2.6 \times 10^6) \times 10^4$. Commutative property of multiplication. $(3 \times 10^6) \times (2.6 \times 10^4) = (3 \times 2.6) \times (10^6 \times 10^4)$. Associative property of multiplication

239 E From condition B, you know that, for the x you are to find, $.6x = 4.2$. By the well-defined property of multiplication, $.6x$ in the product $\frac{1}{.6}(.6x)$ may be replaced by 4.2 without affecting the product.

239 C Condition H can be obtained from condition G by using the difference property of rational numbers.

240 N From condition N, you know that, for the x you are to find, $\frac{3}{2} + {}^-x = 5$. Therefore, by the well-defined property of addition, $(\frac{3}{2} + {}^-x)$ in $({}^-\frac{3}{2}) + (\frac{3}{2} + {}^-x)$ can be replaced by 5 without affecting the sum. Thus, condition O can be obtained from condition N by using the well-defined property of addition.

241 K $\frac{1}{3}a + 7 = {}^-20$.
$(\frac{1}{3}a + 7) + ({}^-7) = {}^-20 + ({}^-7)$.
$\frac{1}{3}a + (7 + {}^-7) = {}^-27$.
$\frac{1}{3}a + 0 = {}^-27$.
$\frac{1}{3}a = {}^-27$.
$3(\frac{1}{3}a) = 3({}^-27)$.
$(3 \cdot \frac{1}{3})a = {}^-81$.
$1a = {}^-81$. $a = {}^-81$.
$\{a | \frac{1}{3}a + 7 = {}^-20\} = \{{}^-81\}$.

243 J ${}^-\frac{11}{4}$, or ${}^-2\frac{3}{4}$, is greater than $^-3$ and is a member of $\{x | x > {}^-3\}$. * Yes. ${}^-3 < {}^-\frac{11}{4}$, and ${}^-\frac{11}{4} < {}^-\frac{5}{2}$; so, ${}^-\frac{11}{4}$ is between $^-3$ and ${}^-\frac{5}{2}$.

243 K Assume that a is the least member of $\{x | x > {}^-3\}$. From the density property of the set of rational numbers, there is a rational number r such that ${}^-3 < r$ and $r < a$. Since ${}^-3 < r$, r is a member of $\{x | x > {}^-3\}$, and r is less than a. This means that a, or any rational number you choose, cannot be the least member of $\{x | x > {}^-3\}$ because there is always a rational number less than a, but greater than $^-3$.

244 J $\{y | y > {}^-\frac{19}{4} \wedge y < {}^-\frac{9}{4}\}$ is the solution set of the compound condition. But $\{y | y > {}^-\frac{19}{4} \wedge y < {}^-\frac{9}{4}\} = \{y | {}^-\frac{19}{4} < y < {}^-\frac{9}{4}\}$. Therefore, $\{y | {}^-\frac{19}{4} < y < {}^-\frac{9}{4}\}$ is the solution set of the compound condition.

244 M The solution set of a compound condition that consists of two simple conditions and the connective "or" contains each solution of either or both simple conditions.

244 O $x \geqq {}^-.9$ is the same condition as $x > {}^-.9 \vee x = {}^-.9$; therefore, their solution sets have the same members. The symbol $\geqq$ means "greater than or equal to."

244 R The condition that it is not the case that $4m - 7 > 5$ * $\overline{R}_n$. This is the set of non-negative rational numbers, which contains all the positive rational numbers and 0.

244 T Yes * The members of $\{m | \sim (m > 3)\}$ are all of the non-negative rational numbers that are either less than or equal to 3. The members of $\overline{R}_n$ that do not satisfy $4m - 7 > 5$ do satisfy $\sim (4m - 7 > 5)$.

247 R $(9.3 \times 10^7) = (2.4 \times 10^5)m$ can be obtained from $(9.3 \times 10^7)/(2.4 \times 10^5) \sim m/1$ by using the definition of equivalent ordered pairs.

247 I Yes. Since $n > {}^-5$ also means ${}^-5 < n$ and $n < 2940$ is equivalent to $3440 - n > 500$, the condition ${}^-5 < n < 2940$ is equivalent to the condition $n > {}^-5 \wedge 3440 - n > 500$. Therefore, the two conditions have the same solution set.

248 P The union * The solution set of a compound condition that includes two simple conditions and the connective "or" contains all those numbers that satisfy either or both of the simple conditions.

248 D Yes. Since $\{x|x < 9.6\}$ contains all the members of the universe that satisfy $^-3.2 + x < 6.4$, the complement of $\{x|x < 9.6\}$, which is $\{x|x < 9.6\}$, contains all the members of the universe that do not satisfy $^-3.2 + x < 6.4$. * $x \geqq 9.6$ and $\sim(^-3.2 + x < 6.4)$ are equivalent conditions.

254 L The square of any positive number is positive because the product of any two positive numbers is positive. The square of any negative number is positive because the product of any two negative numbers is positive. Any positive number is greater than 0.

254 C Yes * $(\frac{2}{3})^2 = \frac{4}{9}$, $(\frac{4}{6})^2 = \frac{16}{36}$, and $\frac{4}{9} = \frac{16}{36}$.

254 F Using the well-defined property of multiplication, since $a = b$, you may replace each a in $a \cdot a$ by b without affecting the product. Therefore, $a \cdot a = b \cdot b$. $a \cdot a = a^2$, and $b \cdot b = b^2$, so $a \cdot a = b \cdot b$ is the same as $a^2 = b^2$.

254 K They are the same; the inverse of the inverse of a given number is the given number. * They are equal.

255 N No * The two numbers may be additive inverses; for example, $(\frac{2}{3})^2 = (^-\frac{2}{3})^2$.

255 U Examples are: $^-7 < 5$, but $(^-7)^2 > 5^2$. $^-20 < ^-10$, but $(^-20)^2 > (^-10)^2$.

255 E There is no positive or negative rational number that, when multiplied by itself, gives a negative rational number.

256 W From the definition of square root, $\sqrt{a}$ is the number that, when used twice as a factor, gives a as the product. Therefore, $(\sqrt{a}) \times (\sqrt{a}) = a$, or $(\sqrt{a})^2 = a$.

256 D 625 * Find the numeral 625 in the column headed by the symbol n^2. Then, to the left of the numeral 625, the numeral 25, printed in blue in the column headed by the letter n, expresses the principal square root of 625.

257 G Yes. The product of any two positive integers is a positive integer. * Because I_p is closed under multiplication, if $x \in I_p$, then $x^2 \in I_p$.

259 O Each rational number can be expressed by a repeating decimal, so, if a number cannot be expressed by a repeating decimal, it must not be a rational number.

261 A The three dots to the right of numeral G indicate that the digits in the decimal go on and on, but the decimal is not a repeating decimal. * No. The decimal for $\sqrt{2}$ is an infinite decimal. $\sqrt{2}$ cannot be named by a terminating decimal.

261 D 5 * .101001000 − 10000100000100000000100 . . .

263 P $\overline{R}$ is an infinite set. * The members of I_p that are not perfect squares form an infinite set. The square root of every positive integer that is not a perfect square is an irrational number.

267 L Each real number has an additive inverse. The difference of two real numbers is equal to the sum of the minuend and the additive inverse of the subtrahend. The sum of two real numbers is a real number.

267 P Each real number has a reciprocal. The quotient of any two nonzero real numbers is the same as the product of the dividend and the reciprocal of the divisor. The product of any two real numbers is a real number.

267 R Yes * The set of real numbers is closed, commutative, and associative under both addition and multiplication; and multiplication distributes over addition.

267 C The square root of a positive integer that is not a perfect square is an irrational number. $\sqrt{2}$ is not an element of $\overline{R}_n$, but $\sqrt{2} \in \overline{R}$.

268 E Statements B, C, and F * No. Inequalities are true statements. Statements A, D, and E are false. * Yes

268 F For each a, it is not true that $a < a$. * For each a, it is not true that $a > a$. * For each a, it is not true that $a \neq a$.

270 K Yes. Since R is a subset of D, the idea of "greater than" is related to

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the idea of "less than" for rational numbers in the same way as these ideas are related for real numbers. * Yes. Since I is a subset of D, the idea of "greater than" is related to the idea of "less than" for integers in the same way as these ideas are related for real numbers.

270 D Yes. By the closure property of addition, $a + c$ and $b + c$ are real numbers. Therefore, $a + c$ and $b + c$ can be treated in the same way with respect to the ideas of "less than" and "greater than" as any other real numbers. * The universe for each variable is D. For each a, b, and c, if $b > a$, then $b + c > a + c$.

271 H Yes. By the closure property of multiplication, ac and bc are real numbers. Therefore, ac and bc can be treated in the same way with respect to the ideas of "less than" and "greater than" as any other real numbers. * The universe for each variable is D. For each a, b, and c, if $b > a$ and $c > 0$, then $bc > ac$.

271 K For the positive-multiplier product property, the third number is greater than 0; while, for the negative-multiplier product property, the third number is less than 0. For the positive-multiplier product property, the product of the first and third numbers is less than the product of the second and third numbers; while,

for the negative-multiplier product property, the product of the first and third numbers is greater than the product of the second and third numbers.
271 M Yes. * The universe for each variable is D. For each a, b, and c, if $b > a$ and $c < 0$, then $bc < ac$.
272 B It is possible to find at least one real number between any two given real numbers. If a and b are any two real numbers, then $\frac{1}{2}(a+b)$ is a real number between a and b. The universe for a and b is D. * The set of real numbers has the density property. If, for any two given members of a set of numbers, you can find another member of the set that is between the two given members, then that set of numbers has the density property.
272 F No. Some points in the line do not correspond to members of R. * No. Some points in the line do not correspond to members of I.
278 P Use the reciprocal of 5 and the well-defined property of multiplication. Since $5x = 10$, then $\frac{1}{5}(5x) = \frac{1}{5}(10)$ and $\frac{1}{5}(5x) = \frac{1}{5}(10)$ is equivalent to $x = 2$. * 2
278 S Use the well-defined property of multiplication. Since $x = 2$, the product $5(x)$ is the same as the product $5(2)$. Therefore, $5x = 10$. * Use the well-defined property of multiplication. Since $2 + 3 = 5$, you can replace 5 in $5x$ by $(2 + 3)$ without affecting the product. * Use the distributive property to obtain $2x + 3x$ from $(2 + 3)x$. * Yes
279 K $3a = 14$. $\frac{1}{3}(3a) = \frac{1}{3}(14)$. Well-defined property of multiplication. $(\frac{1}{3} \cdot 3)a = \frac{14}{3}$. Associative property of multiplication and definition of product. $1a = \frac{14}{3}$. Definition of product. $a = \frac{14}{3}$. Identity-element property of multiplication * $\frac{14}{3}$
279 C Since the difference of any two real numbers $(a - b)$ is also the sum of two real numbers $(a + {}^{-}b)$, multiplication distributes over subtraction. * Yes
279 F ${}^{+}.4$ * Well-defined property of addition
281 J $y < \frac{7}{4}$. $4 \cdot y < 4 \cdot \frac{7}{4}$. Positive-multiplier product property of "less than." $4y < 7$. Definition of product
281 B Yes. Every negative number is less than 0. * Yes * The negative-multiplier product property of "less than"
282 H The sum property of "less than," which can also be expressed in terms of "greater than"
282 J Yes. Associative property of addition * Yes. Definition of sum * Yes. Identity-element property of addition and definition of sum
283 S Yes. $\frac{11}{4} = \frac{11}{4}$. * Yes. $0 < \frac{11}{4}$. * Yes. ${}^{-}\sqrt{2} < \frac{11}{4}$. * Yes. Every negative real number is less than $\frac{11}{4}$.
285 T To the right * To the left
285 U Above * Below
285 X The first quadrant * The third quadrant
285 Y The fourth quadrant * The second quadrant
287 D Examples are (1, 3) and $(\frac{1}{4}, \frac{3}{2})$. * Yes * For each member of D that you use as a replacement for a, there is a b that satisfies the condition. D is an infinite set.
288 I $1 * 3 * 0 * {}^{-}1 * {}^{-}5$ * If a is replaced by 0, b must be greater than 1. If a is replaced by 1, b must be greater than 3. If a is replaced by $\frac{-1}{2}$, b must be greater than 0. If a is replaced by ${}^{-}1$, b must be greater than ${}^{-}1$. If a is replaced by ${}^{-}3$, b must be greater than ${}^{-}5$.
288 N The graph of $\{(a, b) | b < 2a + 1\}$ is a picture of a half-plane bounded by the line whose picture is the graph of $\{(a, b) | b = 2a + 1\}$. The picture of the half-plane is below and to the right of the picture of the line.
289 C Use a straightedge to draw a picture of the line determined by the two points represented by the two dots. * Two points determine exactly one line.
289 H (1, 2) * The dot for point (1, 2). Since the solution set of the compound condition is the intersection of the solution sets of the two simple conditions, the graph of the solution set of the compound condition is the intersection of the graphs of the solution sets of the simple conditions. * $\{(1, 2)\}$
290 L Replace x in $y = 2x$ by 1. * $\{(1, 2)\}$ * Condition A is equivalent to condition E, so (1, 2) also satisfies condition A.
290 M A $x = y + 2 \wedge 2y = x + 1$.
B $x = y + 2 \wedge 2y = y + 2 + 1$.
C $x = y + 2 \wedge y = 3$.
D $x = 3 + 2 \wedge y = 3$.
E $x = 5 \wedge y = 3$.
$\{(x, y) | x = y + 2 \wedge 2y = x + 1\} = \{(5, 3)\}$.
290 B $\{(x, y, z) | x \in \mathrm{D} \wedge y \in \mathrm{D} \wedge z \in \mathrm{D}\}$
293 M Yes * $c > 0$ also means $0 < c$. * Yes. Since $c < 5\frac{1}{4}$ is equivalent to $12\frac{3}{4} - c > 7\frac{1}{2}$, and $0 < c$ is equivalent to $c > 0$, the condition $0 < c < 5\frac{1}{4}$ is equivalent to $12\frac{3}{4} - c > 7\frac{1}{2} \wedge c > 0$. Therefore, $0 < c < 5\frac{1}{4}$ has the same solution set as $12\frac{3}{4} - c > 7\frac{1}{2} \wedge c > 0$.
294 T Yes. $n \geqq 0$ also means $0 \leqq n$. * Yes. Since $0 \leqq n$ is equivalent to $n \geqq 0$, and $n < 235.5$ is equivalent to $n + 114.5 < 350$, the condition $0 \leqq n < 235.5$ is equivalent to $n + 114.5 < 350 \wedge n \geqq 0$. Therefore, the solution set of $0 \leqq n < 235.5$ is the same as the solution set of $n + 114.5 < 350 \wedge n \geqq 0$.
298 O Yes * Condition G was obtained from $60z + 35z = 475$ by using the distributive property. Condition H was obtained from condition G by using the well-defined property of multiplication. $60 + 35 = 95$; so, $(60 + 35)z = 95z$.

304 D 4 * 5 * Since point T is to the right of point X, the coördinate of T is greater than the coördinate of X. This means that the absolute difference of the coördinates of the endpoints of $\overline{ST}$ is greater than the absolute difference of the coördinates of the endpoints of $\overline{SX}$; therefore the measure of $\overline{ST}$ is greater than the measure of $\overline{SX}$. * Since point T is to the left of point Y, the coördinate of T is less than the coördinate of Y. This means that the absolute difference of the coördinates of the endpoints of $\overline{ST}$ is less than the absolute difference of the coördinates of the endpoints of $\overline{SY}$; so the measure of $\overline{ST}$ is less than the measure of $\overline{SY}$.

304 G The segments represented by the two different pieces of chalk may not be congruent.

305 I The metric system has a base of ten. For example, 1 cm. = 10 mm., 1 dm. = 10 cm., 1 m. = 10 dm., and so on.

311 J A physical measure of $3\frac{2}{4}$ in. indicates a unit of length of $\frac{1}{4}$ in. A physical measure of $3\frac{1}{2}$ in. indicates a unit of length of $\frac{1}{2}$ in.

311 K $\frac{1}{12}$ ft. * 1 cm. * $\frac{1}{3}$ yd.

311 L A unit of length of 1 yd. was used to obtain 14 yd., and a unit of length of $\frac{1}{3}$ yd. was used to obtain $14\frac{0}{3}$ yd.

312 T $4\frac{0}{2}$ ft. consists of 8 unit lengths (8 one-half foot units). $4\frac{1}{2}$ ft. consists of 9 unit lengths (9 one-half foot units). We use $4\frac{0}{2}$ ft. rather than $4\frac{1}{2}$ ft. because $4\frac{0}{2}$ ft. consists of an even number of unit lengths.

313 J You can obtain the unit of measure by multiplying the greatest possible error by 2.

313 O 5.75 and 5.85 * $\{x \mid 5.75 \leqq x \leqq 5.85\}$

313 Q $\{x \mid 25.35 \leqq x \leqq 25.45\}$

313 R The tolerance is used to find the limits of the interval. The lesser limit is found by subtracting the tolerance from 25.4. The greater limit is found by adding the tolerance to 25.4. * To find the lesser limit of the interval determined by a physical measure of 25.4, you subtract the greatest possible error from 25.4. To find the greater limit, you add the greatest possible error to 25.4. In other words, both the tolerance and the greatest possible error are used to find the lesser and greater limits of an interval.

313 S The greatest possible error of a physical measure is the tolerance of the interval determined by the physical measure. The numeral 25.4 tells you that the greatest possible error of the physical measure 25.4 is .05. Therefore, the tolerance of $x \approx 25.4$ is .05.

314 E The physical length 135 yd. * The greatest possible error of 135 yd. is less than the greatest possible error of 6.2 mi. The greatest possible error of 135 yd. is $\frac{1}{2}$ yd. The greatest possible error of 6.2 mi. is .05 mi. $\frac{1}{2}$ yd. is less than .05 mi. Because the greatest possible error of 135 yd. is less than the greatest possible error of 6.2 mi., 135 yd. is a more precise measure than 6.2 mi.

316 J Only the first three digits, 7, 5, and 0, are necessary to express 750 ten-mile units of length.

317 B Only the first four digits, 9, 2, 8, and 7, are needed to express the number of 10,000-mile units in 92,870,000 mi.

317 D Yes. Because the unit is 1000 mi., you need the first 5 digits to express the number of 1000-mile units in 92,870,000 mi. Therefore, you should use these 5 digits in expressing the number from 1 to 10 in the scientific notation for 92,870,000.

319 F 3 km., 7.1 km., and 2.8 km. * Yes * The greatest possible error of 12.9 is .05. The lesser limit of the interval determined by 12.9 is 12.9 − .05, or 12.85. The greater limit is 12.9 + .05, or 12.95. The interval determined by a physical measure is a closed interval.

319 G 3 km. has a greatest possible error of .5 km., which is greater than the greatest possible error of either 7.1 km. or 2.8 km. * Yes

319 K The difference of 11.32 and 7.8 is 3.52. 7.8 yd. is less precise than 11.32 yd.; therefore, you should round 3.52 to 3.5. A good approximation of the difference of 11.32 yd. and 7.8 yd. is 3.5 yd.

320 Q $71\frac{3}{4}$ ft. $71\frac{8}{12}$ is nearer to $71\frac{3}{4}$ than to $71\frac{2}{4}$.

320 D The mathematical measure of the length is greater than or equal to 3.245 ft., and the mathematical measure of the width is greater than or equal to 2.45 ft. * Yes

320 E The mathematical measure of the length is less than or equal to 3.255 ft., and the mathematical measure of the width is less than or equal to 2.55 ft. * Yes

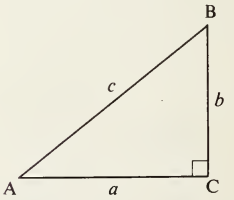
321 I The numeral 2.5 has fewer significant digits than the numeral 3.25.

325 A A triangle with no two of its sides congruent to each other is a scalene triangle. No two sides of $\triangle ABC$ are congruent to each other because each side has a different measure. No two sides of $\triangle DEF$ are congruent to each other because each side has a different measure.

325 B The corresponding sides are $\overline{AB}$ and $\overline{DE}$, $\overline{BC}$ and $\overline{EF}$, and $\overline{AC}$ and $\overline{DF}$. The corresponding angles are $\angle A$ and $\angle D$, $\angle B$ and $\angle E$, and $\angle C$ and $\angle F$.

325 D No. $\overline{AB}$ is not congruent to $\overline{DE}$ because they have different measures. $\overline{BC}$ is not congruent to $\overline{EF}$ because they have different measures. $\overline{AC}$ is not congruent to $\overline{DF}$ because they have different measures. * No. If the two

triangles were congruent, then the corresponding sides would be congruent.
326 K The rate pair for sides MP and RS is 2.3/2.3. The rate pair for sides PQ and ST is 1.8/1.8. The rate pair for sides MQ and RT is 2.0/2.0. * Yes. Each of these rate pairs is equivalent to 1/1. * Yes. The corresponding angles are congruent, and the measures of the corresponding sides are proportional.
327 B If the corresponding angles of two triangles are congruent, then the two triangles are similar.
327 F $\angle S^\circ = 70$. * $\angle R^\circ = 70$. * For any triangle, the sum of the measures of the angles in degrees is 180. * Yes. If two angles have the same measure, then they are congruent.
328 P The measures of corresponding sides of similar triangles are proportional. a and d and b and e represent the measures of corresponding sides of $\triangle UVW$ and $\triangle XYZ$.
328 R The definition of equivalent rate pairs. The definition not only tells you that if $a/d \sim b/e$, then $a \cdot e = b \cdot d$; but it also tells you that, if $a \cdot e = d \cdot b$, then $a/b \sim d/e$. * Yes
329 U $b/e \sim c/f$. Measures of corresponding sides of similar triangles are proportional. $b \times f = c \times e$. Definition of equivalent rate pairs. $b \times f = e \times c$. Commutative property of multiplication. $b/c \sim e/f$. Definition of equivalent rate pairs
329 V $a/d \sim c/f$. Measures of corresponding sides of similar triangles are proportional. $a \times f = c \times d$. Definition of equivalent rate pairs. $a \times f = d \times c$. Commutative property of multiplication. $a/c \sim d/f$. Definition of equivalent rate pairs
330 B {A, B} * Side AB
330 C The sum of the measures of the angles of a triangle in degrees is 180. If one of the angles has a measure of 90, then each of the other two angles has a measure less than 90. An angle whose measure is less than 90 is an acute angle.
331 G An acute angle * Side AC * Side BC
331 I For any right triangle, the sum of the measures of the two acute angles is 90. This is true because the sum of the measures of the angles of a triangle in degrees is 180, and the measure of the right angle is 90.
331 E Yes. For two similar triangles, the rate pair that represents the comparison of the measures of any two sides of one triangle is equivalent to the rate pair that represents the comparison of the measures of the corresponding sides of the other triangle. * Yes * Yes
333 F There are infinitely many points in $\overrightarrow{DG}$, other than point D, that are an endpoint of a segment that is perpendicular to the line that includes $\overrightarrow{DJ}$ and whose other endpoint is in $\overrightarrow{DJ}$. Point D and the two endpoints of each of these segments determine a right triangle that includes a subset of $\angle D$ and a subset of $\overrightarrow{DG}$ as a hypotenuse. * Yes
334 G Yes * Any right triangle that includes $\angle D$ is similar to any other right triangle that includes $\angle D$. For any two similar triangles, the rate pair that represents the comparison of the measures of any two sides of one triangle is equivalent to the rate pair that represents the comparison of the measures of the corresponding sides of the other triangle.
335 J Each of the triangles DFI and DEH is similar to $\triangle DGJ$. This is true because each right triangle that has $\angle D$ as an acute angle is similar to every other right triangle that has $\angle D$ as an acute angle. For any two similar triangles, the rate pair that represents the comparison of the measures of any two sides of one triangle is equivalent to the rate pair that represents the comparison of the measures of the corresponding sides of the other triangle. Thus, 9/15, 6/10, and 3/5 are members of {3/5, . . .}.
332 M Each of the triangles DEH, DFI, and DGJ is similar to each of the other triangles. For any two similar triangles, the rate pair that represents the comparison of the measures of any two sides of one triangle is equivalent to the rate pair that represents the comparison of the measures of the corresponding sides of the other triangle. Thus, 12/15, 8/10, and 4/5 are members of {4/5, . . .}.
334 E Sin 62 ≈ .8829. * Cos 28 ≈ .8829. * In the right triangle represented below, $\angle A$ is the complement of $\angle B$. From the diagram, you can see that $\sin \angle A^\circ/1 \sim a/c$ and $\cos \angle B^\circ/1 \sim a/c$. From $\sin \angle A^\circ/1 \sim a/c$, you know that $\sin \angle A^\circ$ is the quotient of a and c. From $\cos \angle B^\circ/1 \sim a/c$, you know that $\cos \angle B^\circ$ is the quotient of a and c. Therefore, $\sin \angle A^\circ = \cos \angle B^\circ$.



335 H Yes. As the measure of the angle increases, the tangent increases. * Yes. As the measure of the angle increases, the sine increases. * No. As the measure of the angle increases, the cosine decreases.
339 J For two similar triangles, the rate pair

that represents the comparison of the measures of any two sides of one triangle is equivalent to the rate pair that represents the comparison of the measures of the corresponding sides of the other triangles. * Yes

339 L Each of the measures, 39 in. and 69 in., is expressed by a numeral that has two significant digits. Therefore, the numeral that expresses the answer cannot have more than two significant digits.

341 O $(\cos x)/1$. Because you know the measures of the side adjacent to $\angle J$ and of the hypotenuse of $\triangle JKL$, you know a rate pair that is equivalent to $(\cos x)/1$. Therefore, you can use a condition for equivalence to find $\cos x$. Because you do not know the measure of the side opposite $\angle J$, you do not have enough information to find either $\sin x$ or $\tan x$.

341 V $(\sin 77)/1$ or $(\tan 77)/1$ * To find $m(\overline{KL})$, you must use a rate pair that is equivalent to a rate pair that has $m(\overline{KL})$ as one of its components. Both $(\sin 77)/1$ and $(\tan 77)/1$ are equivalent to a rate pair that has $m(\overline{KL})$ as one of its components, but $(\cos 77)/1$ is not.

347 G By using the definition of equivalent ordered pairs * By using the commutative property of multiplication and the symmetric property of equality

347 J Each member of $\{(x, y) | y = 15x\}$ is an ordered pair of numbers, and each member is equivalent to each of the other members.

348 M Yes * Because $y = 15x$ for each x and y, $(x, 15x)$ is the same ordered pair as (x, y). In other words, each ordered pair of numbers whose second component is 15 times as great as its first component is a member of $\{(x, y) | y = 15x\}$.

350 E No * The members of set M are not equivalent to each other. For example, (1, 16) is not equivalent to (2, 64). In a proportional relation, each element must be equivalent to each of the other elements; so set M is not a proportional relation.

351 K The condition that y is equal to the product of 16 and x^2 * Every member of M is a member of $\{(x, y) | y = 16x^2\}$. There are also ordered pairs, other than those in M, that are members of $\{(x, y) | y = 16x^2\}$.

354 G If set S were determined by a condition of the form $y = kx$, then there would be one number, represented by k, by which you could multiply each first component of the members of set S to obtain the second component; but there is no such number. In other words, for each member of S, the second component is not equal to the product of a constant of variation and the first component.

354 K If set S were determined by a condition of the form $y = kx^2$, then there would be one number by which you could multiply the square of each first component of the members of set S to obtain the second component; but there is no such number. For each member of S, the second component is not equal to the product of a constant of variation and the square of the first component.

356 F

x	-3	-2	-1	1	2	3
y	$-\frac{2}{3}$	-1	-2	2	1	$\frac{2}{3}$

356 I You cannot replace x in $y = \frac{2}{x}$ by 0 because you cannot obtain a quotient of 2 and 0; the symbol $\frac{2}{0}$ is meaningless. If you replace y in $y = \frac{2}{x}$ by 0, you obtain $0 = \frac{2}{x}$. There is no real number that satisfies $0 = \frac{2}{x}$.

357 M d is less than b. * In the first quadrant for the locus of the solution set of $y = \frac{2}{x}$, as the first coördinate increases, the second coördinate decreases.

359 J You cannot replace x in $y = \frac{16}{x^2}$ by 0 because the symbol $\frac{16}{0}$ is meaningless. You cannot replace y by 0 because $0 = \frac{16}{x^2}$ does not have a solution in the set of real numbers.

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360 I Yes * The square of any real number is either 0 or a positive real number. * y must be a negative number. This is true because, for any number other than 0, x^2 is positive. Therefore, $-3x^2$, which is the product of a negative number and a positive number, is a negative number.

361 L The same y satisfies each condition.

361 O Points in the first and second quadrants have a positive second coördinate. Because y is never positive, no points with positive second coördinates are in the locus.

362 K x cannot be 0 in $y = \frac{4}{x^2}$. Therefore, x^2 must be positive, and the quotient of 4 and x^2 must be positive. * No

366 D Condition B can be obtained from condition A by using the well-defined property of multiplication. Condition C can be obtained from condition B by using the associative property of multiplication. Condition D can be obtained from condition C by using the reciprocal property together with the definition of multiplication. Condition E can be obtained from condition D by using the identity-element property of multiplication.

367 L x is a variable for the number of months.

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368 R For example: $(^-7)^2 = 49$. $\sqrt{49} = 7$. Therefore, $\sqrt{(^-7)^2} = 7$, and $\sqrt{(^-7)^2} \neq {}^-7$.

369 G x is a variable for the speed in miles per hour and must be a positive number. The universe for x and y is D. For each x, if $x \geqq 0$, then $\sqrt{x^2} = x$. For each x and y, if $x \geqq 0$ and $y \geqq 0$, and if $x = y$, then $\sqrt{x} = \sqrt{y}$.

369 I With twice as many pipes, it would take one half as long to fill the reservoir. * With one third as many pipes, it would take three times as long to fill the reservoir.

379 E You know $\angle A°$ and the measure of the hypotenuse of $\triangle ABC$. You want to find the measure of the side adjacent to $\angle A$.

379 K You want to find the measure of the side adjacent to $\angle A$. From the information given in problem A, you know the measure of the side opposite $\angle A$. From your answer to problem A, you know $\angle A°$, and you can find the tangent of the measure of $\angle A$ from the table of tangents, sines, and cosines. * {28.58}

381 C From the well-defined property of multiplication, you know that, because $rp = x$, the product of x and t is the same as the product of rp and t. * Yes

381 D You should replace p by 1500.

382 L The replacement for p is 0. * No

382 M The replacement for p is negative. * No

384 R $C = \frac{5}{9}(F - 32)$. $\frac{9}{5}C = \frac{9}{5} \cdot \frac{5}{9}(F - 32)$. Well-defined property of multiplication. $\frac{9}{5}C = F - 32$. Reciprocal property and identity-element property of multiplication. $\frac{9}{5}C + 32 = F - 32 + 32$. Well-defined property of addition. $\frac{9}{5}C + 32 = F$. Additive-inverse property. $F = \frac{9}{5}C + 32$. Symmetric property of equality

386 H Replace p in $p = 4s$ by the given perimeter. Then find the replacement for s that satisfies the condition you obtain.

387 C Yes. The segment that represents the width is perpendicular to the lines that include a pair of opposite sides of the rectangle. * Yes * The formula $A = ab$ is a formula for finding the area of a parallelogram, and a rectangle is a parallelogram. You can use the width of the rectangle as the replacement for a because the width is the measure of the altitude. You can use the length as the replacement for b because the length is the measure of the base.

387 D Replace A in $A = lw$ by the area of the rectangle and w by the width of the rectangle. Then find the l that satisfies the condition you obtain.

388 I $\overline{AB}$ is not perpendicular to the lines that include the parallel sides of the trapezoid. * $\overline{CD}$ is not perpendicular to the lines that include the parallel sides of the trapezoid.

389 K a represents the measure of the altitude of $\triangle JKM$. b_1 represents the measure of the base of $\triangle JKM$. Thus, the area of $\triangle JKM$ is represented by $\frac{1}{2}ab_1$. * a represents the measure of the altitude of $\triangle KLM$. b_2 represents the measure of the base of $\triangle KLM$. Thus, the area of $\triangle KLM$ is represented by $\frac{1}{2}ab_2$.

397 A No. Because the two copies are new, they look alike; and because they are on the same shelf, one is not easier to reach than the other. Therefore, it does not seem likely that you would choose one copy in preference to the other.

397 B Because the pieces of pie look alike, the chance of someone's choosing one of the pieces is the same for each piece.

397 C Yes. Because there are 2 pieces of lemon candy for each piece of mint candy, it seems more likely that you would take a piece of lemon candy rather than a piece of mint candy.

400 I For the experiment described in D1, a favorable outcome is the selection of the blue marble. For the experiment described in D2, a favorable outcome is for the pointer to stop on the numeral 4. For the experiment described in D5, a favorable outcome is for Mrs. Andrews to pick up a hard-boiled egg from the shelf.

400 J For the experiment described in D1, the probability is the quotient of 1 and 2. For the experiment described in D2, the probability is the quotient of 1 and 6. For the experiment described in D5, the probability is the quotient of 3 and 10.

400 N Yes. The quotient of any positive real number and itself is 1.

400 O The probability of a favorable outcome is 0. The quotient of 0 and any positive real number is 0.

400 P No. The greatest number of favorable outcomes is the total number of possible outcomes. * No. For the probability to be greater than 1, there would have to be more favorable outcomes than possible outcomes.

400 Q No. The least possible number of favorable outcomes is 0. * The quotient of any non-negative real number

h \ a	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)

and a positive real number is a non-negative real number.

402 C There are 4 ways a blue tag may be drawn and 5 ways a red tag may be drawn. Therefore, there are 4×5, or 20, ways a blue and a red tag may be drawn.

403 M Yes. The sample space represented in D2 is obtained by matching each member of set B, in turn, with each member of set R. Similarly, the Cartesian set $B \times R$ is obtained by matching each member of set B, in turn, with each member of set R.

403 C See the diagram at the bottom of the page. * Yes * 24

403 G In this experiment, each sample point is a single object rather than an ordered pair of objects.

404 B Yes. Each element of set A is contained in $B \times R$. * Yes

404 D 20 * Yes. $B \times R$ contains all the possible outcomes of the experiment, and each member of $B \times R$ is an outcome of the experiment.

406 B $\{(b, r) | 1 \leqq b \leqq 4 \wedge 1 \leqq r \leqq 5\}$.$U = I_p \times I_p$

407 J Yes * Since set S has 20 elements and $\{(2, 3)\}$ has 1 element, the probability of $\{(2, 3)\}$ is 1 in 20, or $\frac{1}{20}$. * Since $A \cap B = \{(2, 3)\}$, and the probability of $\{(2, 3)\}$ is $\frac{1}{20}$, the probability of $A \cap B$ is $\frac{1}{20}$.

407 M 8 * The probability of $A \cup B$ is 8 in 20, or $\frac{2}{5}$.

408 A Event X is $\{(1, 3), (2, 2), (3, 1)\}$. * The sum of the numbers named on a blue tag and a red tag is 4.

408 B Event Y is $\{(1, 4), (2, 4), (3, 4), (4, 4)\}$. * The number 4 is named on the red tag.

408 C No * The least number named on a tag is 1. If a number greater than 3 is named on one tag, the sum of the numbers named on the two tags is greater than 4.

408 G 0 * The empty set * Since $X \cap Y$ has no members, the probability of $X \cap Y$ is 0 in 20, or 0.

409 F Yes * D and $\overline{D}$ are events of a finite sample space that have no elements in common, and their intersection is the empty set.

410 O Yes * You know that, if you consider the universe as an event, the probability of this event is 1. Since the union of two complementary events is the universe, then the probability of an event or its complement is 1.

410 P Yes * Find the difference of 1 and the probability of the given event.

413 V To obtain the probability of $R \cap T$, find the product of the probability of R and the probability of T.

413 W The probability that you will draw a blue tag with a numeral for an even number is $\frac{2}{4}$, or $\frac{1}{2}$. The probability that you will draw a red tag with a numeral for an odd number is $\frac{3}{5}$. Because these two events are independent events, the probability of their intersection is $\frac{1}{2} \times \frac{3}{5}$, or $\frac{3}{10}$.

413 E Yes * No. The probability of G and H is not equal to the product of the probability of G and the probability of H; hence, they are not independent events.

413 F Mutually exclusive events are disjoint sets. The intersection of any two disjoint sets is the empty set.

415 B No * Because a tack is not well balanced, it does not seem that the two outcomes of the experiment are equally likely.

416 I .25 × .25, or about 0625

416 J Use x as a variable for the number of times U will occur in the next 100 tosses. You can use the condition $.25/1 \sim x/100$ to determine the number of times U will occur in the next 100 tosses. 25 satisfies $.25/1 \sim x/100$; therefore, it is likely that U will occur 25 times in the next 100 tosses.

417 G For every 883,342 persons of age 40, $883342 - 677771$ will die by age 60. Therefore, the probability that Mr. Colfax will die by age 60 is $\frac{883342 - 677771}{883342}$, or $\frac{205571}{883342}$.

423 J From the data given in D7, you know that the total number of disks drawn was 200. You can use a condition for equivalence to find the probability of drawing a disk of a certain color in one drawing. For example, the probability of drawing a red disk in one drawing can be found by using the condition $64/200 \sim x/1$. The solution of this condition is .32. If 400 more drawings were to be made, you can use the condition $.32/1 \sim x/400$ to predict the number of times a red disk would be drawn. Thus, you could predict that 128 red disks would be drawn in the next 400 drawings. Using a similar method for each of the other colors, you could predict that 76 blue disks, 148 or 152 white disks, and 44 or 48 green disks would be drawn in the next 400 drawings.

424 O He might use a histogram because it is very easy to tell at a glance that more viewers watch program B than any of the other programs. With a histogram, it would not be necessary to compare the exact number of viewers of each program as you must with a frequency array.

427 M Since 87 is the first measure given in the problem, replace x_1 by 87; since 92 is the second measure, replace x_2 by 92; since 78 is the third measure, replace x_3 by 78; since 90 is the fourth measure, replace x_4 by 90; since 86 is the fifth measure, replace x_5 by 86; since 94 is the last measure, replace x_n by 94. Since there are 6 measures, replace n by 6.

428 F 23 students * The

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score of the twelfth student would be the middle score because 11 students have either the same score or a lower score and 11 students have either the same score or a higher score.

428 G If you consider the students whose scores were 75 as the first and second students, the student whose score was 78 as the third student, the students whose scores were 83 as the fourth through ninth students; then you find that the 12th student is one of the students whose score was 85.

Squares and square roots

n	n^2	$\sqrt{n}$	n	n^2	$\sqrt{n}$	n	n^2	$\sqrt{n}$	n	n^2	$\sqrt{n}$
1	1	1.000	56	3136	7.483	111	12,321	10.536	166	27,556	12.884
2	4	1.414	57	3249	7.550	112	12,544	10.583	167	27,889	12.923
3	9	1.732	58	3364	7.616	113	12,769	10.630	168	28,224	12.961
4	16	2.000	59	3481	7.681	114	12,996	10.677	169	28,561	13.000
5	25	2.236	60	3600	7.746	115	13,225	10.724	170	28,900	13.038
6	36	2.449	61	3721	7.810	116	13,456	10.770	171	29,241	13.077
7	49	2.646	62	3844	7.874	117	13,689	10.817	172	29,584	13.115
8	64	2.828	63	3969	7.937	118	13,924	10.863	173	29,929	13.153
9	81	3.000	64	4096	8.000	119	14,161	10.909	174	30,276	13.191
10	100	3.162	65	4225	8.062	120	14,400	10.954	175	30,625	13.229
11	121	3.317	66	4356	8.124	121	14,641	11.000	176	30,976	13.266
12	144	3.464	67	4489	8.185	122	14,884	11.045	177	31,329	13.304
13	169	3.606	68	4624	8.246	123	15,129	11.091	178	31,684	13.342
14	196	3.742	69	4761	8.307	124	15,376	11.136	179	32,041	13.379
15	225	3.873	70	4900	8.367	125	15,625	11.180	180	32,400	13.416
16	256	4.000	71	5041	8.426	126	15,876	11.225	181	32,761	13.454
17	289	4.123	72	5184	8.485	127	16,129	11.269	182	33,124	13.491
18	324	4.243	73	5329	8.544	128	16,384	11.314	183	33,489	13.528
19	361	4.359	74	5476	8.602	129	16,641	11.358	184	33,856	13.565
20	400	4.472	75	5625	8.660	130	16,900	11.402	185	34,225	13.601
21	441	4.583	76	5776	8.718	131	17,161	11.446	186	34,596	13.638
22	484	4.690	77	5929	8.775	132	17,424	11.489	187	34,969	13.675
23	529	4.796	78	6084	8.832	133	17,689	11.533	188	35,344	13.711
24	576	4.899	79	6241	8.888	134	17,956	11.576	189	35,721	13.748
25	625	5.000	80	6400	8.944	135	18,225	11.619	190	36,100	13.784
26	676	5.099	81	6561	9.000	136	18,496	11.662	191	36,481	13.820
27	729	5.196	82	6724	9.055	137	18,769	11.705	192	36,864	13.856
28	784	5.292	83	6889	9.110	138	19,044	11.747	193	37,249	13.892
29	841	5.385	84	7056	9.165	139	19,321	11.790	194	37,636	13.928
30	900	5.477	85	7225	9.220	140	19,600	11.832	195	38,025	13.964
31	961	5.568	86	7396	9.274	141	19,881	11.874	196	38,416	14.000
32	1024	5.657	87	7569	9.327	142	20,164	11.916	197	38,809	14.036
33	1089	5.745	88	7744	9.381	143	20,449	11.958	198	39,204	14.071
34	1156	5.831	89	7921	9.434	144	20,736	12.000	199	39,601	14.107
35	1225	5.916	90	8100	9.487	145	21,025	12.042	200	40,000	14.142
36	1296	6.000	91	8281	9.539	146	21,316	12.083	201	40,401	14.177
37	1369	6.083	92	8464	9.592	147	21,609	12.124	202	40,804	14.213
38	1444	6.164	93	8649	9.644	148	21,904	12.166	203	41,209	14.248
39	1521	6.245	94	8836	9.695	149	22,201	12.207	204	41,616	14.283
40	1600	6.325	95	9025	9.747	150	22,500	12.247	205	42,025	14.318
41	1681	6.403	96	9216	9.798	151	22,801	12.288	206	42,436	14.353
42	1764	6.481	97	9409	9.849	152	23,104	12.329	207	42,849	14.387
43	1849	6.557	98	9604	9.899	153	23,409	12.369	208	43,264	14.422
44	1936	6.633	99	9801	9.950	154	23,716	12.410	209	43,681	14.457
45	2025	6.708	100	10,000	10.000	155	24,025	12.450	210	44,100	14.491
46	2116	6.782	101	10,201	10.050	156	24,336	12.490	211	44,521	14.526
47	2209	6.856	102	10,404	10.100	157	24,649	12.530	212	44,944	14.560
48	2304	6.928	103	10,609	10.149	158	24,964	12.570	213	45,369	14.595
49	2401	7.000	104	10,816	10.198	159	25,281	12.610	214	45,796	14.629
50	2500	7.071	105	11,025	10.247	160	25,600	12.649	215	46,225	14.663
51	2601	7.141	106	11,236	10.296	161	25,921	12.689	216	46,656	14.697
52	2704	7.211	107	11,449	10.344	162	26,244	12.728	217	47,089	14.731
53	2809	7.280	108	11,664	10.392	163	26,569	12.767	218	47,524	14.765
54	2916	7.348	109	11,881	10.440	164	26,896	12.806	219	47,961	14.799
55	3025	7.416	110	12,100	10.488	165	27,225	12.845	220	48,400	14.832

Tangents, sines, and cosines

$\angle A^\circ = x.$	$\tan x$	$\sin x$	$\cos x$	$\angle A^\circ = x.$	$\tan x$	$\sin x$	$\cos x$
1	.0175	.0175	.9998	46	1.0355	.7193	.6947
2	.0349	.0349	.9994	47	1.0724	.7314	.6820
3	.0524	.0523	.9986	48	1.1106	.7431	.6691
4	.0699	.0698	.9976	49	1.1504	.7547	.6561
5	.0875	.0872	.9962	50	1.1918	.7660	.6428
6	.1051	.1045	.9945	51	1.2349	.7771	.6293
7	.1228	.1219	.9925	52	1.2799	.7880	.6157
8	.1405	.1392	.9903	53	1.3270	.7986	.6018
9	.1584	.1564	.9877	54	1.3764	.8090	.5878
10	.1763	.1736	.9848	55	1.4281	.8192	.5736
11	.1944	.1908	.9816	56	1.4826	.8290	.5592
12	.2126	.2079	.9781	57	1.5399	.8387	.5446
13	.2309	.2250	.9744	58	1.6003	.8480	.5299
14	.2493	.2419	.9703	59	1.6643	.8572	.5150
15	.2679	.2588	.9659	60	1.7321	.8660	.5000
16	.2867	.2756	.9613	61	1.8040	.8746	.4848
17	.3057	.2924	.9563	62	1.8807	.8829	.4695
18	.3249	.3090	.9511	63	1.9626	.8910	.4540
19	.3443	.3256	.9455	64	2.0503	.8988	.4384
20	.3640	.3420	.9397	65	2.1445	.9063	.4226
21	.3839	.3584	.9336	66	2.2460	.9135	.4067
22	.4040	.3746	.9272	67	2.3559	.9205	.3907
23	.4245	.3907	.9205	68	2.4751	.9272	.3746
24	.4452	.4067	.9135	69	2.6051	.9336	.3584
25	.4663	.4226	.9063	70	2.7475	.9397	.3420
26	.4877	.4384	.8988	71	2.9042	.9455	.3256
27	.5095	.4540	.8910	72	3.0777	.9511	.3090
28	.5317	.4695	.8829	73	3.2709	.9563	.2924
29	.5543	.4848	.8746	74	3.4874	.9613	.2756
30	.5774	.5000	.8660	75	3.7321	.9659	.2588
31	.6009	.5150	.8572	76	4.0108	.9703	.2419
32	.6249	.5299	.8480	77	4.3315	.9744	.2250
33	.6494	.5446	.8387	78	4.7046	.9781	.2079
34	.6745	.5592	.8290	79	5.1446	.9816	.1908
35	.7002	.5736	.8192	80	5.6713	.9848	.1736
36	.7265	.5878	.8090	81	6.3138	.9877	.1564
37	.7536	.6018	.7986	82	7.1154	.9903	.1392
38	.7813	.6157	.7880	83	8.1443	.9925	.1219
39	.8098	.6293	.7771	84	9.5144	.9945	.1045
40	.8391	.6428	.7660	85	11.4301	.9962	.0872
41	.8693	.6561	.7547	86	14.3007	.9976	.0698
42	.9004	.6691	.7431	87	19.0811	.9986	.0523
43	.9325	.6820	.7314	88	28.6363	.9994	.0349
44	.9657	.6947	.7193	89	57.2900	.9998	.0175
45	1.0000	.7071	.7071				

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